ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 2 (2025), 209-224
DOI: 10.62072/acm.2025.080206
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

A K-ANALOGUE OF THE LAMBDA GAMMA FUNCTION AND ITS PROPERTIES

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ABSTRACT. This paper establishes a k-analogue of the lambda gamma function and introduces a k-Riemann zeta function and a k-lambda Riemann zeta function. In addition, the paper establishes a relationship between the k-analogue of the lambda gamma function, the k-Riemann zeta function and the k-lambda Riemann zeta function. Finally, the paper establishes some properties of the k-analogue of the lambda gamma function similar to existing properties satisfied by the classical gamma function, the k-analogue of the gamma function and the lambda gamma function.

1. INTRODUCTION

The gamma function was first introduced and studied by Euler around $(1729 - 1730)$. Others such as Carl Friedrich and Karl Weierstrass further studied the gamma function.

The gamma function is defined as [1, p. 64]

$$\begin{aligned}\Gamma(x) &= \int_0^\infty u^{x-1}e^{-u}du,\\ &= \lim_{m\to\infty}\frac{m^x m!}{x(x+1)(x+2)\dots(x+m)}.\end{aligned}$$

Due to the usefulness of the gamma function in many areas of mathematics, researchers have developed interest in generalizing the function. Some of the important generalizations and of interest to this work are the k-analogue of the gamma function and the recent lambda analogue of the gamma function.

The k-analogue of the gamma function is defined as [3]

2020 *Mathematics Subject Classification.* 33B15, 26A48, 26A51, 37C30.

Key words and phrases. k-analogue of the gamma function; Lambda gamma function; k-analogue of the lambda gamma function; k-Riemann zeta function; (λ, k)-Riemann zeta function.

Received: May 10, 2025. Accepted: June 20, 2025. Published: June 30, 2025.

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$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt, \tag{1.1}$$

$$= \lim_{m\to\infty} \frac{m! k^m (mk)^{\frac{x}{k}-1}}{(x)_{m,k}}. \tag{1.2}$$

Nantomah and Ege introduced and studied a new generalization known as the lambda gamma function. It is defined as [6]

$$\Gamma_\lambda(x) = \int_0^\infty t^{x-1} e^{-\lambda t} dt, \tag{1.3}$$

$$= \lambda^{-x}\Gamma(x), \tag{1.4}$$

$$= \lim_{m\to\infty} \frac{m^x m! \lambda^{-x}}{x(x+1)(x+2)\dots(x+m)}. \tag{1.5}$$

Following the introduction of the lambda gamma function, other researchers have so far established some new properties of the function as in [7], [4].

Motivated by the k-analogue of the gamma function and the lambda gamma function, this paper introduces a new generalization known as the k-analogue of the lambda gamma function and its associated properties. This new generalization unifies the k-analogue of the gamma function and the lambda gamma function. Indeed both functions can be obtained as special cases of the new generalization.

2. Preliminaries

The following are some lemmas that are used in establishing the results.

Lemma 2.1. [3]. *For $k > 0$ and $m \in \mathbb{N}$, the k-Pochhammer factorial, denoted by $(x)_{m,k}$ is defined as*

$$(x)_{m,k} = x(x+k)(x+2k)\dots(x+(m-1)k). \tag{2.1}$$

Lemma 2.2. [1, p. 73]. *For $x > 0$, the gamma function satisfies the identity*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2.2}$$

Lemma 2.3. [1, p. 70]. *For $x > 0$, the gamma function satisfies the identity*

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2x). \tag{2.3}$$

Lemma 2.4. [1, p. 71]. *For $x > 0$, the gamma function satisfies the product representation*

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{m=1}^{\infty}\left(1+\frac{x}{m}\right)e^{-\frac{x}{m}}. \tag{2.4}$$

Lemma 2.5. [1, p. 94]. *For $t > 0$ and $u > 0$, the Frullani's integral representation for $\ln t$ is given as*

$$\ln t = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-ut}}{u}\right) du. \tag{2.5}$$

Lemma 2.6. [5]. *Let $f : I \to \mathbb{R}$ be convex. Then f satisfies the condition*

$$f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y), \tag{2.6}$$

for all $\alpha \in (0,1)$ and $x, y \in I$.

Lemma 2.7. [8]. *For $x > 0$ and $k > 0$, the k-digamma function satisfies the identity*

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{m=1}^{\infty} \frac{x}{mk(x+mk)}. \tag{2.7}$$

Lemma 2.8. [2, p. 41]. *A function $f : I \to \mathbb{R}$ is said to be log-convex if*

$$f\left(\frac{u}{a} + \frac{v}{b}\right) \leq [f(u)]^{\frac{1}{a}} [f(v)]^{\frac{1}{b}},$$

for all $u, v \in I$, where $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$.

Lemma 2.9. [7]. *For $x > 1$ and $\lambda > 0$, the lambda analogue of the Riemann zeta function is defined as*

$$\zeta_\lambda(x) = \sum_{m=1}^{\infty} \frac{\lambda^x}{m^x} \tag{2.8}$$

and satisfies the identity

$$\zeta_\lambda(x) = \lambda^x \zeta(x). \tag{2.9}$$

3. Main Results

Definition 3.1. Let $x > 0$, $k > 0$ and $\lambda > 0$. Then the $k-$analogue of the lambda gamma function is defined as

$$\Gamma_{\lambda,k}(x) = \int_0^\infty t^{x-1} e^{-\frac{\lambda t^k}{k}} dt. \tag{3.1}$$

Theorem 3.1. *The $k-$analogue of the lambda gamma function satisfies the identities*

$$\Gamma_{\lambda,k}(x) = \lim_{m\to\infty} \frac{m! k^m (mk)^{\frac{x}{k}-1} \lambda^{-\frac{x}{k}}}{(x)_{m,k}} \tag{3.2}$$

$$= \lambda^{-\frac{x}{k}} \Gamma_k(x). \tag{3.3}$$

Proof. It is observed that

$$\Gamma_{\lambda,k}(x) = \int_0^\infty t^{x-1} \lim_{m\to\infty} \left(1 - \frac{\lambda t^k}{mk}\right)^m dt. \tag{3.4}$$

By using (3.4) with the substitution $s = \frac{\lambda t^k}{k}$, where $t = \left(\frac{ks}{\lambda}\right)^{\frac{1}{k}}$, $dt = \frac{k^{\frac{1}{k}-1} s^{\frac{1}{k}-1}}{\lambda^{\frac{1}{k}}} ds$ and the limits of integration are such that at $t = 0 \implies s = 0$ and at $t = \infty \implies s = \infty$. Now, we have

$$\begin{aligned}
\Gamma_{\lambda,k}(x) &= \lim_{m\to\infty} \frac{1}{m^m} \int_0^m \left(\frac{ks}{\lambda}\right)^{\frac{x-1}{k}} \frac{k^{\frac{1}{k}-1} s^{\frac{1}{k}-1}}{\lambda^{\frac{1}{k}}} (m-s)^m ds \\
&= \lim_{m\to\infty} \frac{1}{m^m} \int_0^m \frac{s^{\frac{x}{k}-1} k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}} (m-s)^m ds \\
&= \lim_{m\to\infty} \frac{k^{\frac{x}{k}-1}}{m^m \lambda^{\frac{x}{k}}} \int_0^m s^{\frac{x}{k}-1} (m-s)^m ds \\
&= \lim_{m\to\infty} \frac{k^{\frac{x}{k}-1}}{m^m \lambda^{\frac{x}{k}}} Q(x).
\end{aligned}$$

Integrating $Q(x)$ $m-$times by parts gives

$$Q(x) = \frac{m! k^{m+1} m^{\frac{x}{k}} m^m}{x(x+k)(x+2k)\dots(x+(m-1)k)(x+mk)}.$$

Thus,

$$\begin{aligned}
\Gamma_{\lambda,k}(x) &= \lim_{m\to\infty}\left(\frac{k^{\frac{x}{k}-1}}{m^m\lambda^{\frac{x}{k}}}\right)\left(\frac{m!k^{m+1}m^{\frac{x}{k}}m^m}{x(x+k)(x+2k)\dots(x+(m-1)k)(x+mk)}\right)\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}}\lambda^{-\frac{x}{k}}}{x(x+k)(x+2k)\dots(x+(m-1)k)(x+mk)}\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}}{x(x+k)(x+2k)\dots(x+(m-1)k)}\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}}{(x)_{m,k}},
\end{aligned}$$

which proves (3.2).
Next, by using identity (1.2) we have

$$\begin{aligned}
\Gamma_{\lambda,k}(x) &= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}}{(x)_{m,k}}\\
&= \lambda^{-\frac{x}{k}}\lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}}{(x)_{m,k}}\\
&= \lambda^{-\frac{x}{k}}\Gamma_k(x), \qquad (3.5)
\end{aligned}$$

which proves (3.3). □

Theorem 3.2. *For $x > 0$, $\lambda > 0$ and $k > 0$, the $k-$analogue of the lambda gamma function satisfies the identities*

$$\Gamma_{\lambda,k}(k) = \frac{1}{\lambda}, \qquad (3.6)$$

$$\begin{aligned}
\Gamma_{\lambda,k}(x+k) &= \frac{x}{\lambda}\Gamma_{\lambda,k}(x), \qquad (3.7)\\
&= \frac{x}{\lambda^{\frac{x}{k}+1}}\Gamma_k(x). \qquad (3.8)
\end{aligned}$$

Proof. By substituting $x = k$ into (3.3), we obtain (3.6).
Next, by using (3.2) and replacing x with $x + k$ we have

$$\begin{aligned}
\Gamma_{\lambda,k}(x+k) &= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}}\lambda^{-\frac{x}{k}}\lambda^{-1}}{(x+k)_{m,k}}\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}}\lambda^{-\frac{x}{k}}\lambda^{-1}}{(x+k)(x+2k)(x+3k)\dots(x+mk)}\\
&= \lim_{m\to\infty}\frac{xm!k^m(mk)^{\frac{x}{k}}\lambda^{-\frac{x}{k}}\lambda^{-1}}{x(x+k)(x+2k)(x+3k)\dots(x+mk)}\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}\frac{x}{\lambda}}{x(x+k)(x+2k)(x+3k)\dots(x+(m-1)k)}\\
&= \frac{x}{\lambda}\left(\lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}}{(x)_{m,k}}\right)\\
&= \frac{x}{\lambda}\Gamma_{\lambda,k}(x). \qquad (3.9)
\end{aligned}$$

Next, using identity (3.5) with (3.9), we obtain (3.8) which completes the proof. □

Theorem 3.3. *The $k-$analogue of the lambda gamma function satisfies the identity*

$$\Gamma_{\lambda,k}(x) = k^{\frac{x}{k}-1}\Gamma_\lambda\left(\frac{x}{k}\right). \tag{3.10}$$

Proof. Using (3.1) with the substitution $u = \frac{t^k}{k}$. We have

$$\begin{aligned}
\Gamma_{\lambda,k}(x) &= \int_0^\infty (uk)^{\frac{x-1}{k}} e^{-\lambda u}(uk)^{-\frac{k-1}{k}} du \\
&= \int_0^\infty (uk)^{\frac{x-1}{k}-\frac{k-1}{k}} e^{-\lambda u} du \\
&= \int_0^\infty (uk)^{\frac{x}{k}-1} e^{-\lambda u} du \\
&= \int_0^\infty u^{\frac{x}{k}-1} k^{\frac{x}{k}-1} e^{-\lambda u} du \\
&= k^{\frac{x}{k}-1}\int_0^\infty u^{\frac{x}{k}-1} e^{-\lambda u} du \\
&= k^{\frac{x}{k}-1}\Gamma_\lambda\left(\frac{x}{k}\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $x > 0$, $\lambda > 0$ and $k > 0$. Then the $k-$analogue of the lambda gamma function satisfies the identity*

$$\Gamma_{\lambda,k}(x) = \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\Gamma\left(\frac{x}{k}\right). \tag{3.11}$$

Proof. Using (3.1) with the substitution $s = \frac{\lambda t^k}{k}$, we obtain

$$\begin{aligned}
\Gamma_{\lambda,k}(x) &= \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\int_0^\infty s^{\frac{x}{k}-1} e^{-s} ds \\
&= \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\Gamma\left(\frac{x}{k}\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *The k-analogue of the lambda gamma function satisfies the identity*

$$\Gamma_{\lambda,k}(x+k) = \frac{xk^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}+1}}\Gamma\left(\frac{x}{k}\right). \tag{3.12}$$

Proof. By substituting (3.11) into (3.7), we obtain (3.12), which completes the proof. □

Theorem 3.6. *For $x > 0, \lambda > 0$ and $k > 0$, the $k-$anlogue of the lambda gamma function has a product representation of the form*

$$\frac{1}{\Gamma_{\lambda,k}(x)} = x\lambda^{\frac{x}{k}} k^{-\frac{x}{k}} e^{\frac{x}{k}\gamma}\prod_{m=1}^{\infty}\left(1+\frac{x}{mk}\right)e^{-\frac{x}{mk}}. \tag{3.13}$$

Proof. Using (3.11) and (2.4), we have

$$\begin{aligned}\frac{1}{\Gamma_{\lambda,k}(x)} &= \frac{\lambda^{\frac{x}{k}}}{k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)}\\
&= \left(\frac{\lambda^{\frac{x}{k}}}{k^{\frac{x}{k}-1}}\right)\left(\frac{x}{k}e^{\frac{x\gamma}{k}}\prod_{m=1}^{\infty}\left[1+\frac{x}{mk}\right]e^{-\frac{x}{mk}}\right)\\
&= \frac{x\lambda^{\frac{x}{k}}}{k^{\frac{x}{k}}}e^{\frac{x\gamma}{k}}\prod_{m=1}^{\infty}\left(1+\frac{x}{mk}\right)e^{-\frac{x}{mk}}\\
&= x\lambda^{\frac{x}{k}}k^{-\frac{x}{k}}e^{\frac{x\gamma}{k}}\prod_{m=1}^{\infty}\left(1+\frac{x}{mk}\right)e^{-\frac{x}{mk}}.\end{aligned}$$

This completes the proof. □

Theorem 3.7. *Let $x > 0$, $k > 0$ and $\lambda > 0$. Then the $k-$analogue of the lambda gamma function satisfies the identity*

$$\Gamma_{\lambda,k}(x)\Gamma_{\lambda,k}(k-x) = \frac{\pi}{\lambda k \sin\frac{\pi x}{k}}. \tag{3.14}$$

Proof. Using (3.11) and (2.2), we proceed as

$$\begin{aligned}\Gamma_{\lambda,k}(x)\Gamma_{\lambda,k}(k-x) &= \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\Gamma\left(\frac{x}{k}\right)\frac{k^{\frac{k-x}{k}-1}}{\lambda^{\frac{k-x}{k}}}\Gamma\left(\frac{k-x}{k}\right)\\
&= \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\frac{k^{\frac{k-x}{k}-1}}{\lambda^{\frac{k-x}{k}}}\Gamma\left(\frac{x}{k}\right)\Gamma\left(\frac{k-x}{k}\right)\\
&= \frac{1}{\lambda k}\Gamma\left(\frac{x}{k}\right)\Gamma\left(\frac{k-x}{k}\right)\\
&= \frac{\pi}{\lambda k \sin\frac{\pi x}{k}},\end{aligned}$$

which completes the proof. □

Lemma 3.8. *For $x > 0$, $\lambda > 0$ and $k > 0$, the identity*

$$\Gamma_{\lambda,k}(2x) = \frac{k^{\frac{2x}{k}-1}}{\lambda^{\frac{2x}{k}}}\Gamma\left(\frac{2x}{k}\right). \tag{3.15}$$

is valid.

Proof. Using (3.11) with the substitution $x \to 2x$, we obtain (3.15), which completes the proof. □

Theorem 3.9. *Let $x > 0$, $\lambda > 0$ and $k > 0$. Then the $k-$analogue of the lambda gamma function satisfies the identity*

$$\Gamma_{\lambda,k}(x)\Gamma_{\lambda,k}\left(x+\frac{k}{2}\right) = \frac{2^{1-\frac{2x}{k}}}{k}\sqrt{\frac{\pi k}{\lambda}}\Gamma_{\lambda,k}(2x). \tag{3.16}$$

Proof. Using (3.11), Lemma 3.15 and together with Lemma 2.3, we begin as

$$\begin{aligned}
\Gamma_{\lambda,k}(x)\Gamma_{\lambda,k}\left(x+\frac{k}{2}\right) &= \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\Gamma\left(\frac{x}{k}\right)\frac{k^{\frac{2x+k}{2k}-1}}{\lambda^{\frac{2x+k}{2k}}}\Gamma\left(\frac{2x+k}{2k}\right)\\
&= \frac{k^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}}\frac{k^{\frac{2x+k}{2k}-1}}{\lambda^{\frac{2x+k}{2k}}}\Gamma\left(\frac{x}{k}\right)\Gamma\left(\frac{2x+k}{2k}\right)\\
&= \frac{k^{\frac{2x}{k}-2}k^{\frac{1}{2}}}{\lambda^{\frac{2x}{k}}\lambda^{\frac{1}{2}}}\Gamma\left(\frac{x}{k}\right)\Gamma\left(\frac{2x+k}{2k}\right)\\
&= \frac{k^{\frac{2x}{k}-2}}{\lambda^{\frac{2x}{k}}}\left(\frac{k}{\lambda}\right)^{\frac{1}{2}}\Gamma\left(\frac{x}{k}\right)\Gamma\left(\frac{2x+k}{2k}\right)\\
&= \frac{k^{\frac{2x}{k}-2}}{\lambda^{\frac{2x}{k}}}\left(\frac{k}{\lambda}\right)^{\frac{1}{2}}2^{1-\frac{2x}{k}}\sqrt{\pi}\Gamma\left(\frac{2x}{k}\right)\\
&= \frac{k^{\frac{2x}{k}-2}}{\lambda^{\frac{2x}{k}}}\left(\frac{k}{\lambda}\right)^{\frac{1}{2}}2^{1-\frac{2x}{k}}\sqrt{\pi}\frac{\lambda^{\frac{2x}{k}}}{k^{\frac{2x}{k}-1}}\Gamma_{\lambda,k}(2x)\\
&= \left(\frac{k}{\lambda}\right)^{\frac{1}{2}}\frac{2^{1-\frac{2x}{k}}}{k}\sqrt{\pi}\Gamma_{\lambda,k}(2x)\\
&= \frac{2^{1-\frac{2x}{k}}}{k}\sqrt{\frac{\pi k}{\lambda}}\Gamma_{\lambda,k}(2x).
\end{aligned}$$

This completes the proof. □

Theorem 3.10. *Let $x>0$, $k>0$ and $\lambda>0$. Then the $k-$analogue of the lambda gamma function satisfies the identity*

$$\frac{\Gamma_{\lambda,k}(x)}{\lambda^m} = \frac{\Gamma_{\lambda,k}(x+mk)}{(x)_{m,k}}. \tag{3.17}$$

Proof. Using (3.2), we have

$$\begin{aligned}
\Gamma_{\lambda,k}(x+mk) &= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x+mk}{k}-1}\lambda^{-\frac{x+mk}{k}}}{(x+mk)_{m,k}}\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}(mk)^m\lambda^{-\frac{x}{k}}\lambda^{-m}}{(x+mk)_{m,k}}\\
&= \lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}}{(x)_{m,k}}\frac{(x)_{m,k}(mk)^m\lambda^{-m}}{(x+mk)_{m,k}}\\
&= \Gamma_{\lambda,k}(x)\lim_{m\to\infty}\frac{(x)_{m,k}(mk)^m\lambda^{-m}}{(x+mk)_{m,k}}\\
&= \Gamma_{\lambda,k}(x)\frac{(x)_{m,k}}{\lambda^m}.
\end{aligned}$$

Rearranging yields the relation

$$\frac{\Gamma_{\lambda,k}(x)}{\lambda^m} = \frac{\Gamma_{\lambda,k}(x+mk)}{(x)_{m,k}},$$

which completes the proof. □

Theorem 3.11. *Let $x > k$ and $k > 0$. Then the $k-$Riemann zeta function is defined as*

$$\zeta_k(x) = \sum_{m=1}^{\infty} \frac{1}{m^{\frac{x}{k}}}. \tag{3.18}$$

and satisfies the integral identities

$$\zeta_k(x) = \frac{1}{\Gamma_k(x)} \int_0^{\infty} \frac{\xi^{x-1}}{e^{\frac{\xi^k}{k}} - 1} d\xi, \tag{3.19}$$

$$= \frac{k^{\frac{x}{k}-1}}{\Gamma_k(x)} \int_0^{\infty} \frac{s^{\frac{x}{k}-1}}{e^s - 1} ds. \tag{3.20}$$

Proof. By using (1.1) with the substitution $t^k = m\xi^k$ where $t^{k-1}dt = m\xi^{k-1}d\xi$, we have

$$\Gamma_k(x) = \int_0^{\infty} m^{\frac{x}{k}} \xi^{x-1} e^{-\frac{m\xi^k}{k}} d\xi. \tag{3.21}$$

Now rearranging (3.21) gives

$$\frac{1}{m^{\frac{x}{k}}} = \frac{1}{\Gamma_k(x)} \int_0^{\infty} \xi^{x-1} e^{-\frac{m\xi^k}{k}} d\xi. \tag{3.22}$$

By summing (3.22) from $m = 1$ to ∞, we obtained

$$\sum_{m=1}^{\infty} \frac{1}{m^{\frac{x}{k}}} = \frac{1}{\Gamma_k(x)} \int_0^{\infty} \xi^{x-1} \sum_{m=1}^{\infty} e^{-\frac{m\xi^k}{k}} d\xi. \tag{3.23}$$

By considering

$$\sum_{m=1}^{\infty} e^{-\frac{m\xi^k}{k}} = e^{-\frac{\xi^k}{k}} + e^{-\frac{2\xi^k}{k}} + e^{-\frac{3\xi^k}{k}} + \dots, \tag{3.24}$$

we obtained

$$\sum_{m=1}^{\infty} e^{-\frac{m\xi^k}{k}} = \frac{1}{e^{\frac{\xi^k}{k}} - 1}. \tag{3.25}$$

Substituting (3.25) into (3.22) gives (3.19) which completes the proof.
Next, let $s = \frac{\xi^k}{k}$ in (3.19). Then $\xi = (ks)^{\frac{1}{k}}$ and $d\xi = \frac{1}{(ks)^{\frac{1}{k}}} ds$ implies that

$$\zeta_k(x) = \frac{1}{\Gamma_k(x)} \int_0^{\infty} \frac{(ks)^{\frac{x-1}{k}}}{e^s - 1} \frac{1}{(ks)^{\frac{k-1}{k}}} ds$$

$$= \frac{1}{\Gamma_k(x)} \int_0^{\infty} \frac{k^{\frac{x}{k}-1} s^{\frac{x}{k}-1}}{e^s - 1} ds$$

$$= \frac{k^{\frac{x}{k}-1}}{\Gamma_k(x)} \int_0^{\infty} \frac{s^{\frac{x}{k}-1}}{e^s - 1} ds,$$

which gives identity (3.20). $\square$

Theorem 3.12. *For $x > k$ and $\lambda > 0$, the $k-$analogue of the lambda gamma function satisfies the identities*

$$\Gamma_{\lambda,k}(x) = \frac{1}{\zeta_k(x)}\int_0^\infty \frac{u^{x-1}}{e^{\frac{\lambda u^k}{k}}-1}du, \tag{3.26}$$

$$= \frac{k^{\frac{x}{k}-1}}{\zeta_k(x)}\int_0^\infty \frac{r^{\frac{x}{k}-1}}{e^{\lambda r}-1}dr, \tag{3.27}$$

$$= \frac{\lambda^{-\frac{x}{k}}k^{\frac{x}{k}-1}}{\zeta_k(x)}\int_0^\infty \frac{s^{\frac{x}{k}-1}}{e^{s}-1}ds, \tag{3.28}$$

$$= \frac{\lambda^{-\frac{x}{k}}k^{\frac{x}{k}-1}(\frac{x}{k}-1)}{\zeta_k(x)}\int_0^\infty \frac{\ln(w+1)}{w(w+1)}dw. \tag{3.29}$$

Proof. By using (3.1) with the substitution, $t^k = mu^k$ where $m \in \mathbb{N}$, we have

$$\begin{aligned}\Gamma_{\lambda,k}(x) &= \int_0^\infty (m^{\frac{1}{k}}u)^{x-1}e^{-\frac{\lambda m u^k}{k}}\frac{mu^{k-1}}{(m^{\frac{1}{k}}u)^{k-1}}du\\ &= \int_0^\infty m^{\frac{x-1}{k}}u^{x-1}e^{-\frac{\lambda m u^k}{k}}\frac{m}{m^{\frac{k-1}{k}}}du\\ &= \int_0^\infty m^{\frac{x}{k}}u^{x-1}e^{-\frac{\lambda m u^k}{k}}du\\ &= m^{\frac{x}{k}}\int_0^\infty u^{x-1}e^{-\frac{\lambda m u^k}{k}}du\end{aligned}$$

Rearranging gives

$$\frac{1}{m^{\frac{x}{k}}} = \frac{1}{\Gamma_{\lambda,k}(x)}\int_0^\infty u^{x-1}e^{-\frac{\lambda m u^k}{k}}du, \tag{3.30}$$

and summing (3.30), we obtain

$$\sum_{m=1}^\infty \frac{1}{m^{\frac{x}{k}}} = \frac{1}{\Gamma_{\lambda,k}(x)}\int_0^\infty u^{x-1}\sum_{m=1}^\infty e^{-\frac{\lambda m u^k}{k}}du \tag{3.31}$$

Now,

$$\sum_{m=1}^\infty e^{-\frac{\lambda m u^k}{k}} = e^{-\frac{m u^k}{k}} + e^{-\frac{2m u^k}{k}} + e^{-\frac{3m u^k}{k}} + \dots \tag{3.32}$$

$$= \frac{e^{-\frac{\lambda u^k}{k}}}{1-e^{-\frac{\lambda u^k}{k}}}$$

$$= \frac{1}{e^{\frac{\lambda u^k}{k}}-1}. \tag{3.33}$$

By substituting (3.33) into (3.31) gives (3.26).
Next, Using (3.26) with the substitution, $r = \frac{u^k}{k}$, we have

$$\begin{aligned}\Gamma_{\lambda,k}(x) &= \frac{1}{\zeta_k(x)}\int_0^\infty \frac{(r^{\frac{1}{k}}k^{\frac{1}{k}})^{x-1}}{(e^{\lambda r}-1)(r^{\frac{1}{k}}k^{\frac{1}{k}})^{k-1}}dr,\\ &= \frac{1}{\zeta_k(x)}\int_0^\infty \frac{r^{\frac{x}{k}-1}k^{\frac{x}{k}-1}}{e^{\lambda r}-1}dr,\\ &= \frac{k^{\frac{x}{k}-1}}{\zeta_k(x)}\int_0^\infty \frac{r^{\frac{x}{k}-1}}{e^{\lambda r}-1}dr,\end{aligned}$$

which gives (3.27).
Next, by using (3.27) and substituting $s = \lambda r$, we have

$$\begin{aligned}\Gamma_{\lambda,k}(x) &= \frac{k^{\frac{x}{k}-1}}{\zeta_k(x)} \int_0^\infty \frac{\left(\frac{s}{k}\right)^{\frac{x}{k}-1}}{\lambda(e^s-1)} ds \\ &= \frac{k^{\frac{x}{k}-1}}{\zeta_k(x)} \int_0^\infty \frac{s^{\frac{x}{k}-1}}{\lambda^{\frac{x}{k}}(e^s-1)} ds \\ &= \frac{\lambda^{-\frac{x}{k}} k^{\frac{x}{k}-1}}{\zeta_k(x)} \int_0^\infty \frac{s^{\frac{x}{k}-1}}{(e^s-1)} ds,\end{aligned}$$

which yields (3.28).
Finally, let $w = e^s - 1$. Then

$$\begin{aligned}\Gamma_{\lambda,k}(x) &= \frac{\lambda^{-\frac{x}{k}} k^{\frac{x}{k}-1}}{\zeta_k(x)} \int_0^\infty \frac{\ln[w+1]^{\frac{x}{k}-1}}{we^{\ln(w+1)}} dw \\ &= \frac{\lambda^{-\frac{x}{k}} k^{\frac{x}{k}-1}}{\zeta_k(x)} \int_0^\infty \frac{\ln[w+1]^{\frac{x}{k}-1}}{w(w+1)} dw \\ &= \frac{\lambda^{-\frac{x}{k}} k^{\frac{x}{k}-1}}{\zeta_k(x)} \int_0^\infty \frac{\left(\frac{x}{k}-1\right)\ln(w+1)}{w(w+1)} dw \\ &= \frac{\lambda^{-\frac{x}{k}} k^{\frac{x}{k}-1}\left(\frac{x}{k}-1\right)}{\zeta_k(x)} \int_0^\infty \frac{\ln(w+1)}{w(w+1)} dw,\end{aligned}$$

gives (3.29), which completes the proof. □

Definition 3.2. Let $x > k$ and $\lambda > 0$. Then the $(\lambda, k)-$Riemann zeta function is defined as

$$\zeta_{\lambda,k}(x) = \sum_{m=1}^{\infty} \frac{\lambda^{\frac{x}{k}}}{m^{\frac{x}{k}}}, \tag{3.34}$$

Theorem 3.13. *The $(\lambda, k)-$ Riemann zeta function satisfies the identity*

$$\zeta_{\lambda,k}(x) = \lambda^{\frac{x}{k}} \zeta_k(x). \tag{3.35}$$

Proof. The proof of (3.35) follows directly from (3.34). □

Remark. *By substituting $k = 1$ into* (3.35)*, we obtain the results of Lemma 2.9.*

Theorem 3.14. *For $x > k$ and $\lambda > 0$, the $(\lambda, k)-$ Riemann zeta function satisfies the identities*

$$\begin{aligned}\zeta_{\lambda,k}(x) &= \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{u^{x-1}}{e^{\frac{u^k}{k}}-1} du, &(3.36)\\ &= \frac{k^{\frac{x}{k}-1}}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{s^{x-1}}{e^s-1} ds, &(3.37)\\ &= \frac{\left(\frac{x}{k}-1\right) k^{\frac{x}{k}-1}}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{\ln(w+1)}{w(w+1)} dw. &(3.38)\end{aligned}$$

Proof. By using (3.1) with the substitution $\lambda t^k = mu^k$ for $m \in \mathbb{N}$, we have

$$\begin{aligned}
\Gamma_{\lambda,k}(x) &= \int_0^\infty \left(\frac{m^{\frac{1}{k}}u}{\lambda^{\frac{1}{k}}}\right)^{x-1} e^{-\frac{mu^k}{k}} \frac{mu^{k-1}}{\lambda\left(\frac{m^{\frac{1}{k}}u}{\lambda^{\frac{1}{k}}}\right)^{k-1}} du \\
&= \int_0^\infty \frac{m^{\frac{x-1}{k}} u^{x-1} m \lambda^{\frac{k-1}{k}}}{\lambda^{\frac{x-1}{k}} \lambda m^{\frac{k-1}{k}}} e^{-\frac{mu^k}{k}} du \\
&= \int_0^\infty m^{\frac{x}{k}} u^{x-1} \lambda^{-\frac{x}{k}} e^{-\frac{mu^k}{k}} du \\
&= \frac{m^{\frac{x}{k}}}{\lambda^{\frac{x}{k}}} \int_0^\infty u^{x-1} e^{-\frac{mu^k}{k}} du.
\end{aligned}$$

By rearranging, we obtain

$$\frac{\lambda^{\frac{x}{k}}}{m^{\frac{x}{k}}} = \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty u^{x-1} e^{-\frac{mu^k}{k}} du \tag{3.39}$$

By summing both sides of (3.39) from $m = 1$ to $m = \infty$, we have

$$\sum_{m=1}^\infty \frac{\lambda^{\frac{x}{k}}}{m^{\frac{x}{k}}} = \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty u^{x-1} \sum_{m=1}^\infty e^{-\frac{mu^k}{k}} du \tag{3.40}$$

Now,

$$\sum_{m=1}^\infty e^{-\frac{mu^k}{k}} = e^{-\frac{u^k}{k}} + e^{-\frac{2u^k}{k}} + e^{-\frac{3u^k}{k}} + \dots \tag{3.41}$$

$$\begin{aligned}
&= \frac{e^{-\frac{u^k}{k}}}{1 - e^{-\frac{u^k}{k}}} \\
&= \frac{1}{e^{\frac{u^k}{k}} - 1},
\end{aligned} \tag{3.42}$$

and by substituting (3.42) into (3.40) gives

$$\zeta_{\lambda,k}(x) = \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{u^{x-1}}{e^{\frac{u^k}{k}} - 1} du,$$

which proves (3.36).

Next, by substituting $s = \frac{u^k}{k}$ in (3.36), we have

$$\begin{aligned}
\zeta_{\lambda,k}(x) &= \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{(s^{\frac{1}{k}} k^{\frac{1}{k}})^{x-1}}{(e^s - 1)(s^{\frac{1}{k}} k^{\frac{1}{k}})^{k-1}} ds \\
&= \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{s^{\frac{x-1}{k}} k^{\frac{x-1}{k}}}{(e^s - 1) s^{\frac{k-1}{k}} k^{\frac{k-1}{k}}} ds \\
&= \frac{1}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{s^{\frac{x}{k}-1} k^{\frac{x}{k}-1}}{e^s - 1} ds \\
&= \frac{k^{\frac{x}{k}-1}}{\Gamma_{\lambda,k}(x)} \int_0^\infty \frac{s^{\frac{x}{k}-1}}{e^s - 1} ds,
\end{aligned}$$

which proves (3.37).
Next, we let $w = e^s - 1$ in (3.27) and proceed as

$$\begin{aligned}\zeta_{\lambda,k}(x) &= \frac{k^{\frac{x}{k}-1}}{\Gamma_{\lambda,k}(x)}\int_0^\infty \frac{\ln{(w+1)}^{\frac{x}{k}-1}}{w(w+1)}dw\\ &= \frac{k^{\frac{x}{k}-1}}{\Gamma_{\lambda,k}(x)}\int_0^\infty \frac{\left(\frac{x}{k}-1\right)\ln{(w+1)}}{w(w+1)}dw\\ &= \frac{k^{\frac{x}{k}-1}\left(\frac{x}{k}-1\right)}{\Gamma_{\lambda,k}(x)}\int_0^\infty \frac{\ln{(w+1)}}{w(w+1)}dw,\end{aligned}$$

which yields (3.38). This completes the proof. □

Lemma 3.15. *For $x > 0$, $\lambda > 0$ and $k > 0$, the identity*

$$\int_0^\infty t^{x-1}e^{-\lambda(1+\frac{ku}{\lambda})\frac{t^k}{k}}\,dt = \frac{\Gamma_{\lambda,k}(x)}{(1+\frac{ku}{\lambda})^{\frac{x}{k}}} \tag{3.43}$$

is valid.

Proof. By using the left hand side integral of (3.43) with the substitution, $s^k = t^k(1+\frac{ku}{\lambda})$, we have

$$\begin{aligned}\int_0^\infty t^{x-1}e^{-\lambda(1+\frac{ku}{\lambda})\frac{t^k}{k}}\,dt &= \int_0^\infty \frac{s^{x-1}}{(1+\frac{ku}{\lambda})^{\frac{x-1}{k}}}e^{\frac{-\lambda s^k}{k}}\frac{s^{k-1}}{\left(\frac{s}{(1+\frac{ku}{\lambda})^{\frac{1}{k}}}\right)^{k-1}(1+\frac{ku}{\lambda})}ds\\ &= \int_0^\infty s^{x-1}\frac{(1+\frac{ku}{\lambda})^{\frac{k-1}{k}}}{(1+\frac{ku}{\lambda})^{\frac{x-1}{k}+1}}e^{-\frac{\lambda s^k}{k}}\,ds\\ &= \int_0^\infty s^{x-1}\left((1+\frac{ku}{\lambda}\right)^{-\frac{x}{k}}e^{-\frac{\lambda s^k}{k}}\,ds\\ &= \frac{1}{\left(1+\frac{ku}{\lambda}\right)^{\frac{x}{k}}}\int_0^\infty s^{x-1}e^{-\frac{\lambda s^k}{k}}\,ds\\ &= \frac{\Gamma_{\lambda,k}(x)}{\left(1+\frac{ku}{\lambda}\right)^{\frac{x}{k}}},\end{aligned}$$

which completes the proof. □

Definition 3.3. Let $x > 0$, $\lambda > 0$ and $k > 0$. The $(\lambda, k)-$digamma function denoted by $\psi_{\lambda,k}(x)$, is define as

$$\psi_{\lambda,k}(x) = \frac{d}{dx}\ln\Gamma_{\lambda,k}(x) = \frac{\Gamma'_{\lambda,k}(x)}{\Gamma_{\lambda,k}(x)}. \tag{3.44}$$

Theorem 3.16. *The* $(\lambda, k)-$*digamma function satisfies the identities*

$$\psi_{\lambda,k}(x) = \frac{\ln k - \ln\lambda - \gamma}{k} - \frac{1}{x} + \sum_{m=1}^{\infty} \frac{x}{mk(x+mk)}, \tag{3.45}$$

$$= -\frac{\ln\lambda}{k} + \psi_k(x), \tag{3.46}$$

$$= \frac{1}{k}\ln\left(\frac{k}{\lambda}\right) + \psi\left(\frac{x}{k}\right), \tag{3.47}$$

$$= \int_0^\infty \left(e^{-\frac{\lambda z}{k}} - (1+z)^{-\frac{x}{k}}\right)\frac{dz}{kz}, \tag{3.48}$$

$$= \int_0^\infty \left(-\frac{e^{\frac{\lambda z}{k}}}{z} - \frac{e^{-z\left(\frac{x}{k}-1\right)}}{e^z - 1}\right)\frac{dz}{k},\ x > k, \tag{3.49}$$

$$= \int_0^\infty \left(\frac{e^{-\frac{\lambda z}{k}}}{z} - \frac{e^{-\frac{zx}{k}}}{1-e^{-z}}\right)\frac{dz}{k}, \tag{3.50}$$

$$= \int_0^1 \left(\frac{e^{-\frac{\lambda(1-z)}{kz}}}{z} - \frac{z^{\frac{x}{k}}}{z}\right)\frac{dz}{k(1-z)}. \tag{3.51}$$

Proof. Using (3.2), we have

$$\begin{aligned}
\ln\Gamma_{\lambda,k}(x) &= \lim_{m\to\infty} \ln \frac{m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}}}{x(x+k)(x+2k)(x+3k)\ldots(x+(m-1)k)} \\
&= \ln m!k^m(mk)^{\frac{x}{k}-1}\lambda^{-\frac{x}{k}} - \ln x(x+k)(x+2k)(x+3k)\ldots(x+(m-1)k) \\
&= \lim_{m\to\infty}\ln m!k^m + \left(\frac{x}{k}-1\right)\lim_{m\to\infty}\ln(mk) - \frac{x}{k}\ln\lambda - \lim_{m\to\infty}[\ln x + \cdots + \ln(x+(m-1)k)]
\end{aligned}$$

Now, differentiating with respect to x gives

$$\frac{\Gamma'_{\lambda,k}(x)}{\Gamma_{\lambda,k}(x)} = \frac{\ln k}{k} + \frac{\ln m}{k} - \frac{\ln\lambda}{k} - \sum_{m=0}^{\infty}\frac{1}{x+mk}, \tag{3.52}$$

which implies that

$$\begin{aligned}
\psi_{\lambda,k}(x) &= \frac{\ln k}{k} + \frac{\ln m}{k} - \frac{\ln\lambda}{k} - \sum_{m=0}^{\infty}\frac{1}{x+mk} \\
&= \frac{\ln k}{k} + \frac{\ln m}{k} + \sum_{m=1}^{\infty}\frac{1}{mk} - \sum_{m=1}^{\infty}\frac{1}{mk} - \frac{1}{x} - \frac{\ln\lambda}{k} - \sum_{m=1}^{\infty}\frac{1}{x+mk} \\
&= \frac{\ln k}{k} - \left(\sum_{m=1}^{\infty}\frac{1}{mk} - \frac{\ln m}{k}\right) - \frac{\ln\lambda}{k} - \frac{1}{x} - \sum_{m=1}^{\infty}\left(\frac{1}{x+mk} - \sum_{m=1}^{\infty}\frac{1}{mk}\right) \\
&= -\frac{\ln\lambda}{k} + \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{m=1}^{\infty}\frac{x}{mk(x+mk)},
\end{aligned}$$

which proves (3.45).
Next, by using (3.45) with (2.7) proves (3.46)
Finally, by using (3.11), we have

$$\ln\Gamma_{\lambda,k}(x) = \left(\frac{x}{k}-1\right)\ln k - \frac{x}{k}\ln\lambda + \ln\Gamma\left(\frac{x}{k}\right). \tag{3.53}$$

and by differentiating (3.53), we obtain

$$\begin{aligned}\psi_{\lambda,k}(x) &= \frac{1}{k}\ln k - \frac{1}{k}\ln\lambda + \psi\left(\frac{x}{k}\right),\\ &= \frac{1}{k}\ln\left(\frac{x}{\lambda}\right) + \psi\left(\frac{x}{k}\right),\end{aligned}$$

which completes the proof.

Next, by using (3.1), (2.5) and (3.43), we begin with

$$\begin{aligned}\Gamma'_{\lambda,k}(x) &= \int_0^\infty t^{x-1}\ln t e^{-\frac{\lambda t^k}{k}}dt\\ &= \frac{1}{k}\int_0^\infty t^{x-1}\ln t^k e^{-\frac{\lambda t^k}{k}}dt\\ &= \frac{1}{k}\int_0^\infty t^{x-1}\int_0^\infty \frac{e^{-u}-e^{-ut^k}}{u}e^{-\frac{\lambda t^k}{k}}dudt\\ &= \frac{1}{k}\int_0^\infty t^{x-1}e^{-\frac{\lambda t^k}{k}}\left(\int_0^\infty \frac{e^{-u}}{u}du - \int_0^\infty \frac{e^{-ut^k}}{u}du\right)dt\\ &= \frac{1}{k}\int_0^\infty \left[e^{-u}\int_0^\infty t^{x-1}e^{-\frac{\lambda t^k}{k}}dt - \int_0^\infty t^{x-1}e^{-(1+\frac{ku}{\lambda})\frac{\lambda t^k}{k}}dt\right]\frac{du}{u}\\ &= \frac{1}{k}\int_0^\infty \left[e^{-u}\Gamma_{\lambda,k}(x) - \int_0^\infty t^{x-1}e^{-(1+\frac{ku}{\lambda})\frac{\lambda t^k}{k}}dt\right]\frac{du}{u}\\ &= \frac{1}{k}\int_0^\infty \left(e^{-u}\Gamma_{\lambda,k}(x) - \frac{\Gamma_{\lambda,k}(x)}{\left(1+\frac{ku}{\lambda}\right)^{\frac{x}{k}}}\right)\frac{du}{u}\end{aligned}$$

Now, rearranging gives

$$\frac{\Gamma'_{\lambda,k}(x)}{\Gamma_{\lambda,k}(x)} = \psi_{\lambda,k}(x) = \int_0^\infty \left(e^{-u} - \left(1+\frac{ku}{\lambda}\right)^{-\frac{x}{k}}\right)\frac{du}{ku},$$

and by substituting $z = \frac{ku}{\lambda}$ yields (3.48).

Next, we split (3.48) as

$$\psi_{\lambda,k}(x) = \int_0^\infty \frac{e^{\frac{\lambda z}{k}}}{zk}dz - \int_0^\infty \frac{dz}{zk(1+z)^{\frac{x}{k}}}$$

and substitute $1+z = e^u$, in the second integral to obtain

$$\begin{aligned}\int_0^\infty \frac{dz}{zk(1+z)^{\frac{x}{k}}} &= \int_0^\infty \frac{e^u du}{(e^u-1)e^{\frac{ux}{k}}}\\ &= \int_0^\infty \frac{e^{-u(\frac{x}{k}-1)}}{k(e^u-1)}du.\end{aligned}$$

Now, substituting (3.54) into (3.48) gives (3.49) and (3.50).

Next, by substituting $u = \frac{1}{1+z}$ in (3.48), we have

$$\psi_{\lambda,k}(x) = -\frac{1}{k}\int_1^0 \left(e^{\frac{\lambda(1-u)}{ku}} - u^{\frac{x}{k}}\right)\frac{du}{u(1-u)} \tag{3.54}$$

$$= \int_0^1 \left(\frac{e^{\frac{-\lambda(1-u)}{ku}}}{u} - \frac{u^{\frac{x}{k}}}{u}\right)\frac{du}{k(1-u)}, \tag{3.55}$$

which gives (3.51). This completes the proof. $\square$

Theorem 3.17. *The* $(\lambda, k)-$*digamma function satisfy the identities*

$$\psi_{\lambda,k}(x+k) = \frac{1}{x} + \psi_{\lambda,k}(x), \tag{3.56}$$

$$= \frac{1}{x} - \frac{\ln\lambda}{k} + \psi_k(x). \tag{3.57}$$

Proof. By using (3.7), we obtain

$$\ln\Gamma_{\lambda,k}(x+k) = \ln\frac{x}{k} + \ln\Gamma_{\lambda,k}(x) \tag{3.58}$$

Now, differentiating (3.58) with respect to x gives

$$\frac{\Gamma'_{\lambda,k}(x+k)}{\Gamma_{\lambda,k}(x)} = \frac{1}{x} + \frac{\Gamma'_{\lambda,k}(x)}{\Gamma_{\lambda,k}(x)},$$

which implies that

$$\psi_{\lambda,k}(x+k) = \frac{1}{x} + \psi_{\lambda,k}(x).$$

This proves (3.56).
Next, by using (3.8), we obtain

$$\ln\Gamma_{\lambda,k}(x+k) = \ln\frac{x}{k} - \frac{x}{k}\ln\lambda + \ln\Gamma_{\lambda,k}(x) \tag{3.59}$$

Differentiating (3.59) gives

$$\psi_{\lambda,k}(x+k) = \frac{1}{x} - \frac{\ln\lambda}{k} + \psi_k(x), \tag{3.60}$$

which completes the proof. $\square$

Remark. *The results of* (3.56) *indicates that the* $k-$*analogue of the lambda gamma function is increasing on* $(0,\infty)$.

Theorem 3.18. *The* $(\lambda, k)-$*digamma function satisfies the identity*

$$\psi_{\lambda,k}(x+k) = \int_0^\infty \left(\frac{e^{-\frac{\lambda z}{k}}}{z} - \frac{e^{-\frac{zx}{k}}}{e^z - 1}\right)\frac{dz}{k}. \tag{3.61}$$

Proof. By using (3.48) and replacing x with $x+k$, we obtain (3.61). $\square$

Theorem 3.19. *The k-analogue of the lambda gamma function is logarithmically convex.*

Proof. By Lemma 2.8, we have

$$\begin{aligned}
\Gamma_{\lambda,k}\left(\frac{u}{a}+\frac{v}{b}\right) &= \int_0^\infty t^{\frac{u}{a}+\frac{v}{b}-1} e^{-\frac{\lambda t^k}{k}}\,dt \\
&= \int_0^\infty t^{\frac{u}{a}} t^{\frac{v}{b}} t^{-\frac{1}{a}} t^{-\frac{1}{b}} e^{-\frac{\lambda t^k}{ak}} e^{-\frac{\lambda t^k}{bk}}\,dt \\
&= \int_0^\infty \left(t^{u-1}e^{-\frac{\lambda t^k}{k}}\right)^{\frac{1}{a}} \left(t^{v-1}e^{-\frac{\lambda t^k}{k}}\right)^{\frac{1}{b}} dt \\
&\le \left(\int_0^\infty t^{u-1}e^{-\frac{\lambda t^k}{k}}\,dt\right)^{\frac{1}{a}} \left(\int_0^\infty t^{v-1}e^{-\frac{\lambda t^k}{k}}\,dt\right)^{\frac{1}{b}} \\
&= \Gamma^{\frac{1}{a}}_{\lambda,k}(u)\Gamma^{\frac{1}{b}}_{\lambda,k}(v)
\end{aligned}$$

This completes the proof. $\square$

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