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SOME NEW COINCIDENCE POINT RESULTS IN B-METRIC SPACES USING A SIMULATION FUNCTION

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ABSTRACT. In this paper, we prove a coincidence point theorem in the context of b-metric spaces. The result is achieved by extending the known conditions of existence and uniqueness through the use of simulation functions. An example is also provided to support the obtained result.

1. INTRODUCTION AND PRELIMINARIES

The fixed point theory in metric spaces is located at the crossroads of nonlinear functional analysis and topology. It has its roots in Liouville (1837) when it was applied for solving differential equations and further developed in Picards work in 1890. A turning point was Banach's celebrated principle of contraction (1922) of complete normed spaces—subsequently called Banach spaces—which created an abstract basis for Picard's method of iteration. Afterward, Caccioppoli (1931) gave a first generalization of Banach's theorem which has motivated a great deal of research related to extension and generalization of fixed point-theorems.
Informed by recent findings, notably the works of E. Karapinar and others (see [3], [4], [5], [6], and [8]), In this study we intent to apply some well-known results concerning Metric spaces via simulation function. These results can have useful applications based from the following novel proofs of existence, and uniqueness of fixed points on b-metric spaces. Using pre-existing results we can combine and then extend the results to b-metric spaces and thereby obtain other fixed-point and coincidence point theorems.

Definition 1.1. [2] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, +\infty[$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied
(1) $d(x, y) = 0$ if and only if $x = y$
(2) $d(x, y) = d(y, x)$
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

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A triplet (X, d, s), is called a b-metric space. We observe that a metric space is included in the class of b-metric spaces. In fact, the notions of convergent sequence, Cauchy sequence and complete space are defined as in metric spaces.

Next, we give some examples of b-metric spaces.

Example 1.2. [7]
Let $X = [0, 1]$ and $d : X \times X \to [0, +\infty[$ be defined by

$$d(x, y) = (x - y)^2,$$

for all $x, y \in X$.

Example 1.3. [7]
Let $C_b(X) = \{f : X \to \mathbb{R} : \|f\|_\infty = \sup_{x\in X} |f(x)| < +\infty\}$ and let

$$\|f\| = \sqrt[3]{\|f^3\|_\infty}.$$

The function $d : C_b(X) \times C_b(X) \to [0, +\infty[$ defined by

$$d(f, g) = \|f - g\|, \quad \text{for all } f, g \in C_b(X)$$

is a b-metric with constant $s = \sqrt[3]{4}$.

Definition 1.4. [8] A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions
(ζ_1) $\zeta(0, 0) = 0$;
(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Let $\mathcal{Z}$ denote the family of all simulation functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$. Due to the axiom (ζ_2), we have $\zeta(t, t) < 0$ for all $t > 0$.

Example 1.5. [6] Let $\phi_i : [0, \infty) \to [0, \infty)$ be continuous functions with $\phi_i(t) = 0$ if and only if, $t = 0$. For $i = 1, 2$, we define the mapping ζ_1:
$\zeta_1(t, s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in [0, \infty)$, where $\phi_1(t) < t \le \phi_2(t)$ for all $t > 0$.
ζ_1 is a simulation function.

Example 1.6. [8] The functions $f, g : [0, \infty)^2 \to (0, \infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.
We define the mapping ζ_2: $\zeta_2(t, s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0, \infty)$, which is also a simulation function .

Definition 1.7. [10] A mapping $G : [0, +\infty)^2 \to \mathbb{R}$ has the property C_G, if there exists an $C_G \ge 0$ such that
(1) $G(s, t) > C_G$ implies $s > t$;
(2) $G(t, t) \le C_G$, for all $t \in [0, +\infty)$.

Definition 1.8. [10] A C_G-simulation function is a mapping $\zeta : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following
(a) For all $t, s > 0$ we have $\zeta(t, s) < G(s, t)$, where $G : [0, +\infty)^2 \to \mathbb{R}$ is a function that has the property C_G;
(b) If $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \zeta\left(t_n, s_n\right) < C_G.$$

Let $\mathcal{Z}_G$ be the family of all C_G-simulation functions $\zeta : [0, +\infty)^2 \to \mathbb{R}$.

Remark. *Each simulation function as in Definition 1.4 is also a C_G-simulation function as in Definition 1.8, but the converse is not true. For this claim we can use the Example of the function $G(s,t) = s - t$ (see* [6]*).*

Lemma 1.1. [9] *Let (X, d) be a complete b-metric space and let $\{u_n\}$ be a sequence in X.Assume that there exist $r \in [0, 1)$ satisfying*

$$d\left(u_{n+1}, u_{n+2}\right) \leq rd\left(u_n, u_{n+1}\right)$$

for any $n \in N$. Then $\{u_n\}$ is Cauchy sequence in X.

For the following, Let f and g be self maps of a b-metric space (X, d).

Definition 1.9. [6] If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.
if also $x = w$, then x is called a common fixed point of f and g.

Definition 1.10. [1] The mappings f and g are compatible if and only if
whenever $\{x_n\}$ is a sequence in x such that

$$\lim_{n\to\infty} f\left(x_n\right) = \lim_{n\to\infty} g\left(x_n\right) = t$$

for some $t \in X$, we have

$$\lim_{n\to\infty} d\left(gf\left(x_n\right), fg\left(x_n\right)\right) = 0.$$

Definition 1.11. [1] The pair (f, g) is weakly compatible if f and g commute at their coincidence points, meaning
for all $x \in X$ such that $w = fx = gx$ we have

$$gfx = fgx.$$

Remark. [1] *If the pair (f, g) is compatible, then it is also weakly compatible, but the converse is not true.*

Definition 1.12. [10] A sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subseteq X$ is a Picard-Jungck sequence of the pair (f, g)(based on $x_0 \in X$) if $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

Remark. [9] *If we have $f(X) \subset g(X)$ or $g(X) \subset f(X)$, then it is certain that a Picard-Jungck sequence exists for the pair (f, g), but the converse is not true.*

Theorem 1.2. [10] *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is a unique common fixed point of f and g.*

Definition 1.13. [10] A mapping f is called a $(\mathcal{Z}_G, g)$-contraction if there exists $\zeta \in \mathcal{Z}_G$ such that

$$\zeta(d(fx, fy), d(gx, gy)) \geq C_G \tag{1.1}$$

for all $x, y \in X$ with $gx \neq gy$.
In the case, $g = i_X$ (identity mapping on X) and $C_G = 0$ we get what is called a $\mathcal{Z}$-contraction.

2. MAIN RESULTS

Now in this section, we prove some results on the existence and uniqueness of the common fixed point by using simulation functions in the framework of b-metric spaces.

Let us consider F the family of mappings $H : [0,+\infty) \to [0,+\infty)$ that satisfies the following conditions

$$\exists r \in [0,1) \text{ so that } 0 < H(t) \leq r.t \quad \text{for all} \quad t \in (0,+\infty) \text{ and } H(0) = 0. \tag{2.1}$$

Theorem 2.1. *Let (X,d,s) be a b-metric space, $f,g : X \to X$ be self-mappings, assuming that there exists $\zeta \in \mathcal{Z}_G$ and $H \in F$such that*

$$\zeta(d(fx,fy),H(d(gx,gy))) \geq C_G \tag{2.2}$$

for all $x,y \in X$ with $gx \neq gy$.
Suppose also that there exists a Picard-Jungck sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of (f,g).
Also assume that at least one of the following conditions hold
(i) $(g(X),d)$ is complete and f and g are weakly compatible;
(ii) (X,d) is complete, g is continuous and (f,g) is compatible.
Then f and g have a unique common fixed point.

Proof. First of all we shall prove that the point of coincidence of f and g is unique (if it exists). Suppose, on the contrary, that z_1 and z_2 are distinct points of coincidence of f and g. Then, by definition of a point of coincidence, there exist two points v_1 and v_2 $(v_1 \neq v_2)$ such that $fv_1 = gv_1 = z_1$ and $fv_2 = gv_2 = z_2$. Applying condition (2.2) we obtain

$$\begin{aligned} C_G \leq \zeta\left(d\left(fv_1,fv_2\right),H(d\left(gv_1,gv_2\right))\right) &= \zeta\left(d\left(z_1,z_2\right),H(d\left(z_1,z_2\right))\right) \\ &< G\left(H(d\left(z_1,z_2\right)),d\left(z_1,z_2\right)\right). \end{aligned}$$

According to definition 1.7 this would mean that

$$H(d\left(z_1,z_2\right)) > d\left(z_1,z_2\right)$$

which is a contradiction with the properties of H.
To prove that f and g have a point of coincidence, consider a Picard-Jungck sequence $\{x_n\}$ such that $y_n = fx_n = gx_{n+1}$ where $n \in \mathbb{N}\cup\{0\}$.
If $y_k = y_{k+1}$ for some $k \in \mathbb{N}\cup\{0\}$, then $gx_{k+1} = y_k = y_{k+1} = fx_{k+1}$ and f and g have a point of coincidence.
Otherwise, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}\cup\{0\}$. By substituting $x = x_{n+1}, y = x_{n+2}$ in (2.2) we obtain

$$\begin{aligned} C_G &\leq \zeta\left(d\left(fx_{n+1},fx_{n+2}\right),H(d\left(gx_{n+1},gx_{n+2}\right))\right) \\ &= \zeta\left(d\left(y_{n+1},y_{n+2}\right),H(d\left(y_n,y_{n+1}\right))\right) \\ &< G\left(H(d\left(y_n,y_{n+1}\right)),d\left(y_{n+1},y_{n+2}\right)\right). \end{aligned}$$

Using Definition 1.7, we have $H(d\left(y_n,y_{n+1}\right)) > d\left(y_{n+1},y_{n+2}\right)$,
Hence, for all $n \in \mathbb{N}\cup\{0\}$ we get

$$d\left(y_{n+1},y_{n+2}\right) < r.d\left(y_n,y_{n+1}\right).$$

Now, suppose that condition (i) holds, i.e., $(g(X),d)$ is complete, then according to Lemma 1.1 the sequence $\{y_n\}$ is a Cauchy sequence in $(g(X),d)$ .Then there exists $v \in X$ such that $gx_n \to gv$ as $n \to \infty$. We shall prove that $fv = gv$.
It is clear that we can suppose $y_n \neq fv, gv$ for all $n \in \mathbb{N}\cup\{0\}$. Therefore, by (2.2) we

have

$$C_G \leq \zeta\left(d\left(fx_n, fv\right), H(d\left(gx_n, gv\right))\right) < G\left(H(d\left(gx_n, gv\right)), d\left(fx_n, fv\right)\right).$$

By definition 1.7, we get

$$d\left(fx_n, fv\right) < H(d\left(gx_n, gv\right)) < d\left(gx_n, gv\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It follows that $fx_n \rightarrow fv$ as $n \rightarrow \infty$, hence, $fv = gv$ is a (unique) point of coincidence of f and g.
Furthermore, since f and g are weakly compatible , it follows from theorem 1.2 that they possess a unique common fixed point.
Finally, suppose that condition (ii) holds. Since (X, d) is complete, Lemma 1.1 guarantees that the sequence $\{y_n\}$ is a Cauchy sequence in (X, d) . which means that there exists $v \in X$ such that $fx_n \rightarrow v$ as $n \rightarrow \infty$. As g is continuous, $g\left(fx_n\right) \rightarrow gv$ as $n \rightarrow \infty$. Consider

$$\begin{aligned} C_G &\leq \zeta\left(d\left(f\left(gx_n\right), fv\right), H(d\left(g\left(gx_n\right), gv\right))\right) \\ &< G\left(H(d\left(g\left(gx_n\right), gv\right)), d\left(f\left(gx_n\right), fv\right)\right). \end{aligned}$$

Using Definition (1.7) and continuity of g, we have

$$H(d\left(g\left(gx_n\right), gv\right)) > d\left(f\left(gx_n\right), fv\right)$$

this would mean that

$$d\left(f\left(gx_n\right), fv\right) < r.d\left(g\left(gx_n\right), gv\right) = r.d\left(g\left(fx_{n-1}\right), gv\right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It implies that $d\left(f\left(gx_n\right), fv\right) \rightarrow 0$, as $n \rightarrow \infty$. Further, as f and g are compatible, we have

$$d(fv, gv) \leq s\Big(d\big(fv, f(gx_n)\big) + s^2\big(d(f(gx_n), g(fx_n)) + d(g(fx_n), gv)\big)\Big)$$

and

$$\begin{aligned} s \cdot \Big(d(fv, f(gx_n)) + s^2\big(d(f(gx_n), g(fx_n)) + d(g(fx_n), gv)\big)\Big) &\rightarrow s \cdot 0 + s^2(0+0) \\ &= 0. \end{aligned}$$

Hence, the result is established in both cases, i.e. the mappings f and g have a unique point of coincidence.
Since f and g are compatible then they are also weakly compatible which means that according to theorem 1.2 $f$and g possess a unique common fixed point. This completes the proof. □

Theorem 2.2. *Let us consider the same hypotheses as theorem 2.1, this time assume that at least one of the following conditions hold*
(i) $f(X) \subset g(X)$, $(f(X), d)$ is complete and f and g are weakly compatible.
(ii) (X, d) is complete, g is continuous and (f, g) is compatible.
Then f and g have a unique common fixed point.

Proof. It is clear that we will use the same proof as theorem 2.1 to conclude that the sequence $\{y_n\}$ is a Cauchy sequence in $(f(X), d)$, since $(f(X), d)$ is complete (condition (i)), then according to Lemma 1.1 the sequence $\{y_n\}$ is a Cauchy sequence in $(f(X), d)$. This means that there exists $u \in X$ such that $fx_n \rightarrow fu$ as $n \rightarrow \infty$, this would also mean that

$$fx_{n-1} \rightarrow fu \text{ as } n \rightarrow \infty$$

therefore

$$gx_n \to fu \text{ as } n \to \infty$$

and we know that $f(X) \subset g(X)$ so
There exists $v \in X$ such that $gx_n \to gv$ as $n \to \infty$.
Now we can use a similar method as we did with condition (i) in theorem 2.1 to prove that $fv = gv$ is a (unique) point of coincidence of f and g.
Further, since f and g are weakly compatible, then according to theorem 1.2 they have a unique common fixed point, the proof remains the same for condition (ii). □

Theorem 2.3. *Let us consider the same hypotheses as theorems 2.1 and 2.2, also assume that at least one of the following conditions hold*
(i) $f(X) \subset g(X)$ *and* $(f(X), d)$ *is complete.*
(ii) $(g(X), d)$ *is complete.*
Then f *and* g *have a unique point of coincidence.*

Proof. The proof of this theorem has already been established in the previous two theorems, however, this theorem does not require the compatibility or weak compatibility of the pair (f, g). □

Corollary 2.4. *Any one of the theorems 2.1, 2.2 and 2.3, could be used to demonstrate the existence and uniqueness of the fixed point of a single function* f *if we consider that* $g = i_X$*, it is sufficient to prove that the pair* (f, g) *has a unique point of coincidence to conclude this result.*

Remark. *We can obtain the main result in* [10] *if we consider* $H = i_{[0,+\infty)}$ *and* $s = 1$ *in theorem 2.1 .*
Also, we can obtain the main result in [9] *if we consider* $s = 1$ *and* $C_G = 0$ *in theorem 2.2.*

Example 2.1. Let $X = [0, 1]$ be endowed with the b-metric $d(x, y) = (x - y)^2$ for all $x, y \in [0, 1]$ with $s = 2$. Define the mappings $f, g : [0, 1] \to [0, +\infty)$ given, for all $x \in [0, 1]$, by

$$fx = x + 2, \quad gx = 4x + e^{2x}.$$

We want to solve the nonlinear equation

$$x + 2 = 4x + e^{2x}.$$

To this end, we apply Theorem 2.3 using the simulation function $\zeta(t, s) = \frac{9}{10}(s - t)$ for $s, t \in [0, +\infty)$, with the constant $C_F = 0$, and the following mapping with the property C_F

$$F(s, t) = s - t,$$

and H is given by

$$H(t) = \frac{1}{2} t \text{ if } t \in [0; +\infty).$$

Now, we compute

$$\begin{aligned}
\zeta(d(fx, fy), H(d(gx, gy))) &= \frac{9}{10}\left(H(d(gx, gy)) - d(fx, fy)\right) \\
&= \frac{9}{10}\left(\frac{1}{2}\left(4(x-y) + \left(e^{2x} - e^{2y}\right)\right)^2 - (x-y)^2\right) \\
&= \frac{9}{10}\left(7(x-y)^2 + \frac{1}{2}\left(e^{2x} - e^{2y}\right)^2 + 4(x-y)\left(e^{2x} - e^{2y}\right)\right) \\
&\geq 0.
\end{aligned}$$

Moreover, since $g(X) = [1, 4 + e^2]$, by theorem 2.3 (condition (ii)) the result follows. Hence, the nonlinear equation

$$x + 2 = 4x + e^{2x}.$$

has a unique solution, which corresponds to the common fixed point of f and g.

3. Conclusions

In this work, we combined and extended two previous results to the setting of a b-metric space, thus we established new existence and uniqueness theorems for fixed and coincidence points. These findings highlight the versatility of simulation methods in generalized metric frameworks.

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