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THEORETICAL RESULTS ON NEW HARDY-HILBERT-TYPE INEQUALITIES

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Abstract. Hardy-Hilbert-type integral inequalities lie at the heart of mathematical analysis. They have been the subject of much research. In this article, we make a contribution to the field by examining two new two-parameter modifications of the classical Hardy-Hilbert integral inequality. We derive the closed-form expression of the optimal constant for each modification. We also present supplementary results, including one-function and primitive variants. All proofs are provided in full, with each step justified, to ensure the article is self-contained.

1. Introduction

The Hardy-Hilbert integral inequality, first introduced by G. H. Hardy in [11], is considered a classic of mathematical analysis. It provides an upper bound on the weighted double integral of the product of two functions in terms of their unweighted integral norms. This inequality is particularly useful in the fields of Fourier analysis, operator theory and integral inequality theory. Its formal statement is given below. Let $p > 1$, $q = p/(p-1)$ (i.e., the Hölder conjugate of p), and $f, g : [0, \infty) \mapsto [0, \infty)$ be two (non-negative) functions such that

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \int_0^{\infty} g^q(y)dy < \infty.$$

Then we have

$$\int_0^{\infty}\int_0^{\infty} \frac{1}{x+y} f(x)g(y)dxdy \leq \frac{\pi}{\sin(\pi/p)} \left[\int_0^{\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{\infty} g^q(y)dy\right]^{1/q}. \tag{1.1}$$

By identification, the kernel function associated with the double integral is given by $k(x, y) = 1/(x + y)$. It exhibits a symmetric decay that plays a central role in the inequality. The constant $\pi/\sin(\pi/p)$ is optimal, i.e., the inequality cannot hold with a smaller constant for two functions f and g satisfying the integrability conditions. In the particular case when

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$p = 2$, the inequality reduces to the classical Hilbert integral inequality, with the constant simplified to π, as follows:

$$\int_0^\infty \int_0^\infty \frac{1}{x+y} f(x)g(y)dxdy \leq \pi \left[\int_0^\infty f^2(x)dx\right]^{1/2} \left[\int_0^\infty g^2(y)dy\right]^{1/2}.$$

Over the years, the Hardy-Hilbert integral inequality has inspired substantial research, leading to a multitude of improvements and extensions. Key contributions can be found in [12, 17, 21, 19, 16, 20, 13, 14, 9, 15, 2, 3, 4, 1, 5, 6, 7]. A selection of these advances are highlighted in the reference book [22].

In particular, a significant generalization was proposed in [18]. This involves introducing a power parameter $\alpha > 0$ into the kernel function. A formal statement is given below. Let p, q, and $f, g : [0, \infty) \mapsto [0, \infty)$ be as before, with the same integrability assumptions, and $\alpha > 2 - \min(p, q)$. Then we have

$$\begin{aligned}&\int_0^\infty \int_0^\infty \frac{1}{(x+y)^\alpha} f(x)g(y)dxdy \\ &\leq B_{eta}\left(1 - \frac{2-\alpha}{p}, 1 - \frac{2-\alpha}{q}\right) \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}, \qquad (1.2)\end{aligned}$$

where $B_{eta}(a, b)$ denotes the classical beta function at $a, b > 0$, defined by

$$B_{eta}(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

The kernel function associated with the double integral is thus given by $k(x, y) = 1/(x + y)^\alpha$. Setting $\alpha = 1$ in Equation (1.2) yields the classical inequality in Equation (1.1), since

$$B_{eta}\left(1 - \frac{1}{p}, 1 - \frac{1}{q}\right) = \frac{\pi}{\sin(\pi/p)},$$

using the well-known identity for the beta function in terms of the sine function.

The study of Hardy-Hilbert-type integral inequalities is a dynamic field of research, driven by continuous advancements in analysis. To illustrate this, we present two recent results from [8]. Let p, q and $f, g : [0, \infty) \mapsto [0, \infty)$ be as before, with the same integrability assumptions, and $\beta > 0$. The first main result in [8] is then given by

$$\begin{aligned}&\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{|x-y|^{\beta-1}}{(x+y)^{\beta+1}} f(x)g(y)dxdy \\ &\leq \frac{1}{\beta} \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}.\end{aligned}$$

This one-parameter inequality can be viewed as a modified Hardy-Hilbert integral inequality. The kernel function associated with the double integral is thus given by $k(x, y) = x^{1/p} y^{1/q} |x - y|^{\beta-1}/(x + y)^{\beta+1}$. It is asymmetric except for the case $p = 2$. A notable feature of this result is the optimality and simplicity of the constant $1/\beta$, which makes it particularly useful for obtaining sharp estimates and norm inequalities.

In [8], this result is complemented by a multiplicative analogue, in which the kernel function involves products rather than sums. A formal statement is presented below. Let p, q and $f, g : [0, \infty) \mapsto [0, \infty)$ be as before, with the same integrability assumptions, and

$\beta > 0$. Then we have

$$\int_0^{\infty} \int_0^{\infty} x^{1/p} y^{1/q} \frac{|1 - xy|^{\beta-1}}{(1 + xy)^{\beta+1}} f(x) g(y) dx dy$$
$$\leq \frac{1}{\beta} \left[\int_0^{\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{\infty} g^q(y) dy \right]^{1/q}.$$

Once again, $1/\beta$ is optimal, and its simplicity makes it suitable for a variety of analytical applications.

In this article, we draw inspiration from these results to explore new Hardy-Hilbert-type integral inequalities, considering original two-parameter kernel functions. Specifically, we aim to determine a sharp constant σ in the framework described below. Let p, q and $f, g : [0, \infty) \mapsto [0, \infty)$ be as before, with the same integrability assumptions. Then we have

$$\int_0^{\infty} \int_0^{\infty} x^{1/p} y^{1/q} \frac{(x + y)^{\beta-1}}{(\alpha x + y)^{\beta+1}} f(x) g(y) dx dy$$
$$\leq \sigma \left[\int_0^{\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{\infty} g^q(y) dy \right]^{1/q}.$$

The main novelty of this inequality lies in its consideration of a special two-parameter asymmetric kernel function, i.e., $k(x, y) = x^{1/p} y^{1/q} (x+y)^{\beta-1}/(\alpha x+y)^{\beta+1}$. In particular, the parameter α modulates the asymmetry of the ratio function. Furthermore, the constant σ we derive is optimal and expressed in closed form, making it practical for applications.

Following the spirit in [8], we also study the multiplicative analogue of this inequality. It features a two-parameter kernel function based on the product xy, i.e., $k(x, y) = x^{1/p} y^{1/q} (1 + xy)^{\beta-1}/(\alpha + xy)^{\beta+1}$. This inequality is stated formally below. Let p, q and $f, g : [0, \infty) \mapsto [0, \infty)$ be as before, with the same integrability assumptions. Then we have

$$\int_0^{\infty} \int_0^{\infty} x^{1/p} y^{1/q} \frac{(1 + xy)^{\beta-1}}{(\alpha + xy)^{\beta+1}} f(x) g(y) dx dy$$
$$\leq \sigma \left[\int_0^{\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{\infty} g^q(y) dy \right]^{1/q}.$$

The constant σ is the same as in the previous inequality and is again shown to be optimal.

In addition to these inequalities, we present several auxiliary results, including variants involving only a single function, or the primitives of the functions f and g. Complete proofs containing all the technical details are provided. These results extend the classical theory and could be useful in various areas of analysis, particularly in the study of weighted inequalities, convolution-type estimates and integral operators with asymmetric or multiplicative kernel functions.

The rest of the article is divided into the following sections: Section 2 lists all the results in full generality, and emphasizes some special cases. The proofs are given in Section 3. Section 4 provides a conclusion.

2. Results

2.1. **An intermediary proposition.** The proposition below is an integral result involving two parameters. The first identity is stated in [10, Formula 3.195] but without proof. This gap is filled, along with another integral identity, in the second part of the proposition.

Proposition 2.1. *For any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have*

$$\int_0^\infty \frac{(1+x)^{\beta-1}}{(\alpha+x)^{\beta+1}}dx = \begin{cases} \dfrac{1-\alpha^{-\beta}}{\beta(\alpha-1)} & \text{if } \beta \neq 0,\ \alpha \in (0,\infty)\backslash\{1\}, \\ \\ \dfrac{\log(\alpha)}{\alpha-1} & \text{if } \beta = 0,\ \alpha \in (0,\infty)\backslash\{1\}, \\ \\ 1 & \text{if } \alpha = 1, \end{cases}$$

which is also equal to the following modified integral:

$$\int_0^\infty \frac{(1+x)^{\beta-1}}{(\alpha x+1)^{\beta+1}}dx.$$

We recall that, for all the results in this article, the proofs are given in Section 3 .

To highlight a one-parameter example, let us consider the case $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$. Then Proposition 2.1 gives

$$\int_0^\infty \frac{1}{(\alpha+x)(1+x)}dx = \int_0^\infty \frac{1}{(\alpha x+1)(1+x)}dx = \frac{\log(\alpha)}{\alpha-1}.$$

This example will be reused throughout the study, primarily due to its originality and simplicity.

2.2. **First main results.** We are now ready to present our first main result. It can be viewed as a new, two-parameter, Hardy-Hilbert-type integral inequality with a tractable constant. We will discuss the nature of this constant after the statement.

Theorem 2.2. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^\infty f^p(x)dx < \infty, \quad \int_0^\infty g^q(y)dy < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^\infty \int_0^\infty x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)g(y)dxdy$$
$$\leq C_{\alpha,\beta}\left[\int_0^\infty f^p(x)dx\right]^{1/p}\left[\int_0^\infty g^q(y)dy\right]^{1/q},$$

where

$$C_{\alpha,\beta} = \begin{cases} \dfrac{1-\alpha^{-\beta}}{\beta(\alpha-1)} & \text{if } \beta \neq 0,\ \alpha \in (0,\infty)\backslash\{1\}, \\ \\ \dfrac{\log(\alpha)}{\alpha-1} & \text{if } \beta = 0,\ \alpha \in (0,\infty)\backslash\{1\}, \\ \\ 1 & \text{if } \alpha = 1. \end{cases} \tag{2.1}$$

The proof is based on suitably decomposing the double integral, applying the Hölder integral inequality, making an adapted change of variables, and using Proposition 2.1. This last argument also explains the expression of the constant $C_{\alpha,\beta}$.

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0, \infty)\backslash\{1\}$, Theorem 2.2 gives

$$\int_0^{\infty}\int_0^{\infty} x^{1/p}y^{1/q}\frac{1}{(\alpha x + y)(x + y)}f(x)g(y)dxdy \leq \frac{\log(\alpha)}{\alpha - 1}\left[\int_0^{\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} g^q(y)dy\right]^{1/q}.$$

An important feature of our result is the optimality of the constant $C_{\alpha,\beta}$, which plays a central role in the inequality. This optimality is formalized in the proposition below.

Proposition 2.3. *The constant $C_{\alpha,\beta}$ in Equation* (2.1) *is the optimal one for the inequality in Theorem 2.2.*

The proof is based on choosing suitable functions that allow us to use a process of reasoning by contradiction.

Theorem 2.2 thus provides a generalized Hardy–Hilbert-type integral inequality with the kernel function

$$k(x, y) = x^{1/p}y^{1/q}\frac{(x + y)^{\beta-1}}{(\alpha x + y)^{\beta+1}},$$

along with an explicit, optimal constant $C_{\alpha,\beta}$. It is worth noting that this kernel function is homogeneous with degree -1; for any $\lambda > 0$, using $1/p + 1/q = 1$, we have

$$\begin{aligned} k(\lambda x, \lambda y) &= (\lambda x)^{1/p}(\lambda y)^{1/q}\frac{(\lambda x + \lambda y)^{\beta-1}}{(\alpha\lambda x + \lambda y)^{\beta+1}} \\ &= \frac{\lambda^{1/p+1/q}\lambda^{\beta-1}}{\lambda^{\beta+1}}x^{1/p}y^{1/q}\frac{(x + y)^{\beta-1}}{(\alpha x + y)^{\beta+1}} \\ &= \frac{1}{\lambda}k(x, y). \end{aligned}$$

However, it is the asymmetric nature of the kernel function that primarily offers greater flexibility in applications. Furthermore, the closed-form expression of the sharp constant renders the result theoretically significant and practically tractable. The next subsection illustrates these points by presenting further results.

2.3. **Additional results.** The proposition below presents a reformulation of Theorem 2.2 without the presence of p (and q) in the double integral.

Proposition 2.4. *Let $p > 1$, $q = p/(p - 1)$, and $f, g : [0, \infty) \mapsto [0, \infty)$ such that*

$$\int_0^{\infty}\frac{1}{x}f^p(x)dx < \infty, \quad \int_0^{\infty}\frac{1}{y}g^q(y)dy < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^{\infty}\int_0^{\infty}\frac{(x + y)^{\beta-1}}{(\alpha x + y)^{\beta+1}}f(x)g(y)dxdy \leq C_{\alpha,\beta}\left[\int_0^{\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{\infty}\frac{1}{y}g^q(y)dy\right]^{1/q},$$

where $C_{\alpha,\beta}$ is given by Equation (2.1)*.*

Note that the associated kernel function is simply given by

$$k(x, y) = \frac{(x + y)^{\beta-1}}{(\alpha x + y)^{\beta+1}},$$

without the presence of p. It is asymmetric for $\alpha \neq 1$ and homogeneous with degree -2. Weighted integral norms of f and g are considered in the upper bound, which explains the modified integrability conditions.

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$, Proposition 2.4 implies that

$$\int_0^{\infty}\int_0^{\infty}\frac{1}{(\alpha x+y)(x+y)}f(x)g(y)dxdy$$
$$\leq \frac{\log(\alpha)}{\alpha-1}\left[\int_0^{\infty}\frac{1}{x}f^p(x)dx\right]^{1/p}\left[\int_0^{\infty}\frac{1}{y}g^q(y)dy\right]^{1/q}.$$

The result below can be seen as a primitive modification of Theorem 2.2. In a sense, it combines the features of Theorem 2.2 and the classical Hardy integral inequality.

Proposition 2.5. *Let $p > 1$, $q = p/(p-1)$, $f, g : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \int_0^{\infty} g^q(y)dy < \infty,$$

and $F, G : [0,\infty) \mapsto [0,\infty)$ defined by

$$F(x) = \int_0^{x} f(t)dt, \quad G(y) = \int_0^{y} g(t)dt.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^{\infty}\int_0^{\infty} x^{1/p-1}y^{1/q-1}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}F(x)G(y)dxdy$$
$$\leq C_{\alpha,\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} g^q(y)dy\right]^{1/q},$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

The proof is primarily based on the thorough application of Theorem 2.2 and the Hardy integral inequality.

Noting that $q/(q-1) = p$, the main constant can in fact be expressed in a more condensed form, as follows:

$$C_{\alpha,\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right) = C_{\alpha,\beta}\frac{p^2}{p-1}.$$

Furthermore, as this constant is based on the product of three optimal constants, it can be argued that it is sharp. However, its optimality is not fully demonstrated here.

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$, Proposition 2.5 implies that

$$\int_0^{\infty}\int_0^{\infty} x^{1/p-1}y^{1/q-1}\frac{1}{(\alpha x+y)(x+y)}F(x)G(y)dxdy$$
$$\leq \frac{\log(\alpha)}{\alpha-1}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{\infty} f^p(x)dx\right]^{1/p}\left[\int_0^{\infty} g^q(y)dy\right]^{1/q}.$$

The proposition below is an integral inequality based on Theorem 2.2, but it depends on only one function.

Proposition 2.6. *Let $p > 1$, $q = p/(p-1)$, and $f : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^{\infty} f^p(x)dx < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^{\infty} \left[\int_0^{\infty} x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f(x)dx\right]^p dy \leq C_{\alpha,\beta}^p \int_0^{\infty} f^p(x)dx,$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$, Proposition 2.6 gives

$$\int_0^{\infty} \left[\int_0^{\infty} x^{1/p} y^{1/q} \frac{1}{(\alpha x+y)(x+y)} f(x)dx\right]^p dy \leq \left[\frac{\log(\alpha)}{\alpha-1}\right]^p \int_0^{\infty} f^p(x)dx.$$

We can interpret this result by the boundedness of the following integral operator:

$$O(f)(y) = \int_0^{\infty} x^{1/p} y^{1/q} \frac{1}{(\alpha x+y)(x+y)} f(x)dx$$

from $L_p([0,\infty))$ to itself, with

$$\|O\|_{L_p \to L_p} = \sup_{f \in L_p([0,\infty))} \frac{1}{\|f\|_p} \|O(f)\|_p \leq \frac{\log(\alpha)}{\alpha-1},$$

where $L_p([0,\infty)) = \{h : [0,\infty) \mapsto [0,\infty);\ \|h\|_p < \infty\}$ and $\|\cdot\|_p$ denotes that the standard L_p norm, i.e., $\|h\|_p = \left[\int_0^{\infty} |h(x)|^p dx\right]^{1/p}$.

2.4. **Second main results.** A multiplicative version of Theorem 2.2 is proposed in the theorem below.

Theorem 2.7. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^{\infty} f^p(x)dx < \infty, \quad \int_0^{\infty} g^q(y)dy < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^{\infty} \int_0^{\infty} x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq C_{\alpha,\beta} \left[\int_0^{\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{\infty} g^q(y)dy\right]^{1/q},$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

The proof is based on suitably decomposing the double integral, applying the Hölder integral inequality, performing a change of variables, and using Proposition 2.1.

The upper bound is the same as in Theorem 2.2, including the constant $C_{\alpha,\beta}$. The double integral now deals with the following kernel function:

$$k(x,y) = x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}.$$

It is clearly asymmetric due to the product $x^{1/p}y^{1/q}$, but the parameter α does not modulate the asymmetry anymore. We can remark that this kernel function is not homogeneous; we can not find a $\epsilon \in \mathbb{R}$ such that, for any $\lambda > 0$, we have $k(\lambda x, \lambda y) = \lambda^{\epsilon} k(x,y)$.

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$, Theorem 2.7 implies that

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{1}{(\alpha + xy)(1+xy)} f(x) g(y) dx dy \leq \frac{\log(\alpha)}{\alpha - 1} \left[\int_0^\infty f^p(x) dx\right]^{1/p} \left[\int_0^\infty g^q(y) dy\right]^{1/q}.$$

An important feature of our result is the optimality of the constant $C_{\alpha,\beta}$. This is formalized in the proposition below.

Proposition 2.8. *The constant $C_{\alpha,\beta}$ in Equation* (2.1) *is the optimal one for the inequality in Theorem 2.7.*

The proof is based on choosing suitable functions that allow us to use a process of reasoning by contradiction.

Remark. *A variant of Theorem 2.7 cane be proved, as follows: Let $p > 1$, $q = p/(p-1)$, and $f, g : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^\infty f^p(x) dx < \infty, \quad \int_0^\infty g^q(y) dy < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(1+\alpha xy)^{\beta+1}} f(x) g(y) dx dy \leq C_{\alpha,\beta} \left[\int_0^\infty f^p(x) dx\right]^{1/p} \left[\int_0^\infty g^q(y) dy\right]^{1/q},$$

where $C_{\alpha,\beta}$ is given by Equation (2.1)*, and this constant is optimal.*

The parameter α now modulates the product xy, without change on the upper bound.

2.5. **Additional results.** Some additional results to Theorem 2.7 are given in this subsection, beginning with a reformulation in the proposition below.

Proposition 2.9. *Let $p > 1$, $q = p/(p-1)$, and $f, g : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^\infty \frac{1}{x} f^p(x) dx < \infty, \quad \int_0^\infty \frac{1}{y} g^q(y) dy < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^\infty \int_0^\infty \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x) g(y) dx dy \leq C_{\alpha,\beta} \left[\int_0^\infty \frac{1}{x} f^p(x) dx\right]^{1/p} \left[\int_0^\infty \frac{1}{y} g^q(y) dy\right]^{1/q},$$

where $C_{\alpha,\beta}$ is given by Equation (2.1)*.*

This proposition can also be seen as the multiplicative version of Proposition 2.4.

Note that the double integral is now independent of p, and the associated kernel function is given by

$$k(x,y) = \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}},$$

Although it is now symmetric, it is still non-homogeneous.

Weighted integral norms of f and g define the upper bound of the inequality, which is one of its main interests.

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$, Proposition 2.9 implies that

$$\int_0^\infty \int_0^\infty \frac{1}{(\alpha+xy)(1+xy)} f(x)g(y)dxdy \\ \leq \frac{\log(\alpha)}{\alpha-1}\left[\int_0^\infty \frac{1}{x} f^p(x)dx\right]^{1/p}\left[\int_0^\infty \frac{1}{y} g^q(y)dy\right]^{1/q}.$$

The result below can be seen as a primitive version of Theorem 2.7. In a sense, it combines the features of Theorem 2.7 and the classical Hardy integral inequality.

Proposition 2.10. *Let $p > 1$, $q = p/(p-1)$, $f, g : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^\infty f^p(x)dx < \infty, \quad \int_0^\infty g^q(y)dy < \infty,$$

and $F, G : [0,\infty) \mapsto [0,\infty)$ defined by

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^\infty \int_0^\infty x^{1/p-1} y^{1/q-1} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} F(x)G(y)dxdy \\ \leq C_{\alpha,\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^\infty f^p(x)dx\right]^{1/p}\left[\int_0^\infty g^q(y)dy\right]^{1/q},$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

This result can also be viewed as the multiplicative version of Proposition 2.5.

The proposition below offers an integral inequality based on Theorem 2.7, but which depends on only one function.

Proposition 2.11. *Let $p > 1$, $q = p/(p-1)$, and $f : [0,\infty) \mapsto [0,\infty)$ such that*

$$\int_0^\infty f^p(x)dx < \infty.$$

Then, for any $\beta \in \mathbb{R}$ and $\alpha > 0$, we have

$$\int_0^\infty \left[\int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy \leq C_{\alpha,\beta}^p \int_0^\infty f^p(x)dx,$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

This proposition can also be seen as the multiplicative version of Proposition 2.6. Typically, we can interpret this result in terms of the boundedness of a specific integral operator from $L_p([0,\infty))$ to itself.

As a one-parameter special case, for $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$, Proposition 2.6 gives

$$\int_0^\infty \left[\int_0^\infty x^{1/p} y^{1/q} \frac{1}{(\alpha+xy)(1+xy)} f(x)dx\right]^p dy \leq \left[\frac{\log(\alpha)}{\alpha-1}\right]^p \int_0^\infty f^p(x)dx.$$

Last but not least, to the best of our knowledge, all of the integral inequalities presented in this section are new additions to the literature.

3. PROOFS

This section provides detailed proofs of all the stated propositions and theorems.

3.1. **Proofs associated of the main integral formula.**

Proof of Proposition 2.1. Let us first prove the main formula by distinguishing the indicated parameter cases.

- Let us consider the case $\beta \neq 0$ and $\alpha \in (0,\infty)\backslash\{1\}$. Making the change of variables $y = (1+x)/(\alpha+x)$, so that $x = (1-\alpha y)/(y-1)$ and $dy = [(\alpha-1)/(\alpha+x)^2]dx$, and using standard power primitives, we get
$$\int_0^{\infty} \frac{(1+x)^{\beta-1}}{(\alpha+x)^{\beta+1}}dx = \frac{1}{\alpha-1}\int_0^{\infty}\left(\frac{1+x}{\alpha+x}\right)^{\beta-1}\left[\frac{\alpha-1}{(\alpha+x)^2}dx\right]$$
$$= \frac{1}{\alpha-1}\int_{1/\alpha}^{1} y^{\beta-1}dy = \frac{1}{\alpha-1}\left[\frac{1}{\beta}y^{\beta}\right]_{y=1/\alpha}^{y=1} = \frac{1}{\beta(\alpha-1)}\left(1-\frac{1}{\alpha^{\beta}}\right)$$
$$= \frac{1-\alpha^{-\beta}}{\beta(\alpha-1)}.$$
- Let us consider the case $\beta = 0$ and $\alpha \in (0,\infty)\backslash\{1\}$. Using standard fractional decomposition and logarithmic primitives, we get
$$\int_0^{\infty} \frac{(1+x)^{\beta-1}}{(\alpha+x)^{\beta+1}}dx = \int_0^{\infty}\frac{1}{(1+x)(\alpha+x)}dx = \frac{1}{\alpha-1}\int_0^{\infty}\left(\frac{1}{1+x}-\frac{1}{\alpha+x}\right)dx$$
$$= \frac{1}{\alpha-1}\left[\log(1+x)-\log(\alpha+x)\right]_{x=0}^{x\to\infty} = \frac{1}{\alpha-1}\left[\log\left(\frac{1+x}{\alpha+x}\right)\right]_{x=0}^{x\to\infty}$$
$$= \frac{1}{\alpha-1}\left[\log(1)-\log\left(\frac{1}{\alpha}\right)\right] = \frac{\log(\alpha)}{\alpha-1}.$$
- Let us consider the case $\alpha = 1$. We have
$$\int_0^{\infty} \frac{(1+x)^{\beta-1}}{(\alpha+x)^{\beta+1}}dx = \int_0^{\infty}\frac{1}{(1+x)^2}dx = \left[-\frac{1}{1+x}\right]_{x=0}^{x\to\infty} = 0-(-1) = 1.$$

For the second part on the integral equality, making the change of variables $x = 1/y$, we get
$$\int_0^{\infty} \frac{(1+x)^{\beta-1}}{(\alpha x+1)^{\beta+1}}dx = \int_{\infty}^{0}\frac{(1+1/y)^{\beta-1}}{(\alpha/y+1)^{\beta+1}}\left(-\frac{1}{y^2}dy\right)$$
$$= \int_{\infty}^{0}\frac{(y+1)^{\beta-1}}{(\alpha+y)^{\beta+1}}\times\frac{y^{\beta+1}}{y^{\beta-1}}\times\left(-\frac{1}{y^2}dy\right)$$
$$= \int_0^{\infty}\frac{(1+y)^{\beta-1}}{(\alpha+y)^{\beta+1}}dy,$$
which corresponds to the first integral considered. This concludes the proof of Proposition 2.1. □

3.2. **Proofs associated to Theorem 2.2.**

Proof of Theorem 2.2. Decomposing the integrand of the double integral in a suitable way and using the Hölder integral inequality, we derive

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f(x)g(y)dxdy \\
&= \int_0^\infty \int_0^\infty x^{1/p} \frac{(x+y)^{(\beta-1)/p}}{(\alpha x+y)^{(\beta+1)/p}} f(x) \times y^{1/q} \frac{(x+y)^{(\beta-1)/q}}{(\alpha x+y)^{(\beta+1)/q}} g(y)dxdy \\
&\leq \left[\int_0^\infty \int_0^\infty x \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f^p(x)dxdy\right]^{1/p} \\
&\times \left[\int_0^\infty \int_0^\infty y \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} g^q(y)dxdy\right]^{1/q}. \qquad (3.1)
\end{aligned}$$

Using the Fubini-Tonelli integral theorem, making the change of variables $u = y/x$ and applying Proposition 2.1, we get

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f^p(x)dxdy \\
&= \int_0^\infty f^p(x) \left[\int_0^\infty \frac{(1+y/x)^{\beta-1}}{(\alpha+y/x)^{\beta+1}} \times \left(\frac{1}{x}dy\right)\right] dx \\
&= \int_0^\infty f^p(x) \left[\int_0^\infty \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}} du\right] dx = C_{\alpha,\beta} \int_0^\infty f^p(x)dx. \qquad (3.2)
\end{aligned}$$

In a similar way, but with the change of variables $v = x/y$ and the second part of Proposition 2.1, we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty y \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} g^q(y)dxdy \\
&= \int_0^\infty g^q(y) \left[\int_0^\infty \frac{(1+x/y)^{\beta-1}}{(\alpha x/y+1)^{\beta+1}} \times \left(\frac{1}{y}dx\right)\right] dy \\
&= \int_0^\infty g^q(y) \left[\int_0^\infty \frac{(1+v)^{\beta-1}}{(\alpha v+1)^{\beta+1}} dv\right] dy = C_{\alpha,\beta} \int_0^\infty g^q(y)dy. \qquad (3.3)
\end{aligned}$$

Using Equations (3.1), (3.2) and (3.3), and $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f(x)g(y)dxdy \\
&\leq \left[C_{\alpha,\beta} \int_0^\infty f^p(x)dx\right]^{1/p} \left[C_{\alpha,\beta} \int_0^\infty g^q(y)dy\right]^{1/q} \\
&= C_{\alpha,\beta} \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Theorem 2.2. □

Proof of Proposition 2.3. We use an argument by contradiction, supposing that there is a constant $\gamma \in (0, C_{\alpha,\beta})$ such that, for any $f, g : [0, \infty) \mapsto [0, \infty)$ satisfying

$$\int_0^\infty f^p(x)dx < \infty, \quad \int_0^\infty g^q(y)dy < \infty,$$

we have

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq \gamma \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}. \tag{3.4}$$

For any $n \in \mathbb{N}\backslash\{0\}$, we define two functions $f_n, g_n : [0,\infty) \mapsto [0,\infty)$, as follows:

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ x^{-(1+1/n)/p} & \text{if } x \in [1,\infty), \end{cases}$$

and

$$g_n(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{-(1+1/n)/q} & \text{if } y \in [1,\infty). \end{cases}$$

We then determine

$$\int_0^\infty f_n^p(x)dx = \int_1^\infty (x^{-(1+1/n)/p})^p dx = \left[-nx^{-1/n}\right]_{x=1}^{x\to\infty} = n$$

and, similarly,

$$\int_0^\infty g_n^q(y)dy = \int_1^\infty (y^{-(1+1/n)/q})^q dy = \left[-ny^{-1/n}\right]_{y=1}^{y\to\infty} = n.$$

Using this, $1/p + 1/q = 1$ and Equation (3.4), we get

$$\gamma = \gamma \frac{1}{n} n^{1/p} n^{1/q} = \frac{1}{n}\left\{\gamma \left[\int_0^\infty f_n^p(x)dx\right]^{1/p} \left[\int_0^\infty g_n^q(y)dy\right]^{1/q}\right\}$$
$$\geq \frac{1}{n}\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f_n(x)g_n(y)dxdy. \tag{3.5}$$

Now, let us focus on this double integral. Expressing f_n and g_n, making the change of variables $x = uy$, applying the Fubini-Tonelli integral theorem and using $1/p + 1/q = 1$, we obtain

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f_n(x)g_n(y)dxdy$$
$$= \int_1^\infty \int_1^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} x^{-(1+1/n)/p} y^{-(1+1/n)/q} dxdy$$
$$= \int_1^\infty \left[\int_1^\infty \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} x^{-1/(np)} dx\right] y^{-1/(nq)} dy$$
$$= \int_1^\infty \left[\int_{1/y}^\infty \frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}} \times \frac{1}{y^2} u^{-1/(np)} y^{-1/(np)} (ydu)\right] y^{-1/(nq)} dy$$
$$= \int_1^\infty \left[\int_{1/y}^\infty \frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}} u^{-1/(np)} du\right] y^{-(1+1/n)} dy. \tag{3.6}$$

Using the Chasles integral relation with the splitting value $u = 1$, the Fubini-Tonelli integral theorem, simple integral calculus and $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\int_1^{\infty}\left[\int_{1/y}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy\\
&=\int_1^{\infty}\left[\int_{1/y}^{1}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy\\
&+\int_1^{\infty}\left[\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\right]y^{-(1+1/n)}dy\\
&=\int_0^{1}\left[\int_{1/u}^{\infty}y^{-(1+1/n)}dy\right]\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\\
&+\left[\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\right]\left[\int_1^{\infty}y^{-(1+1/n)}dy\right]\\
&=\int_0^{1}(nu^{1/n})\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du+n\left[\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\right]\\
&=n\left[\int_0^{1}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{1/(nq)}du+\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\right]. \qquad (3.7)
\end{aligned}$$

Joining Equations (3.5), (3.6) and (3.7), we establish that

$$\gamma\geq\int_0^{1}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{1/(nq)}du+\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du.$$

Applying the inferior limit with respect to n denoted $\liminf_{n\to\infty}$, the Fatou integral lemma, $\liminf_{n\to\infty}u^{1/(nq)} = 1$ for $u \in (0, 1)$, $\liminf_{n\to\infty}u^{-1/(np)} = 1$ for $u \in [1,\infty)$, the Chasles integral relation and the second part of Proposition 2.1, we obtain

$$\begin{aligned}
\gamma&\geq\liminf_{n\to\infty}\int_0^{1}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{1/(nq)}du+\liminf_{n\to\infty}\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}u^{-1/(np)}du\\
&\geq\int_0^{1}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}\left[\liminf_{n\to\infty}u^{1/(nq)}\right]du+\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}\left[\liminf_{n\to\infty}u^{-1/(np)}\right]du\\
&=\int_0^{1}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}du+\int_{1}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}du=\int_{0}^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha u+1)^{\beta+1}}du=C_{\alpha,\beta}.
\end{aligned}$$

A contradiction is with the assumption $\gamma \in (0, C_{\alpha,\beta})$. Consequently, the constant $C_{\alpha,\beta}$ is optimal. This ends the proof of Proposition 2.3. □

Proof of Proposition 2.4. We can write

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)g(y)dxdy\\
&=\int_0^{\infty}\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}[x^{-1/p}f(x)][y^{-1/q}g(y)]dxdy\\
&=\int_0^{\infty}\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f_\star(x)g_\star(y)dxdy, \qquad (3.8)
\end{aligned}$$

where $f_\star(x) = x^{-1/p}f(x)$ and $g_\star(y) = y^{-1/q}g(y)$.

Applying Theorem 2.2 to $f_\star$ and $g_\star$, we get

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f_\star(x) g_\star(y) dxdy \leq C_{\alpha,\beta} \left[\int_0^\infty f_\star^p(x)dx\right]^{1/p} \left[\int_0^\infty g_\star^q(y)dy\right]^{1/q}, \tag{3.9}$$

where $C_{\alpha,\beta}$ is given by Equation (2.1),

$$\int_0^\infty f_\star^p(x)dx = \int_0^\infty [x^{-1/p} f(x)]^p dx = \int_0^\infty \frac{1}{x} f^p(x)dx \tag{3.10}$$

and

$$\int_0^\infty g_\star^q(y)dy = \int_0^\infty [y^{-1/q} g(y)]^q dy = \int_0^\infty \frac{1}{y} g^q(y)dy. \tag{3.11}$$

Joining Equations (3.8), (3.9), (3.10) and (3.11), we get

$$\int_0^\infty \int_0^\infty \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f(x) g(y) dxdy \leq C_{\alpha,\beta} \left[\int_0^\infty \frac{1}{x} f^p(x)dx\right]^{1/p} \left[\int_0^\infty \frac{1}{y} g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 2.4. □

Proof of Proposition 2.5. We can write

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{1/p-1} y^{1/q-1} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} F(x) G(y) dxdy \\
&= \int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} [x^{-1}F(x)][y^{-1}G(y)] dxdy \\
&= \int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f_\dagger(x) g_\dagger(y) dxdy,
\end{aligned} \tag{3.12}$$

where $f_\dagger(x) = x^{-1}F(x)$ and $g_\dagger(y) = y^{-1}G(y)$.

Applying Theorem 2.2 to $f_\dagger$ and $g_\dagger$, we get

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}} f_\dagger(x) g_\dagger(y) dxdy \leq C_{\alpha,\beta} \left[\int_0^\infty f_\dagger^p(x)dx\right]^{1/p} \left[\int_0^\infty g_\dagger^q(y)dy\right]^{1/q}, \tag{3.13}$$

where $C_{\alpha,\beta}$ is given by Equation (2.1). Using the Hardy integral inequality, we obtain

$$\int_0^\infty f_\dagger^p(x)dx = \int_0^\infty [x^{-1}F(x)]^p dx = \int_0^\infty \frac{1}{x^p} F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{3.14}$$

and

$$\int_0^\infty g_\dagger^q(y)dy = \int_0^\infty [y^{-1}G(y)]^q dy = \int_0^\infty \frac{1}{y^q} G^q(y)dy \leq \left(\frac{q}{q-1}\right)^q \int_0^\infty g^q(y)dy. \tag{3.15}$$

Joining Equations (3.12), (3.13), (3.14) and (3.15), we have

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)g(y)dxdy\\
&\leq C_{\alpha,\beta}\left[\left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)dx\right]^{1/p}\left[\left(\frac{q}{q-1}\right)^q\int_0^{\infty}g^q(y)dy\right]^{1/q}\\
&=C_{\alpha,\beta}\left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left[\int_0^{\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{\infty}g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This concludes the proof of Proposition 2.5. □

Proof of Proposition 2.6. Using the Fubini-Tonelli integral theorem, we can write

$$\begin{aligned}
&\int_0^{\infty}\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^p dy\\
&=\int_0^{\infty}\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]\times\\
&\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^{p-1}dy\\
&=\int_0^{\infty}\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)g_{\diamond}(y)dxdy, \qquad (3.16)
\end{aligned}$$

where

$$g_{\diamond}(y)=\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^{p-1}.$$

Applying Theorem 2.2 to the functions f and $g_{\diamond}$, we get

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)g_{\diamond}(y)dxdy\\
&\leq C_{\alpha,\beta}\left[\int_0^{\infty}f^p(x)dx\right]^{1/p}\left[\int_0^{\infty}g_{\diamond}^q(y)dy\right]^{1/q}, \qquad (3.17)
\end{aligned}$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

Furthermore, since $q(p-1)=p$, we have

$$\begin{aligned}
&\int_0^{\infty}g_{\diamond}^q(y)dy=\int_0^{\infty}\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^{q(p-1)}dy\\
&=\int_0^{\infty}\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^p dy. \qquad (3.18)
\end{aligned}$$

Joining Equations (3.16), (3.17) and (3.18), we get

$$\begin{aligned}
&\int_0^{\infty}\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^p dy\\
&\leq C_{\alpha,\beta}\left[\int_0^{\infty}f^p(x)dx\right]^{1/p}\left\{\int_0^{\infty}\left[\int_0^{\infty}x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^p dy\right\}^{1/q}.
\end{aligned}$$

This and $1 - 1/q = 1/p$ give

$$\left\{\int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^p dy\right\}^{1/p} \leq C_{\alpha,\beta}\left[\int_0^\infty f^p(x)dx\right]^{1/p},$$

so that

$$\int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q}\frac{(x+y)^{\beta-1}}{(\alpha x+y)^{\beta+1}}f(x)dx\right]^p dy \leq C^p_{\alpha,\beta}\int_0^\infty f^p(x)dx.$$

This concludes the proof of Proposition 2.6. □

3.3. **Proofs associated to Theorem 2.7.**

Proof of Theorem 2.7. Decomposing the integrand of the double integral in a suitable way and using the Hölder integral inequality, we derive

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{1/p}y^{1/q}\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}f(x)g(y)dxdy \\
&= \int_0^\infty\int_0^\infty x^{1/p}\frac{(1+xy)^{(\beta-1)/p}}{(\alpha+xy)^{(\beta+1)/p}}f(x)\times y^{1/q}\frac{(1+xy)^{(\beta-1)/q}}{(\alpha+xy)^{(\beta+1)/q}}g(y)dxdy \\
&\leq \left[\int_0^\infty\int_0^\infty x\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}f^p(x)dxdy\right]^{1/p} \\
&\times\left[\int_0^\infty\int_0^\infty y\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}g^q(y)dxdy\right]^{1/q}.
\end{aligned} \tag{3.19}$$

Using the Fubini-Tonelli integral theorem, making the change of variables $u = xy$ and applying Proposition 2.1, we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}f^p(x)dxdy = \int_0^\infty f^p(x)\left[\int_0^\infty \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}\times(xdy)\right]dx \\
&= \int_0^\infty f^p(x)\left[\int_0^\infty \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}du\right]dx = C_{\alpha,\beta}\int_0^\infty f^p(x)dx.
\end{aligned} \tag{3.20}$$

In a similar way, but with the change of variables $v = xy$, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty y\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}g^q(y)dxdy = \int_0^\infty g^q(y)\left[\int_0^\infty \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}\times(ydx)\right]dy \\
&= \int_0^\infty g^q(y)\left[\int_0^\infty \frac{(1+v)^{\beta-1}}{(\alpha+v)^{\beta+1}}dv\right]dy = C_{\alpha,\beta}\int_0^\infty g^q(y)dy.
\end{aligned} \tag{3.21}$$

Using Equations (3.19), (3.20) and (3.21), and $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{1/p}y^{1/q}\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}f(x)g(y)dxdy \\
&\leq \left[C_{\alpha,\beta}\int_0^\infty f^p(x)dx\right]^{1/p}\left[C_{\alpha,\beta}\int_0^\infty g^q(y)dy\right]^{1/q} \\
&= C_{\alpha,\beta}\left[\int_0^\infty f^p(x)dx\right]^{1/p}\left[\int_0^\infty g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This ends the proof of Theorem 2.7. □

Proof of Proposition 2.8. We use an argument by contradiction, supposing that there is a constant $\mu \in (0, C_{\alpha,\beta})$ such that, for any $f, g : [0,\infty) \mapsto [0,\infty)$ satisfying

$$\int_0^\infty f^p(x)dx < \infty, \quad \int_0^\infty g^q(y)dy < \infty,$$

we have

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq \mu \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}. \tag{3.22}$$

For any $n \in \mathbb{N}\backslash\{0\}$, we define two functions $f_n, g_n : [0,\infty) \mapsto [0,\infty)$, as follows:

$$f_n(x) = \begin{cases} x^{(1/n-1)/p} & \text{if } x \in (0,1), \\ 0 & \text{if } x \in [1,\infty), \end{cases}$$

from wich we can add the value $f_n(x) = 0$ if $x = 0$ to cover the domain $[0,\infty)$, and

$$g_n(y) = \begin{cases} 0 & \text{if } y \in [0,1), \\ y^{-(1+1/n)/q} & \text{if } y \in [1,\infty). \end{cases}$$

We then calculate

$$\int_0^\infty f_n^p(x)dx = \int_0^1 (x^{(1/n-1)/p})^p dx = \left[n x^{1/n}\right]_{x=0}^{x=1} = n$$

and, similarly,

$$\int_0^\infty g_n^q(y)dy = \int_1^\infty (y^{-(1+1/n)/q})^q dy = \left[-n y^{-1/n}\right]_{y=1}^{y\to\infty} = n.$$

Using this, $1/p + 1/q = 1$ and Equation (3.22), we obtain

$$\mu = \mu \frac{1}{n} n^{1/p} n^{1/q} = \frac{1}{n} \left\{ \mu \left[\int_0^\infty f_n^p(x)dx\right]^{1/p} \left[\int_0^\infty g_n^q(y)dy\right]^{1/q} \right\}$$
$$\geq \frac{1}{n} \int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f_n(x)g_n(y)dxdy. \tag{3.23}$$

Now, let us concentrate on this double integral. Expressing f_n and g_n, making the change of variables $x = u/y$, applying the Fubini-Tonelli integral theorem and using $1/p + 1/q = 1$, we get

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f_n(x)g_n(y)dxdy$$
$$= \int_1^\infty \int_0^1 x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} x^{(1/n-1)/p} y^{-(1+1/n)/q} dxdy$$
$$= \int_1^\infty \left[\int_0^1 \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} x^{1/(np)} dx\right] y^{-1/(nq)} dy$$
$$= \int_1^\infty \left[\int_0^y \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}} u^{1/(np)} y^{-1/(np)} \frac{1}{y} du\right] y^{-1/(nq)} dy$$
$$= \int_1^\infty \left[\int_0^y \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}} u^{1/(np)} du\right] y^{-(1+1/n)} dy. \tag{3.24}$$

Using the Chasles integral relation with the splitting value $u = 1$, the Fubini-Tonelli integral theorem and $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
&\int_1^{\infty}\left[\int_0^y \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy\\
&=\int_1^{\infty}\left[\int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy\\
&+\int_1^{\infty}\left[\int_1^y \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\right]y^{-(1+1/n)}dy\\
&=\left[\int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\right]\left[\int_1^{\infty} y^{-(1+1/n)}dy\right]\\
&+\int_1^{\infty}\left[\int_u^{\infty} y^{-(1+1/n)}dy\right]\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\\
&=n\left[\int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\right]+\int_1^{\infty}(nu^{-1/n})\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du\\
&=n\left[\int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du+\int_1^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{-1/(nq)}du\right]. \qquad (3.25)
\end{aligned}$$

Joining Equations (3.23), (3.24) and (3.25), we find that

$$\mu \geq \int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du+\int_1^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{-1/(nq)}du.$$

Applying the inferior limit with respect to n, the Fatou integral lemma, $\liminf_{n\to\infty} u^{1/(np)} = 1$ for $u \in (0, 1)$, $\liminf_{n\to\infty} u^{-1/(nq)} = 1$ for $u \in [1,\infty)$, the Chasles integral relation and Proposition 2.1, we obtain

$$\begin{aligned}
\mu &\geq \liminf_{n\to\infty}\int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{1/(np)}du+\liminf_{n\to\infty}\int_1^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}u^{-1/(nq)}du\\
&\geq \int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}\left[\liminf_{n\to\infty} u^{1/(np)}\right]du+\int_1^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}\left[\liminf_{n\to\infty} u^{-1/(nq)}\right]du\\
&=\int_0^1 \frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}du+\int_1^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}du=\int_0^{\infty}\frac{(1+u)^{\beta-1}}{(\alpha+u)^{\beta+1}}du=C_{\alpha,\beta}.
\end{aligned}$$

A contradiction appears with the assumption $\mu \in (0, C_{\alpha,\beta})$. Consequently, the constant $C_{\alpha,\beta}$ is optimal. This completes the proof of Proposition 2.8. □

Proof of Proposition 2.9. We can write

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}f(x)g(y)dxdy\\
&=\int_0^{\infty}\int_0^{\infty}x^{1/p}y^{1/q}\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}[x^{-1/p}f(x)][y^{-1/q}g(y)]dxdy\\
&=\int_0^{\infty}\int_0^{\infty}x^{1/p}y^{1/q}\frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}}f_*(x)g_*(y)dxdy, \qquad (3.26)
\end{aligned}$$

where $f_*(x) = x^{-1/p}f(x)$ and $g_*(y) = y^{-1/q}g(y)$.

Applying Theorem 2.7 to f_* and g_*, we get

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f_*(x) g_*(y) dx dy \leq C_{\alpha,\beta} \left[\int_0^\infty f_*^p(x) dx\right]^{1/p} \left[\int_0^\infty g_*^q(y) dy\right]^{1/q}, \quad (3.27)$$

where $C_{\alpha,\beta}$ is given by Equation (2.1),

$$\int_0^\infty f_*^p(x) dx = \int_0^\infty [x^{-1/p} f(x)]^p dx = \int_0^\infty \frac{1}{x} f^p(x) dx \quad (3.28)$$

and

$$\int_0^\infty g_*^q(y) dy = \int_0^\infty [y^{-1/q} g(y)]^q dy = \int_0^\infty \frac{1}{y} g^q(y) dy. \quad (3.29)$$

Joining Equations (3.26), (3.27), (3.28) and (3.29), we obtain

$$\int_0^\infty \int_0^\infty \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x) g(y) dx dy \leq C_{\alpha,\beta} \left[\int_0^\infty \frac{1}{x} f^p(x) dx\right]^{1/p} \left[\int_0^\infty \frac{1}{y} g^q(y) dy\right]^{1/q}.$$

This concludes the proof of Proposition 2.9. □

Proof of Proposition 2.10. We can write

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{1/p-1} y^{1/q-1} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} F(x) G(y) dx dy \\
&= \int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} [x^{-1} F(x)][y^{-1} G(y)] dx dy \\
&= \int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f_\ddagger(x) g_\ddagger(y) dx dy \quad (3.30)
\end{aligned}$$

where $f_\ddagger(x) = x^{-1} F(x)$ and $g_\ddagger(y) = y^{-1} G(y)$.

Applying Theorem 2.7 to $f_\ddagger$ and $g_\ddagger$, we get

$$\int_0^\infty \int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f_\ddagger(x) g_\ddagger(y) dx dy \leq C_{\alpha,\beta} \left[\int_0^\infty f_\ddagger^p(x) dx\right]^{1/p} \left[\int_0^\infty g_\ddagger^q(y) dy\right]^{1/q}, \quad (3.31)$$

where $C_{\alpha,\beta}$ is given by Equation (2.1). Using the Hardy integral inequality, we obtain

$$\int_0^\infty f_\ddagger^p(x) dx = \int_0^\infty [x^{-1} F(x)]^p dx = \int_0^\infty \frac{1}{x^p} F^p(x) dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x) dx \quad (3.32)$$

and

$$\int_0^\infty g_\ddagger^q(y) dy = \int_0^\infty [y^{-1} G(y)]^q dy = \int_0^\infty \frac{1}{y^q} G^q(y) dy \leq \left(\frac{q}{q-1}\right)^q \int_0^\infty g^q(y) dy. \quad (3.33)$$

Joining Equations (3.30), (3.31), (3.32) and (3.33), we have

$$\int_0^\infty \int_0^\infty \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)g(y)dxdy$$
$$\leq C_{\alpha,\beta} \left[\left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx\right]^{1/p} \left[\left(\frac{q}{q-1}\right)^q \int_0^\infty g^q(y)dy\right]^{1/q}$$
$$= C_{\alpha,\beta} \left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right) \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g^q(y)dy\right]^{1/q}.$$

This concludes the proof of Proposition 2.10. □

Proof of Proposition 2.11. Using the Fubini-Tonelli integral theorem, we can write

$$\int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy$$
$$= \int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right] \times$$
$$\left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^{p-1} dy$$
$$= \int_0^\infty \int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)g_\triangle(y)dxdy, \tag{3.34}$$

where

$$g_\triangle(y) = \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^{p-1}.$$

Applying Theorem 2.7 to the functions f and $g_\triangle$, we get

$$\int_0^\infty \int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)g_\triangle(y)dxdy$$
$$\leq C_{\alpha,\beta} \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left[\int_0^\infty g_\triangle^q(y)dy\right]^{1/q}, \tag{3.35}$$

where $C_{\alpha,\beta}$ is given by Equation (2.1).

Furthermore, since $q(p-1) = p$, we have

$$\int_0^\infty g_\triangle^q(y)dy = \int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^{q(p-1)} dy$$
$$= \int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy. \tag{3.36}$$

Joining Equations (3.34), (3.35) and (3.36), we get

$$\int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy$$
$$\leq C_{\alpha,\beta} \left[\int_0^\infty f^p(x)dx\right]^{1/p} \left\{\int_0^\infty \left[\int_0^\infty x^{1/p}y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy\right\}^{1/q}.$$

This and $1 - 1/q = 1/p$ give

$$\left\{\int_0^\infty \left[\int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy\right\}^{1/p} \leq C_{\alpha,\beta} \left[\int_0^\infty f^p(x)dx\right]^{1/p},$$

so that

$$\int_0^\infty \left[\int_0^\infty x^{1/p} y^{1/q} \frac{(1+xy)^{\beta-1}}{(\alpha+xy)^{\beta+1}} f(x)dx\right]^p dy \leq C_{\alpha,\beta}^p \int_0^\infty f^p(x)dx.$$

This concludes the proof of Proposition 2.11. □

4. Conclusion

Inspired by recent developments, we have derived two new two-parameter modifications of the classical Hardy-Hilbert integral inequality. We have determined a closed-form expression for the optimal constants for each of them, which is necessary for theoretical investigations and potential applications in analysis and related fields. We also obtained supplementary results, including variants involving the primitives of the main functions. Our article is self-contained and accessible to a broad mathematical audience thanks to the detailed, step-by-step proofs that accompany all of our results.

This work suggests several natural directions for further research. For instance, one could consider weighted extensions of the proposed inequalities, or analogous results in higher dimensions or on different domains. Furthermore, as outlined in the article, the application of these results to the boundedness of certain integral operators or convolution-type inequalities warrants further exploration. We hope that these results will stimulate further research into integral inequalities and their many applications in analysis and other fields.

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