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# COMPLETE APPROXIMATION BY SYMMETRIZED AND PERTURBED HYPERBOLIC TANGENT ACTIVATED MULTIDIMENSIONAL CONVOLUTIONS AS POSITIVE LINEAR OPERATORS

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ABSTRACT. In this work are studied in detail the multivariate symmetrized and perturbed hyperbolic tangent activated convolution type operators of three kinds. Here this is done with the method of positive linear operators. Their alternative approximation properties are established by the quantitative convergence to the unit operator using the modulus of continuity. It is also studied the related multivariate simultaneous approximation, as well as the multivariate iterated approximation.

## 1. INTRODUCTION

The author studied extensively the quantitative approximation of positive linear operators to the unit since 1985, see for example [1]-[3], [5]. He originated from the quantitative weak convergence of finite positive measures to the unit Dirac measure, having as a method the geometric moment theory, see [2], and he produced best upper bounds, leading to attained (i.e. sharp Jackson type inequalities), e.g. see [1], [2]. These studies have been gone to all possible directions, univariate and multivariate, though in this work we stay only on the multivariate approach over an infinite domain.

Our multidimensional convolution operators here have a kernel based on the symmetrized and perturbed hyperbolic tangent activation function, which is used frequently in the study of neural networks, and they can be interpreted as positive linear operators.

So here our proving methods come from the theory of positive linear operators.

Thus in Section 2, we discuss about the symmetrized and perturbed hyperbolic tangent activation function in the multivariate setting. We also describe our activated multidimensional convolution type operators with their properties, such as differentiation and iteration.

In Section 3, we derive some auxiliary results which are estimates to our operators, when applied to basic functions and to be used into our main results.

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In Section 4, we present our main explicit results under the lens of positive linear operators and we include their necessary related theory. We treat also the simultaneous and iterated approximation cases under the same spirit.

We are greatly inspired by our earlier works [6], [7].

Furthermore, general motivation comes from the great works [11]-[16].

## 2. BASICS

Initially we follow [4], pp. 455-460.

Our perturbed hyperbolic tangent activation function is

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - qe^{-\lambda x}}{e^{\lambda x} + qe^{-\lambda x}}, \ \ \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

Above $\lambda$ is the parameter and $q$ is the deformation coefficient.

For more read Chapter 18 of [4]: "$q$-deformed and $\lambda$-Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

The Chapters 17 and 18 of [4] motivate our current work.

The proposed "symmetrization method" aims to use half data feed to our multivariate neural networks.

We will employ the following density function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{2}$$

$\forall x \in \mathbb{R}; q, \lambda > 0$.

We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R}; \ q, \lambda > 0, \tag{3}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \ \forall\, x \in \mathbb{R}; \ q, \lambda > 0. \tag{4}$$

Adding (3) and (4) we obtain

$$M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x), \tag{5}$$

a key to this work.

So that

$$\Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2} > 0, \tag{6}$$

is an even function, symmetric with respect to the $y$-axis.

By (18.18) of [4], we have

$$\begin{aligned} &M_{q,\lambda}\left(\tfrac{\ln q}{2\lambda}\right) = \tfrac{\tanh(\lambda)}{2}, \\ &\text{and} \\ &M_{\frac{1}{q},\lambda}\left(-\tfrac{\ln q}{2\lambda}\right) = \tfrac{\tanh(\lambda)}{2}, \ \lambda > 0. \end{aligned} \tag{7}$$

sharing the same maximum at symmetric points.

By Theorem 18.1, p. 458 of [4], we have that

$$\begin{aligned} &\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1, \ \forall\, x \in \mathbb{R}, \lambda, q > 0, \\ &\text{and} \\ &\sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1, \ \forall\, x \in \mathbb{R}, \lambda, q > 0. \end{aligned} \tag{8}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \ \forall\, x \in \mathbb{R}. \tag{9}$$

By Theorem 18.2, p. 459 of [4], we have that

$$\begin{gathered} \int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \\ \text{and} \\ \int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\,dx = 1, \end{gathered} \tag{10}$$

so that

$$\int_{-\infty}^{\infty} \Phi(x)\,dx = 1, \tag{11}$$

therefore $\Phi$ is a density function.

Clearly, then

$$\int_{-\infty}^{\infty} \Phi(nx-u)\,du = 1, \ \ \forall\, n \in \mathbb{N}, x \in \mathbb{R}. \tag{12}$$

An essential property follows:

**Proposition 2.1.** *([6]) It holds ($k \in \mathbb{N}$)*

$$\int_{-\infty}^{\infty} |z|^k \,\Phi(z)\,dz \leq \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}k!}{(2\lambda)^k}\right] < \infty. \tag{13}$$

We mention

**Definition 2.1.** The modulus of continuity here is defined by

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N: \\ \|x-y\|_\infty<\delta}} |f(x)-f(y)|, \ \ \delta>0, \tag{14}$$

where $f:\mathbb{R}^N \to \mathbb{R}$ bounded and continuous, denoted by $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$. Similarly $\omega_1$ is defined for $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions). We have that $f \in C_U\left(\mathbb{R}^N\right)$, iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$.

Denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$.

We make

**Remark.** *We introduce*

$$\widetilde{Z}(x_1,...,x_N) := \widetilde{Z}(x) := \prod_{i=1}^{N} \Phi(x_i), \ \ \ x=(x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{15}$$

*It has the properties:*

*(i)*

$$\widetilde{Z}(x) > 0, \ \forall x \in \mathbb{R}^N; \ \widetilde{Z}(-x) = \widetilde{Z}(x),$$

*(ii)*

$$\int_{\mathbb{R}^N} \widetilde{Z}(x-k) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \widetilde{Z}(x_1-u_1,...,x_N-u_N)\,du_1...du_N =$$

$$\prod_{i=1}^{N} \int_{-\infty}^{\infty} \Phi(x_i-u_i)\,du_i \overset{(12)}{=} 1, \ \forall x \in \mathbb{R}^N, \tag{16}$$

*hence*

*(iii)*
$$\int_{\mathbb{R}^N} \widetilde{Z}(nx-u)\,du = 1, \ \ \forall\, x \in \mathbb{R}^N; \ n \in \mathbb{N}, \tag{17}$$
*and*
*(iv) by (11)*
$$\int_{\mathbb{R}^N} \widetilde{Z}(x)\,dx = 1, \tag{18}$$
*that is $\widetilde{Z}$ is a multivariate density function.*

We mention a useful related result.

**Theorem 2.2.** *([7]) It holds ($k \in \mathbb{N}$)*
$$\int_{\mathbb{R}^N} \|x\|_\infty^k \,\widetilde{Z}(x)\,dx \le N^k \left[\frac{\tanh(\lambda)}{(k+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}k!}{(2\lambda)^k}\right] < \infty. \tag{19}$$
*When $k=0$, (19) is again valid.*

We give

**Definition 2.2.** Let $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$. We define the following activated symmetrized and perturbed hyperbolic tangent multivariate convolution type operators:

The basic one
$$\widetilde{S}_n(f)(x) := \int_{\mathbb{R}^N} f\left(\frac{u}{n}\right)\widetilde{Z}(nx-u)\,du, \ \ \forall\, x \in \mathbb{R}^N, \tag{20}$$
the activated Kantorovich type
$$\widetilde{S}_n^*(f)(x) := n^N \int_{\mathbb{R}^N}\left(\int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t)\,dt\right)\widetilde{Z}(nx-u)\,du, \ \ \forall\, x \in \mathbb{R}^N. \tag{21}$$

Let now $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1 r_2 ... r_N} \ge 0$, such that
$$\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N} w_{r_1 r_2 ... r_N} = 1; \ \ u \in \mathbb{R}^N,$$
and
$$\delta_n(f)(u) := \delta_n(f)(u_1, ..., u_N) := \sum_{r=0}^{\theta} w_r f\left(\frac{u}{n} + \frac{r}{n\theta}\right) =$$
$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N} w_{r_1 r_2 ... r_N} f\left(\frac{u_1}{n} + \frac{r_1}{n\theta_1}, \frac{u_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{u_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{22}$$
where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We define the activated quadrature operators
$$\widehat{S_n}(f)(x) := \widehat{S_n}(f, x_1, ..., x_N) := \int_{\mathbb{R}^N} \delta_n(f)(u)\,\widetilde{Z}(nx-u)\,du = \tag{23}$$
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} \delta_n(f)(u_1, ..., u_N)\left(\prod_{i=1}^{N}\Phi(nx_i - u_i)\right)du_1...du_N, \ \ \forall x \in \mathbb{R}^N.$$

One can rewrite

$$\widetilde{S}_n(f)(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} f\left(\frac{u_1}{n},\frac{u_2}{n},...,\frac{u_N}{n}\right)\left(\prod_{i=1}^{N}\Phi(nx_i-u_i)\right)du_1...du_N, \tag{24}$$

and

$$\widetilde{S}_n^*(f)(x) = n^N\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\left(\int_{\frac{u_1}{n}}^{\frac{u_1+1}{n}}\int_{\frac{u_2}{n}}^{\frac{u_2+1}{n}}...\int_{\frac{u_N}{n}}^{\frac{u_N+1}{n}} f(t_1,...,t_N)\,dt_1...dt_N\right)$$
$$\left(\prod_{i=1}^{N}\Phi(nx_i-u_i)\right)du_1...du_N, \tag{25}$$

$n\in\mathbb{N}$, $\forall\, x\in\mathbb{R}^N$.

In this work we study the approximation properties of the operators $\widetilde{S}_n$, $\widetilde{S}_n^*$ and $\widehat{S_n}$, expecially their convergence to the unit operator $I$, and being treated as positive linear operators.

We notice that

$$\widetilde{S}_n(1) = \widetilde{S}^*(1) = \widehat{S}(1) = 1. \tag{26}$$

From [7] we have that $\widetilde{S}_n$, $\widetilde{S}_n^*$, $\widehat{S_n}$ map $C_B\left(\mathbb{R}^N\right)$ into itself.

Differentiation of our operators follows:

**Remark.** *([7]) Let $i\in\mathbb{N}$ be fixed. Assume that $f\in C^{(i)}\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}$. Here $f_\alpha$ denotes a partial derivative of $f$, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_j\in\mathbb{Z}_+$, $j=1,...,N$, and $|\alpha|:=\sum_{j=1}^{N}\alpha_j=l$, where $l=0,1,...,i$.*

*We write also $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order $l$.*

*We assume that any partial $f_\alpha\in C_B\left(\mathbb{R}^N\right)$, for all $\alpha:|\alpha|=l$, $l=0,1,...,i$.*

*We have that*

$$\left(\widetilde{S}_n(f)\right)_\alpha(x) = \int_{\mathbb{R}^N} f_\alpha\left(x-\frac{z}{n}\right)\widetilde{Z}(z)\,dz = \tag{27}$$
$$\int_{\mathbb{R}^N} f_\alpha\left(\frac{u}{n}\right)\widetilde{Z}(nx-u)\,du = \left(\widetilde{S}_n(f_\alpha)\right)(x),\ \ \forall\, x\in\mathbb{R}^N.$$

*Similarly, we have that*

$$\left(\widetilde{S}_n^*(f)\right)_\alpha(x) = \left(\widetilde{S}_n^*(f_\alpha)\right)(x), \tag{28}$$

*and*

$$\left(\widehat{S_n}(f)\right)_\alpha(x) = \left(\widehat{S_n}(f_\alpha)\right)(x),\ \ \forall\, x\in\mathbb{R}^N; \tag{29}$$

*for all $\alpha:|\alpha|=l$, $l=0,1,...,i$.*

We also need

**Remark.** *([7]) About iterated operators:*

*It holds*

$$\left|\widetilde{S}_n(f)(x)\right| \leq \|f\|_\infty\int_{\mathbb{R}^N}\widetilde{Z}(z)\,dz = \|f\|_\infty,$$

*i.e.*

$$\left\|\widetilde{S}_n(f)\right\|_\infty \leq \|f\|_\infty. \tag{30}$$

*So $\widetilde{S}_n$ is a bounded positive linear operator.*

*Clearly it holds*

$$\left\| \widetilde{S}_n^2 (f) \right\|_\infty = \left\| \widetilde{S}_n \left( \widetilde{S}_n (f) \right) \right\|_\infty \leq \left\| \widetilde{S}_n (f) \right\|_\infty \leq \|f\|_\infty . \tag{31}$$

*And for* $k \in \mathbb{N}$ *we obtain*

$$\left\| \widetilde{S}_n^k (f) \right\|_\infty \leq \left\| \widetilde{S}_n^{k-1} (f) \right\|_\infty \leq \left\| \widetilde{S}_n^{k-2} (f) \right\|_\infty \leq ... \leq \|f\|_\infty , \tag{32}$$

*so the contraction property valid and* $\widetilde{S}_n^k$ *is a bounded linear operator.*

*Let* $r \in \mathbb{N}$*, we observe that*

$$\widetilde{S}_n^r f - f = \left( \widetilde{S}_n^r f - \widetilde{S}_n^{r-1} f \right) + \left( \widetilde{S}_n^{r-1} f - \widetilde{S}_n^{r-2} f \right) + \left( \widetilde{S}_n^{r-2} f - \widetilde{S}_n^{r-3} f \right)$$
$$+ ... + \left( \widetilde{S}_n^2 f - \widetilde{S}_n f \right) + \left( \widetilde{S}_n f - f \right). \tag{33}$$

*Then*

$$\left\| \widetilde{S}_n^r f - f \right\|_\infty \leq \left\| \widetilde{S}_n^r f - \widetilde{S}_n^{r-1} f \right\|_\infty + \left\| \widetilde{S}_n^{r-1} f - \widetilde{S}_n^{r-2} f \right\|_\infty + \left\| \widetilde{S}_n^{r-2} f - \widetilde{S}_n^{r-3} f \right\|_\infty$$
$$+ ... + \left\| \widetilde{S}_n^2 f - \widetilde{S}_n f \right\|_\infty + \left\| \widetilde{S}_n f - f \right\|_\infty =$$
$$\left\| \widetilde{S}_n^{r-1} \left( \widetilde{S}_n f - f \right) \right\|_\infty + \left\| \widetilde{S}_n^{r-2} \left( \widetilde{S}_n f - f \right) \right\|_\infty + ... + \left\| \widetilde{S}_n \left( \widetilde{S}_n f - f \right) \right\|_\infty +$$
$$\left\| \widetilde{S}_n f - f \right\|_\infty \leq r \left\| \widetilde{S}_n f - f \right\|_\infty .$$

*Therefore*

$$\left\| \widetilde{S}_n^r f - f \right\|_\infty \leq r \left\| \widetilde{S}_n f - f \right\|_\infty . \tag{34}$$

*Let now* $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$*, and* $\widetilde{S}_{m_i}$ *as above.*

$$\widetilde{S}_{m_r} \left( \widetilde{S}_{m_{r-1}} \left( ... \widetilde{S}_{m_2} \left( \widetilde{S}_{m_1} f \right) \right) \right) - f = ... =$$
$$\widetilde{S}_{m_r} \left( \widetilde{S}_{m_{r-1}} \left( ... \widetilde{S}_{m_2} \right) \right) \left( \widetilde{S}_{m_1} f - f \right) + \widetilde{S}_{m_r} \left( \widetilde{S}_{m_{r-1}} \left( ... \widetilde{S}_{m_3} \right) \right) \left( \widetilde{S}_{m_2} f - f \right) + \tag{35}$$
$$\widetilde{S}_{m_r} \left( \widetilde{S}_{m_{r-1}} \left( ... \widetilde{S}_{m_4} \right) \right) \left( \widetilde{S}_{m_3} f - f \right) + ... + \widetilde{S}_{m_r} \left( \widetilde{S}_{m_{r-1}} f - f \right) + \widetilde{S}_{m_r} f - f.$$

*Consequently it holds, as in* [4]*, Chapter 2,*

$$\left\| \widetilde{S}_{m_r} \left( \widetilde{S}_{m_{r-1}} \left( ... \widetilde{S}_{m_2} \left( \widetilde{S}_{m_1} f \right) \right) \right) - f \right\|_\infty \leq \sum_{i=1}^{r} \left\| \widetilde{S}_{m_i} f - f \right\|_\infty . \tag{36}$$

The properties (30)-(36) are also valid for the iterations of operators $\widetilde{S}_n^*$ and $\widehat{S_n}$.

**Notation.** *We denote by* $K_n$ *any of the operators* $\widetilde{S}_n$*,* $\widetilde{S}_n^*$ *and* $\widehat{S_n}$*,* $\forall\, n \in \mathbb{N}$.

## 3. Auxiliary Results

Here let $x_0 \in \mathbb{R}^N$, $m, n \in \mathbb{N}$. It follows:

**Proposition 3.1.** *It holds that*

$$\widetilde{S}_n \left( \| \cdot - x_0 \|_\infty^m \right) (x_0) > 0, \tag{37}$$

*and*

$$0 < \left\| \widetilde{S}_n \left( \| \cdot - x_0 \|_\infty^m \right) (x_0) \right\|_\infty \leq \tag{38}$$

$$\frac{N^m}{n^m}\left[\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right](<+\infty)\to 0, \textit{ as } n\to+\infty.$$

*Proof.* We have that

$$0<\widetilde{S}_n\left(\|\cdot-x_0\|_\infty^m\right)(x_0)=\int_{\mathbb{R}^N}\left\|\frac{u}{n}-x_0\right\|_\infty^m\widetilde{Z}(nx_0-u)\,du=$$
$$\frac{1}{n^m}\int_{\mathbb{R}^N}\|nx_0-u\|_\infty^m\widetilde{Z}(nx_0-u)\,du= \tag{39}$$
$$\frac{1}{n^m}\int_{\mathbb{R}^N}\|x\|_\infty^m\widetilde{Z}(x)\,dx\overset{(19)}{\leq}$$
$$\frac{N^m}{n^m}\left[\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right](<+\infty)\to 0, \text{ as } n\to+\infty.$$

∎

**Proposition 3.2.** *It holds that*

$$\widetilde{S}_n^*\left(\|\cdot-x_0\|_\infty^m\right)(x_0)>0, \tag{40}$$

*and*

$$0<\left\|\widetilde{S}_n^*\left(\|\cdot-x_0\|_\infty^m\right)(x_0)\right\|_\infty\leq$$
$$\frac{2^{m-1}}{n^m}\left[1+N^m\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](<+\infty)\to 0, \textit{ as } n\to+\infty. \tag{41}$$

*Proof.* We have that

$$0<\widetilde{S}_n^*\left(\|\cdot-x_0\|_\infty^m\right)(x_0)=\int_{\mathbb{R}^N}\left(n^N\int_{\left[0,\frac{1}{n}\right]^N}\left\|t+\frac{u}{n}-x_0\right\|_\infty^m dt\right)\widetilde{Z}(nx_0-u)\,du\leq$$
$$\int_{\mathbb{R}^N}\left(n^N\int_{\left[0,\frac{1}{n}\right]^N}\left(\left\|\frac{u}{n}-x_0\right\|_\infty+\|t\|_\infty\right)^m dt\right)\widetilde{Z}(nx_0-u)\,du\leq \tag{42}$$
$$\int_{\mathbb{R}^N}\left(n^N\int_{\left[0,\frac{1}{n}\right]^N}\left(\left\|\frac{u}{n}-x_0\right\|_\infty+\frac{1}{n}\right)^m dt\right)\widetilde{Z}(nx_0-u)\,du=$$
$$\int_{\mathbb{R}^N}\left(\left\|\frac{u}{n}-x_0\right\|_\infty+\frac{1}{n}\right)^m\widetilde{Z}(nx_0-u)\,du=$$
$$\frac{1}{n^m}\int_{\mathbb{R}^N}\left(\|nx_0-u\|_\infty+1\right)^m\widetilde{Z}(nx_0-u)\,du\leq \tag{43}$$
$$\frac{2^{m-1}}{n^m}\left(\int_{\mathbb{R}^N}\left(1+\|nx_0-u\|_\infty^m\right)\widetilde{Z}(nx_0-u)\,du\right)\overset{(17)}{=}$$
$$\frac{2^{m-1}}{n^m}\left[1+\int_{\mathbb{R}^N}\|nx_0-u\|_\infty^m\widetilde{Z}(nx_0-u)\,du\right]=$$
$$\frac{2^{m-1}}{n^m}\left[1+\int_{\mathbb{R}^N}\|x\|_\infty^m\widetilde{Z}(x)\,dx\right]\overset{(19)}{\leq}$$

$$\frac{2^{m-1}}{n^m}\left[1+N^m\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](<+\infty)\to 0, \text{ as } n\to+\infty.$$

□

**Proposition 3.3.** *It holds that*

$$\widehat{S_n}\left(\|\cdot-x_0\|_\infty^m\right)(x_0)>0,$$

*and*

$$0<\left\|\widehat{S_n}\left(\|\cdot-x_0\|_\infty^m\right)(x_0)\right\|_\infty\leq$$

$$\frac{2^{m-1}}{n^m}\left[1+N^m\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](<+\infty)\to 0, \text{ as } n\to+\infty. \tag{44}$$

*Proof.* We have that

$$0<\widehat{S_n}\left(\|\cdot-x_0\|_\infty^m\right)(x_0)=\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left\|\frac{u}{n}+\frac{r}{n\theta}-x_0\right\|_\infty^m\right)\widetilde{Z}(nx_0-u)\,du\leq$$

$$\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left(\left\|\frac{u}{n}-x_0\right\|_\infty+\left\|\frac{r}{n\theta}\right\|_\infty\right)^m\right)\widetilde{Z}(nx_0-u)\,du\leq$$

$$\int_{\mathbb{R}^N}\left(\left\|\frac{u}{n}-x_0\right\|_\infty+\frac{1}{n}\right)^m\widetilde{Z}(nx_0-u)\,du \tag{45}$$

(the rest as in the proof of Proposition 3.2)

$$\frac{2^{m-1}}{n^m}\left[1+N^m\left(\frac{\tanh(\lambda)}{(m+1)}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right](<+\infty)\to 0, \text{ as } n\to+\infty.$$

□

## 4. MAIN RESULTS

We need the following:

**Theorem 4.1.** *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of positive linear operators from $C_B\left(\mathbb{R}^N\right)$ into itself, such that $L_n(1)=1$. Assume $0<L_n\left(\|\cdot-x_0\|_\infty\right)(x_0)<+\infty$.*

*Then*

$$|L_n(f)(x_0)-f(x_0)|\leq 2\omega_1\left(f,L_n\left(\|\cdot-x_0\|_\infty\right)(x_0)\right)<+\infty. \tag{46}$$

*Given that $\lim_{n\to+\infty}L_n\left(\|\cdot-x_0\|_\infty\right)(x_0)=0$ and $f$ is uniformly continuous, we get that $\lim_{n\to+\infty}L_n(f)(x_0)=f(x_0)$.*

*Proof.* Here it is $\omega_1(f,h)<\infty$, $h>0$.

By Lemma 7.1.1, p. 208, [2], we get that

$$|f(t)-f(x)|\leq\omega_1(f,h)\left\lceil\frac{\|t-x\|_\infty}{h}\right\rceil, \tag{47}$$

where $\lceil\cdot\rceil$ is the ceiling function.

But it holds

$$\left\lceil\frac{\|t-x\|_\infty}{h}\right\rceil\leq 1+\frac{\|t-x\|_\infty}{h}. \tag{48}$$

Thus we get

$$|f(t) - f(x)| \leq \omega_1(f, h)\left[1 + \frac{\|t - x\|_\infty}{h}\right], \ \forall\, t \in \mathbb{R}^N. \tag{49}$$

The last can be written also as

$$|f(\cdot) - f(x)| \leq \omega_1(f, h)\left[1 + \frac{\|\cdot - x\|_\infty}{h}\right], \quad \text{over } \mathbb{R}^N. \tag{50}$$

Hence we derive

$$|L_n(f)(x_0) - f(x_0)| = |L_n(f(\cdot) - f(x_0))(x_0)| \leq$$
$$L_n(|f(\cdot) - f(x_0)|)(x_0) \overset{(50)}{\leq} \omega_1(f, h)\left[1 + \frac{L_n(\|\cdot - x_0\|_\infty)(x_0)}{h}\right] \tag{51}$$

(choose $h := L_n(\|\cdot - x_0\|_\infty)(x_0) > 0$)

$$= 2\omega_1(f, L_n(\|\cdot - x_0\|_\infty)(x_0)).$$

□

We derive

**Theorem 4.2.** *We have for* $K_n = \widetilde{S}_n$, $\widetilde{S}_n^*$, $\widehat{S_n}$ *that*

$$|K_n(f)(x_0) - f(x_0)| \leq 2\omega_1(f, K_n(\|\cdot - x_0\|_\infty)(x_0)) < +\infty, \tag{52}$$

*where* $x_0 \in \mathbb{R}^N$, $f \in C_B(\mathbb{R}^N)$.

*Given that* $\lim_{n\to+\infty} K_n(\|\cdot - x_0\|_\infty)(x_0) = 0$ *and* $f$ *is uniformly continuous, we get that* $\lim_{n\to+\infty} K_n(f)(x_0) = f(x_0)$.

*Proof.* By Theorem 4.1. □

More specifically we derive:

**Proposition 4.3.** *It holds that (*$f \in C_B(\mathbb{R}^N)$*)*

$$\left\|\widetilde{S}_n(f) - f\right\|_\infty \leq 2\omega_1\left(f, \frac{N}{n}\left[\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right]\right) < +\infty. \tag{53}$$

*If* $f$ *is uniformly continuous, then* $\widetilde{S}_n(f) \to f$*, uniformly as* $n \to +\infty$.

*Proof.* By Proposition 3.1, and Theorem 4.2. □

**Proposition 4.4.** *It holds that (*$f \in C_B(\mathbb{R}^N)$*)*

$$\left\|\widetilde{S}_n^*(f) - f\right\|_\infty \leq 2\omega_1\left(f, \frac{1}{n}\left[1 + N\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right) < +\infty. \tag{54}$$

*If* $f$ *is uniformly continuous, then* $\widetilde{S}_n^*(f) \to f$*, uniformly as* $n \to +\infty$.

*Proof.* By Proposition 3.2, and Theorem 4.2. □

**Proposition 4.5.** *It holds that (*$f \in C_B(\mathbb{R}^N)$*)*

$$\left\|\widehat{S_n}(f) - f\right\|_\infty \leq 2\omega_1\left(f, \frac{1}{n}\left[1 + N\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right) < +\infty. \tag{55}$$

*If $f$ is uniformly continuous, then $\widehat{S_n}(f) \to f$, uniformly as $n \to +\infty$.*

*Proof.* By Proposition 3.3, and Theorem 4.2. ☐

Next, we study our operators under convexity:

**Theorem 4.6.** *Let $(L_n)_{n\in\mathbb{N}}$ positive linear operators from $C_B(\mathbb{R}^N)$ into itself: $L_n(1) = 1$. Let $f \in C_B(\mathbb{R}^N)$ and the convex function $|f(\cdot) - f(x_0)|$ over $\mathbb{R}^N$, $x_0 \in \mathbb{R}^N$. Assume that $0 < L_n(\|\cdot - x_0\|_\infty)(x_0) < +\infty$.*

*Then*

$$|L_n(f)(x_0) - f(x_0)| \leq \omega_1\left(f, L_n(\|\cdot - x_0\|_\infty)(x_0)\right). \tag{56}$$

*The last (56) is attained by $f(t) = \|t - x_0\|_\infty$, $\forall\, t \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, that is sharp. Given that $f$ is uniformly continuous and $L_n(\|\cdot - x_0\|_\infty)(x_0) \to 0$, as $n \to \infty$, then $L_n(f)(x_0) \to f(x_0)$, as $n \to \infty$.*

*Proof.* Based on Lemma 8.1.1, p. 243, [2], we obtain:

Let $(V, \|\cdot\|)$ be a real normed vector space and $f : V \to \mathbb{R}$ such that $|f(t) - f(x_0)|$ is convex in $t \in V$; $x_0 \in V$. Then

$$|f(t) - f(x_0)| \leq \frac{\omega_1(f, h)}{h}\|t - x_0\|, \ \ \forall\, t \in V, \ h > 0. \tag{57}$$

That is, it holds here

$$|f(\cdot) - f(x_0)| \leq \frac{\omega_1(f, h)}{h}\|\cdot - x_0\|, \ \ \text{over } \mathbb{R}^N. \tag{58}$$

Then, the proof of (56) goes as in the proof of Theorem 4.1.

Inequality (56) is sharp, namely it is attained by $f(t) = \|t - x_0\|_\infty$, $\forall\, t \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$.

We have

$$|L_n(f, x_0) - f(x_0)| = L_n(\|\cdot - x_0\|_\infty)(x_0). \tag{59}$$

Next we observe

$$\omega_1\left(f, L_n(\|\cdot - x_0\|_\infty)(x_0)\right) = \omega_1\left(\|\cdot - x_0\|_\infty, L_n(\|\cdot - x_0\|_\infty)(x_0)\right) = \sup_{\left\{\begin{array}{c} t_1, t_2 \in \mathbb{R}^N : \\ : \|t_1 - t_2\|_\infty \leq L_n(\|\cdot - x_0\|_\infty)(x_0) \end{array}\right.} \left|\|t_1 - x_0\|_\infty - \|t_2 - x_0\|_\infty\right| \leq \tag{60}$$

$$\sup_{\left\{\begin{array}{c} t_1, t_2 \in \mathbb{R}^N : \\ : \|t_1 - t_2\|_\infty \leq L_n(\|\cdot - x_0\|_\infty)(x_0) \end{array}\right.} \|t_1 - t_2\|_\infty = L_n(\|\cdot - x_0\|_\infty)(x_0).$$

So that

$$\omega_1\left(f, L_n(\|\cdot - x_0\|_\infty)(x_0)\right) = L_n(\|\cdot - x_0\|_\infty)(x_0).$$

Thus (56) it is attained. ☐

Consequently we obtain:

**Theorem 4.7.** *Let $f \in C_B(\mathbb{R}^N)$ and the convex function $|f(\cdot) - f(x_0)|$ over $\mathbb{R}^N$, $x_0 \in \mathbb{R}^N$.*

*Then*

$$|K_n(f)(x_0) - f(x_0)| \leq \omega_1\left(f, K_n(\|\cdot - x_0\|_\infty)(x_0)\right) < +\infty. \tag{61}$$

*The last (61) is attained by $f(t) = \|t - x_0\|_\infty$, $\forall\, t \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$, that is sharp. Given that $f$ is uniformly continuous, we derive that $K_n(f)(x_0) \to f(x_0)$, as $n \to \infty$.*

*Proof.* By Theorem 4.6. □

More specifically we derive:

**Proposition 4.8.** *Let $f \in C_B\left(\mathbb{R}^N\right)$ such that $|f(\cdot) - f(x_0)|$ is convex over $\mathbb{R}^N$, $x_0 \in \mathbb{R}^N$.*

*Then*

$$\left|\widetilde{S}_n(f)(x_0) - f(x_0)\right| \leq \omega_1\left(f, \frac{N}{n}\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right) < +\infty. \quad (62)$$

*If $f$ is uniformly continuous, then $\widetilde{S}_n(f)(x_0) \to f(x_0)$, as $n \to \infty$.*

*Proof.* By Proposition 3.1, and Theorem 4.7. □

**Proposition 4.9.** *Let $f$ be as in Proposition 4.8. It holds that*

$$\left|\widetilde{S}_n^*(f)(x_0) - f(x_0)\right| \leq \omega_1\left(f, \frac{1}{n}\left[1 + N\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right) < +\infty. \quad (63)$$

*If $f$ is uniformly continuous, then $\widetilde{S}_n^*(f)(x_0) \to f(x_0)$, as $n \to \infty$.*

*Proof.* By Proposition 3.2, and Theorem 4.7. □

**Proposition 4.10.** *Let $f$ be as in Proposition 4.8. It holds that*

$$\left|\widehat{S_n}(f)(x_0) - f(x_0)\right| \leq \omega_1\left(f, \frac{1}{n}\left[1 + N\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right) < +\infty. \quad (64)$$

*If $f$ is uniformly continuous, then $\widehat{S_n}(f)(x_0) \to f(x_0)$, as $n \to \infty$.*

*Proof.* By Proposition 3.3, and Theorem 4.7. □

We make

**Remark.** *We consider $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$ and $\left(\mathbb{R}^N\right)^j$ denote the $j$-fold product space $\mathbb{R}^N \times ... \times \mathbb{R}^N$ endowed with the max-norm $\|x\|_{(\mathbb{R}^N)^j} := \max_{1 \leq k \leq j} \|x_k\|_\infty$, where $x := (x_1, ..., x_j) \in \left(\mathbb{R}^N\right)^j$. Then the space $L_j := L_j\left(\left(\mathbb{R}^N\right)^j; \mathbb{R}\right)$ of all real valued multilinear continuous functions $g : \left(\mathbb{R}^N\right)^j \to \mathbb{R}$ is a Banach space with norm*

$$\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{(\mathbb{R}^N)^j} = 1\right)} |g(x)| = \sup \frac{|g(x)|}{\|x_1\|_\infty ... \|x_j\|_\infty}. \quad (65)$$

*Let $x_0 \in \mathbb{R}^N$ be fixed, and $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous and bounded function whose Fréchet derivatives*

$$f^{(j)} : \mathbb{R}^N \to L_j = L_j\left(\left(\mathbb{R}^N\right)^j; \mathbb{R}\right)$$

*exist and are continuous and bounded for all $1 \leq j \leq m$, $m \in \mathbb{N}$. Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in \left(\mathbb{R}^N\right)^j = \underbrace{\mathbb{R}^N \times ... \times \mathbb{R}^N}_{-j-}$.*

*Then, by Taylor's formula, see* [10], *p. 124 and* [9], *we get*

$$f(x)=\sum_{j=0}^{m}\frac{f^{(j)}(x_0)(x-x_0)^j}{j!}+R_m(x,x_0),\ \text{all } x\in\mathbb{R}^N, \tag{66}$$

*where*

$$R_m(x,x_0):=\int_0^1\frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0+u(x-x_0))-f^{(m)}(x_0)\right)(x-x_0)^m\,du. \tag{67}$$

*Here we assume that* $f^{(j)}(x_0)=0$, $j=1,...,m$, *then*

$$f(x)-f(x_0)=\int_0^1\frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0+u(x-x_0))-f^{(m)}(x_0)\right)(x-x_0)^m\,du. \tag{68}$$

*Consider*

$$w:=\omega_1\left(f^{(m)},h\right)=\sup_{\substack{x,y\in\mathbb{R}^N\\ :\|x-y\|_\infty\le h}}\left\|f^{(m)}(x)-f^{(m)}(y)\right\|,\ h>0. \tag{69}$$

*Notice that* $w<+\infty$.

*We obtain that*

$$\begin{gathered}\left\|\left(f^{(m)}(x_0+u(x-x_0))-f^{(m)}(x_0)\right)(x-x_0)^m\right\|\le\\ \left\|f^{(m)}(x_0+u(x-x_0))-f^{(m)}(x_0)\right\|\|x-x_0\|^m\le\\ w\|x-x_0\|^m\left\lceil\frac{u\|x-x_0\|}{h}\right\rceil,\end{gathered} \tag{70}$$

*by Lemma 7.1.1, p. 208,* [2].

*Therefore for all* $x\in\mathbb{R}^N$ :

$$\begin{gathered}|f(x)-f(x_0)|\le w\|x-x_0\|^m\int_0^1\left\lceil\frac{u\|x-x_0\|_\infty}{h}\right\rceil\frac{(1-u)^{m-1}}{(m-1)!}du\\ =w\Phi_m(\|x-x_0\|_\infty),\end{gathered} \tag{71}$$

*by a change of variable, where*

$$\Phi_m(t):=\int_0^{|t|}\left\lceil\frac{s}{h}\right\rceil\frac{(|t|-s)^{m-1}}{(m-1)!}ds=\frac{1}{m!}\left(\sum_{j=0}^{\infty}(|t|-j\cdot h)_+^m\right), \tag{72}$$

$t\in\mathbb{R}$; $(x)_+=\max\{x,0\}$.

*For a full list of properties of the spline function* $\Phi_m$ *see Remark 7.1.3 of* [2], *p. 210.*

*By (7.1.18) of* [2], *p. 210, we have*

$$\Phi_m(x)\le\left(\frac{|x|^{m+1}}{(m+1)!h}+\frac{|x|^m}{2m!}+\frac{h|x|^{m-1}}{8(m-1)!}\right),\ \forall x\in\mathbb{R}. \tag{73}$$

*Consequently it holds* ($x\in\mathbb{R}^N$)

$$|f(x)-f(x_0)|\le\omega_1\left(f^{(m)},h\right)\left(\frac{\|x-x_0\|_\infty^{m+1}}{(m+1)!h}+\frac{\|x-x_0\|_\infty^m}{2m!}+\frac{h\|x-x_0\|_\infty^{m-1}}{8(m-1)!}\right). \tag{74}$$

*So that*

$$\left|f(\cdot)-f(x_0)\right| \leq \omega_1\left(f^{(m)},h\right)\left(\frac{\|\cdot-x_0\|_\infty^{m+1}}{(m+1)!h}+\frac{\|\cdot-x_0\|_\infty^{m}}{2m!}+\frac{h\|\cdot-x_0\|_\infty^{m-1}}{8(m-1)!}\right), \tag{75}$$

*over* $\mathbb{R}^N$.

*Let* $(L_n)_{n\in\mathbb{N}}$ *be a sequence of positive linear operators from* $C_B\left(\mathbb{R}^N\right)$ *into itself such that* $L_n(1)=1$, $\forall\, n\in\mathbb{N}$.

*By* $|f|\leq|f|$, *iff* $-|f|\leq f\leq|f|$, *then* $-L_n(|f|)\leq L_n(f)\leq L_n(|f|)$, *and* $|L_n(f)|\leq L_n(|f|)$.

*Hence*

$$\left|L_n\left(f(\cdot)-f(x_0)\right)(x_0)\right| \leq L_n\left(\left|f(\cdot)-f(x_0)\right|\right)(x_0) \overset{(75)}{\leq} \omega_1\left(f^{(m)},h\right)$$

$$\left[\frac{L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)}{(m+1)!h}+\frac{L_n\left(\|\cdot-x_0\|_\infty^{m}\right)(x_0)}{2m!}+\frac{hL_n\left(\|\cdot-x_0\|_\infty^{m-1}\right)(x_0)}{8(m-1)!}\right]=:(\xi)\,. \tag{76}$$

We need the following Hölder's type inequality for positive linear operators.

**Theorem 4.11.** *(*[8]*) Let* $L$ *be a positive linear operator from* $C_B\left(\mathbb{R}^N\right)$ *into itself and* $f,g\in C_B\left(\mathbb{R}^N\right)$, *furthermore let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Assume that* $L\left(\left(|f(\cdot)|^p\right)\right)(s_*)$, $L\left(\left(|g(\cdot)|^q\right)\right)(s_*)>0$ *for some* $s_*\in\mathbb{R}^N$. *Then*

$$L\left(|f(\cdot)g(\cdot)|\right)(s_*)\leq\left(L\left(\left(|f(\cdot)|^p\right)\right)(s_*)\right)^{\frac{1}{p}}\left(L\left(\left(|g(\cdot)|^q\right)\right)(s_*)\right)^{\frac{1}{q}}. \tag{77}$$

We continue with

**Remark.** *Let* $g=1$ *and* $L_n:L_n(1)=1$, *then by (77) we get:*

$$L_n\left(\|\cdot-x_0\|_\infty^{m}\right)(x_0)\leq\left(L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}, \tag{78}$$

*and*

$$L_n\left(\|\cdot-x_0\|_\infty^{(m-1)}\right)(x_0)\leq\left(L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right)^{\left(\frac{m-1}{m+1}\right)}, \tag{79}$$

*where* $m\in\mathbb{N}$ *and* $L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)>0$.

*Continuing from (76):*

$$(\xi)\leq\omega_1\left(f^{(m)},h\right)\left[\frac{L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)}{(m+1)!h}+\right. \tag{80}$$

$$\left.\frac{\left(L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m}{m+1}}}{2m!}+\frac{h\left(L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{m-1}{m+1}}}{8(m-1)!}\right]$$

*(assume* $h:=\left(L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}}>0$*)*

$$=\omega_1\left(f^{(m)},\left(L_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right)^{\frac{1}{m+1}}\right)$$

$$\left[\frac{h^m}{(m+1)!}+\frac{h^m}{2m!}+\frac{h^m}{8(m-1)!}\right] \tag{81}$$

$$= \omega_1 \left( f^{(m)}, \left( L_n \left( \|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right)^{\frac{1}{m+1}} \right)$$

$$\left( L_n \left( \|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right)^{\frac{m}{m+1}} \left[ \frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!} \right]. \tag{82}$$

We have proved that

**Theorem 4.12.** *Let $f \in C_B(\mathbb{R}^N)$, $x_0 \in \mathbb{R}^N$, Fréchet derivatives $f^{(j)}$ continuous and bounded with $f^{(j)}(x_0) = 0$, $j = 1, ..., m$. Then*

$$|L_n(f)(x_0) - f(x_0)| \leq \omega_1 \left( f^{(m)}, \left( L_n \left( \|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right)^{\frac{1}{m+1}} \right) \tag{83}$$

$$\left( L_n \left( \|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right)^{\frac{m}{m+1}} \left[ \frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!} \right], \ \forall\, n \in \mathbb{N}.$$

*Here $L_n(1) = 1$, $L_n$ positive linear operators from $C_B(\mathbb{R}^N)$ into itself under the assumption $L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) > 0$.*

*Let $L_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) \to 0$, as $n \to +\infty$, then $\lim_{n \to +\infty} L_n(f)(x_0) = f(x_0)$.*

Next we apply Theorem 4.12 to the operators $K_n$.

**Theorem 4.13.** *Let $f \in C_B(\mathbb{R}^N)$, $x_0 \in \mathbb{R}^N$, Fréchet derivatives $f^{(j)}$ continuous and bounded with $f^{(j)}(x_0) = 0$, $j = 1, ..., m$. Then*

$$|K_n(f)(x_0) - f(x_0)| \leq \omega_1 \left( f^{(m)}, \left( K_n \left( \|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right)^{\frac{1}{m+1}} \right) \tag{84}$$

$$\left( K_n \left( \|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right)^{\frac{m}{m+1}} \left[ \frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!} \right],$$

$\forall\, n \in \mathbb{N}$.

*We have that $\lim_{n \to +\infty} K_n(f)(x_0) = f(x_0)$.*

*Proof.* Here $K_n$ any of $\widetilde{S}_n$, $\widetilde{S}_n^*$ and $\widehat{S_n}$.

By Theorem 4.12, and Propositions 3.1-3.3 we get $K_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) \to 0$, as $n \to +\infty$. ☐

More specifically we derive:

**Proposition 4.14.** *All as in Theorem 4.13, $m \in \mathbb{N}$. Denote by*

$$\psi_{1n}(m) := \frac{N^m}{n^m} \left[ \frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q + \frac{1}{q}\right) e^{2\lambda} m!}{(2\lambda)^m} \right]. \tag{85}$$

*Then*

$$\left| \widetilde{S}_n(f)(x_0) - f(x_0) \right| \leq \omega_1 \left( f^{(m)}, (\psi_{1n}(m+1))^{\frac{1}{m+1}} \right)$$

$$(\psi_{1n}(m+1))^{\frac{m}{m+1}} \left[ \frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!} \right], \tag{86}$$

$\forall\, n \in \mathbb{N}$.

*We have that $\lim_{n \to +\infty} \widetilde{S}_n(f)(x_0) = f(x_0)$.*

*Proof.* By Theorem 4.13 and Proposition 3.1. ☐

**Proposition 4.15.** *All as in Theorem 4.13, $m \in \mathbb{N}$. Denote by*

$$\psi_{2n}(m) := \frac{2^{m-1}}{n^m}\left[1 + N^m\left(\frac{\tanh(\lambda)}{(m+1)} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}m!}{(2\lambda)^m}\right)\right]. \tag{87}$$

*Then*

$$\left\{\begin{array}{l} \left|\widetilde{S}_n^*(f)(x_0) - f(x_0)\right| \\ \left|\widehat{S_n}(f)(x_0) - f(x_0)\right| \end{array}\right\} \leq \omega_1\left(f^{(m)}, (\psi_{2n}(m+1))^{\frac{1}{m+1}}\right)$$
$$(\psi_{2n}(m+1))^{\frac{m}{m+1}}\left[\frac{1}{(m+1)!} + \frac{1}{2m!} + \frac{1}{8(m-1)!}\right], \tag{88}$$

$\forall\, n \in \mathbb{N}$.

*We have that* $\lim_{n\to+\infty} \widetilde{S}_n^*(f)(x_0) = \lim_{n\to+\infty} \widehat{S_n}(f)(x_0) = f(x_0)$.

*Proof.* By Theorem 4.13 and Propositions 3.2, 3.3. □

Next we do simultaneous approximation.

**Theorem 4.16.** *Let $i \in \mathbb{N}$ be fixed and $f \in C^{(i)}\left(\mathbb{R}^N\right)$. Here $f_\alpha$ denotes a partial derivative of $f$, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $j = 1, ..., N$, and $|\alpha| := \sum_{j=1}^{N} \alpha_j = l$, where $l = 0, 1, ..., i$. We assume that $f_\alpha \in C_B\left(\mathbb{R}^N\right)$, for all $\alpha : |\alpha| = l$, $l = 0, 1, ..., i$; $x_0 \in \mathbb{R}^N$. Then*

$$\left|(K_n(f))_\alpha(x_0) - f_\alpha(x_0)\right| \leq 2\omega_1\left(f_\alpha, K_n\left(\|\cdot - x_0\|_\infty\right)(x_0)\right) < +\infty. \tag{89}$$

*Given that $\lim_{n\to+\infty} K_n\left(\|\cdot - x_0\|_\infty\right)(x_0) = 0$ and $f_\alpha$ is uniformly continuous, we get that $\lim_{n\to+\infty} (K_n(f))_\alpha(x_0) = f(x_0)$.*

*Proof.* By Remark 2 and Theorem 4.2. □

**Proposition 4.17.** *All as in Theorem 4.16. Then*

$$\left\|\left(\widetilde{S}_n(f)\right)_\alpha - f_\alpha\right\|_\infty \leq 2\omega_1\left(f_\alpha, \frac{N}{n}\left[\frac{\tanh(\lambda)}{2} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right]\right) < +\infty. \tag{90}$$

*If $f_\alpha$ is uniformly continuous, then $\left(\widetilde{S}_n(f)\right)_\alpha \to f_\alpha$, uniformly as $n \to +\infty$.*

*Proof.* By Remark 2 and Proposition 4.3. □

**Proposition 4.18.** *All as in Theorem 4.16. Then*

$$\left\{\begin{array}{l} \left\|\left(\widetilde{S}_n^*(f)\right)_\alpha - f_\alpha\right\|_\infty \\ \left\|\left(\widehat{S_n}(f)\right)_\alpha - f_\alpha\right\|_\infty \end{array}\right\} \leq 2\omega_1\left(f_\alpha, \frac{1}{n}\left[1 + N\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right) < +\infty. \tag{91}$$

*If $f_\alpha$ is uniformly continuous, then $\left(\widetilde{S}_n^*(f)\right)_\alpha \to f_\alpha$, $\left(\widehat{S_n}(f)\right)_\alpha \to f_\alpha$, uniformly as $n \to +\infty$.*

*Proof.* By Remark 2 and Propositions 4.4, 4.5. □

Next, we do simultaneous convexity approximation.

**Theorem 4.19.** *The notation is as in Theorem 4.16, with a particular $\alpha : |\alpha| = l \in \{0, 1, ..., i\}$, $f_\alpha \in C_B(\mathbb{R}^N)$ and $|f_\alpha(\cdot) - f_\alpha(x_0)|$ is convex over $\mathbb{R}^N$, $x_0 \in \mathbb{R}^N$. Then*

$$|(K_n(f))_\alpha(x_0) - f_\alpha(x_0)| \leq \omega_1\left(f_\alpha, K_n\left(\|\cdot - x_0\|_\infty\right)(x_0)\right) < +\infty. \tag{92}$$

*Given that $f_\alpha$ is uniformly continuous, we derive that $(K_n(f))_\alpha(x_0) \to f_\alpha(x_0)$, as $n \to \infty$.*

*Proof.* By Remark 2 and Theorem 4.7. □

More specifically we obtain:

**Proposition 4.20.** *All as in Theorem 4.19. Then*

$$\left|\left(\widetilde{S}_n(f)\right)_\alpha(x_0) - f_\alpha(x_0)\right| \leq \omega_1\left(f_\alpha, \frac{N}{n}\left[\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right]\right) < +\infty. \tag{93}$$

*If $f_\alpha$ is uniformly continuous, then $\left(\widetilde{S}_n(f)\right)_\alpha(x_0) \to f_\alpha(x_0)$, as $n \to +\infty$.*

*Proof.* By Remark 2, Proposition 4.8 and Theorem 4.19. □

**Proposition 4.21.** *All as in Theorem 4.19. Then*

$$\left\{\begin{array}{c} \left|\left(\widetilde{S}_n^*(f)\right)_\alpha(x_0) - f_\alpha(x_0)\right| \\ \left|\left(\widehat{S_n}(f)\right)_\alpha(x_0) - f_\alpha(x_0)\right| \end{array}\right\} \leq$$

$$\omega_1\left(f_\alpha, \frac{1}{n}\left[1 + N\left(\frac{\tanh(\lambda)}{2} + \frac{\left(q + \frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right) < +\infty. \tag{94}$$

*If $f_\alpha$ is uniformly continuous, then $\left(\widetilde{S}_n^*(f)\right)_\alpha(x_0) \to f_\alpha(x_0)$, $\left(\widehat{S_n}(f)\right)_\alpha(x_0) \to f_\alpha(x_0)$, as $n \to +\infty$.*

*Proof.* By Remark 2 and Propositions 4.9, 4.10 and Theorem 4.19. □

We continue with iterated approximations:

**Remark.** *Here $f \in C_B(\mathbb{R}^N)$; $r \in \mathbb{N}$; $m_1, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$; $K_n = \widetilde{S}_n, \widetilde{S}_n^*, \widehat{S_n}$.*

*As in (34) we get that*

$$\|K_n^r(f) - f\|_\infty \leq r\|K_n(f) - f\|_\infty, \tag{95}$$

*and as in (36) we obtain that*

$$\left\|K_{m_r}\left(K_{m_{r-1}}\left(...K_{m_2}\left(K_{m_1}(f)\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^r \|K_{m_i}(f) - f\|_\infty. \tag{96}$$

Clearly then follows:

**Proposition 4.22.** *Let $f \in C_B(\mathbb{R}^N)$; $r \in \mathbb{N}$; and $m_1, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$. Then*

*1)*

$$\left\|\widetilde{S}_n^r(f) - f\right\|_\infty \leq r\left\|\widetilde{S}_n(f) - f\right\|_\infty \leq \tag{97}$$

$$2r\omega_1\left(f,\frac{N}{n}\left[\frac{\tanh(\lambda)}{2}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right]\right)<+\infty,$$

*and the speed of convergence of $\widetilde{S}_n^r$ to the unit $I$ is not worse than the speed of convergence of $\widetilde{S}_n$ to $I$,*

*2)*

$$\left\|\widetilde{S}_{m_r}\left(\widetilde{S}_{m_{r-1}}\left(...\widetilde{S}_{m_2}\left(\widetilde{S}_{m_1}(f)\right)\right)\right)-f\right\|_\infty\leq\sum_{i=1}^{r}\left\|\widetilde{S}_{m_i}(f)-f\right\|_\infty\leq \tag{98}$$

$$2\sum_{i=1}^{r}\omega_1\left(f,\frac{N}{m_i}\left[\frac{\tanh(\lambda)}{2}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right]\right)\leq$$

$$2r\omega_1\left(f,\frac{N}{m_1}\left[\frac{\tanh(\lambda)}{2}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right]\right)<+\infty. \tag{99}$$

*Again, the speed of convergence of the iterated operator to $I$ is not worse than the speed of convergence of $\widetilde{S}_{m_1}$ to $I$.*

*Proof.* By (95), (96) and Proposition 4.3. □

Similarly we have:

**Proposition 4.23.** *Let $f\in C_B\left(\mathbb{R}^N\right)$; $r\in\mathbb{N}$; and $m_1,...,m_r\in\mathbb{N}:m_1\leq m_2\leq...\leq m_r$. Then*

*1)*

$$\left\{\begin{array}{l}\left\|\widetilde{S}_n^{*r}(f)-f\right\|_\infty\\ \left\|\widehat{S_n}^r(f)-f\right\|_\infty\end{array}\right\}\leq r\left\{\begin{array}{l}\left\|\widetilde{S}_n^{*}(f)-f\right\|_\infty\\ \left\|\widehat{S_n}(f)-f\right\|_\infty\end{array}\right\}\leq \tag{100}$$

$$2r\omega_1\left(f,\frac{1}{n}\left[1+N\left(\frac{\tanh(\lambda)}{2}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right)<+\infty,$$

*and the speed of convergence of $\widetilde{S}_n^{*r}$, $\widehat{S_n}^r$ to $I$ is not worse than the speed of convergence of $\widetilde{S}_n^{*}$, $\widehat{S_n}$ to $I$,*

*2)*

$$\left\{\begin{array}{l}\left\|\widetilde{S}^*_{m_r}\left(\widetilde{S}^*_{m_{r-1}}\left(...\widetilde{S}^*_{m_2}\left(\widetilde{S}^*_{m_1}(f)\right)\right)\right)-f\right\|_\infty\\ \left\|\widehat{S_{m_r}}\left(\widehat{S_{m_{r-1}}}\left(...\widehat{S_{m_2}}\left(\widehat{S_{m_1}}(f)\right)\right)\right)-f\right\|_\infty\end{array}\right\}\leq\sum_{i=1}^{r}\left\{\begin{array}{l}\left\|\widetilde{S}^*_{m_i}(f)-f\right\|_\infty\\ \left\|\widehat{S_{m_i}}(f)-f\right\|_\infty\end{array}\right\}\leq \tag{101}$$

$$2\sum_{i=1}^{r}\omega_1\left(f,\frac{1}{m_i}\left[1+N\left(\frac{\tanh(\lambda)}{2}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right)\leq$$

$$2r\omega_1\left(f,\frac{1}{m_1}\left[1+N\left(\frac{\tanh(\lambda)}{2}+\frac{\left(q+\frac{1}{q}\right)e^{2\lambda}}{2\lambda}\right)\right]\right)<+\infty. \tag{102}$$

*Again, the speed of convergence of the iterated operator to $I$ is not worse than the speed of convergence of $\widetilde{S}^*_{m_1}$, $\widehat{S_{m_1}}$ to $I$.*

*Proof.* By (95), (96) and Propositions 4.4, 4.5. □

We finish our work with multivariate simultaneous iterations.

**Remark.** *Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}\left(\mathbb{R}^N\right)$, with $f_\alpha \in C_B\left(\mathbb{R}^N\right)$, with $\alpha : |\alpha| = l$, $l = 0, 1, ..., i$; $r \in \mathbb{N}$. Then, by (34), we obtain*

$$\left\| \widetilde{S}_n^r\left(f_\alpha\right) - f_\alpha \right\|_\infty \leq r \left\| \widetilde{S}_n\left(f_\alpha\right) - f_\alpha \right\|_\infty . \tag{103}$$

*By (27) and inductively, we obtain*

$$\left\| \left(\widetilde{S}_n^r\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| \left(\widetilde{S}_n\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty , \tag{104}$$

*Similarly, we derive that*

$$\left\| \left(\widetilde{S}_n^{*^r}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| \left(\widetilde{S}_n^*\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty , \tag{105}$$

*and*

$$\left\| \left(\widehat{S_n}^r\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| \left(\widehat{S_n}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty . \tag{106}$$

*Let now $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$. Then, based on (36), we find that*

$$\left\| \left(\widetilde{S}_{m_r}\left(\widetilde{S}_{m_{r-1}}\left(...\widetilde{S}_{m_2}\left(\widetilde{S}_{m_1} f\right)\right)\right)\right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^{r} \left\| \left(\widetilde{S}_{m_{i^*}}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty . \tag{107}$$

*Similarly, we get that*

$$\left\| \left(\widetilde{S}^*_{m_r}\left(\widetilde{S}^*_{m_{r-1}}\left(...\widetilde{S}^*_{m_2}\left(\widetilde{S}^*_{m_1} f\right)\right)\right)\right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^{r} \left\| \left(\widetilde{S}^*_{m_{i^*}}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty , \tag{108}$$

*and*

$$\left\| \left(\widehat{S_{m_r}}\left(\widehat{S_{m_{r-1}}}\left(...\widehat{S_{m_2}}\left(\widehat{S_{m_1}} f\right)\right)\right)\right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^{r} \left\| \left(\widehat{S_{m_{i^*}}}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty . \tag{109}$$

*All the above inequalities (103)-(109) prove that our implied multivariate iterated simultaneous approximations do not have a speed worse than our basic simultanous approximations by the activated convolution operators.*

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