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A FEW OBSERVATIONS AND SOME NEW RESULTS ON AB-ALGEBRAS

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ABSTRACT. The notion of AB-algebras was introduced in 2017 by Hameed and Abbas. Subsequently, this class of algebras was the subject of study by several authors. In this paper, in addition to giving an overview of some of the results and assertions about this class of logical algebras, we comment on a significant number of demonstrations of these assertions, and also state several new assertions and demonstrate their proofs.

1. INTRODUCTION

Since the beginning of the second half of the 20th century, specific algebraic structures have begun to appear, which we classify as so-called logical algebras. Some of the first of these, known as BCK/BCI-algebras, were introduced in 1966 by Y. Imai and K. Iséki ([10]). In 1983, Hu and Li ([9]) introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK- and BCI-algebras. Then a whole series of different logical algebras was determined, such as, for example, BCC-algebras ([13]), BH-algebra ([12]), B-algebra ([15]), BF-algebras ([17]), CI-algebras ([14]) and BI-algebras ([3]. This author has participated in the analysis of some of them (see, for example [16]). A special and very clear insight into the classes of many logical algebras is given in the article [11].

The concept of AB-algebras was introduced in 2017 in the article [6] by A. T. Hameed and B. N. Abbas. This algebraic structure was then the subject of several research texts (see, for example, [2, 5, 7, 8]). While the article [2] discusses homomorphisms between AB-algebras, the papers [5, 8] describe the properties of AB-algebras using fuzzy techniques.

By studying the available literature on this class of logical algebras, we not only notice some incompleteness about these algebras but also prove several new theorems about them,

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thus complementing the internal architecture of AB-algebras and also observing the properties of some of their substructures. As confirmation of the opinion expressed here, the author has designed several observations on the claims presented in previously published texts. This paper is designed as follows:

In the preliminaries section, besides presenting the necessary concepts for Section 3, which is the main part of this paper, we also give a three observation on the original determination of the concept of AB-algebras and some of its properties presented in the original source on this class of logical algebras.

Section 3 contains six subsections. The first of these discusses the internal architecture of AB-algebras. It also provides three observations relating to previously published texts on this class of logical algebras. In the second, the theorem is proved that the direct product of any family of AB-algebras is again an AB-algebra. The third subsection discusses subalgebras and ideals in AB-algebras. Among other things, two criteria for recognizing ideals in AB-algebras are proved. While subsections 4 and 5 discuss the properties of ideals and congruences on AB-algebras as well as their connections, subsection 6 deals with the concept of weak AB-algebras.

2. Preliminaries

The concept of AB-algebras is introduced in [6] by A. T. Hameed and B. N. Abbas as follows:

Definition 2.1. ([6], Definition 2.1) An AB-algebra is a nonempty set A with a constant 0 and a binary operation $*$ satisfying the following axioms:

(AB) $(\forall x, y, z \in A)(((x * y) * (z * y)) * (x * z) = 0)$.

(L) $(\forall x \in A)(0 * x = 0)$.

(M) $(\forall x \in A)(x * 0 = x)$.

We denote this axiomatic system by **AB** and the corresponding algebraic structure by $\mathfrak{A} =: (A, *, 0)$.

Example 2.2. Let $A = \{0, a, b, c, d\}$ be set and let the operation $*$ be determined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

Then $\mathfrak{A} =: (A, *, 0)$ is a AB-algebra ([6], Example 2.3). □

Observation 1. In [6], Proposition 2.5(2) the statement

$$(\forall x, y \in A)((x * y) * y = 0)$$

is stated. However, this formula is not valid in the AB-algebra $\mathfrak{A}$ because, for example, in the previous example, we have $(d * a) * a = c * a = c \neq 0$.

Observation 2. In [6], Proposition 2.5(3) the statement

$$(\forall x, y \in A)(x * y = 0 \implies 0 * x = 0 * y)$$

is stated. However, the hypothesis in this implication is not necessary, because, by (L), the following holds

$$(\forall x, y \in A)(0 = 0 * x = 0 * y = 0).$$

Observation 3. In [6], Remark 2.4, the axiom (iii') should reads

$$(\forall x \in A)(0 \leqslant x)$$

with previously accepting the meaning of the sign $\leqslant$.

The terminology and notation in this text are mostly adapted from the very well-known article [11].

3. The main results

3.1. **AB-algebras.** First, from (L), that is, from (M), we immediately obtain:

Proposition 3.1. *Let* $\mathfrak{A} =: (A, *, 0)$ *be a AB-algebra. Then:*

(1) $0 * 0 = 0$.

Proposition 3.2. *In any AB-algebra* $\mathfrak{A} =: (A, *, 0)$*, the following properties holds:*

(Re) $(\forall x \in A)(x * x = 0)$.

(2) $(\forall x, y \in A)((x * y) * x = 0)$.

Proof. If we put $z = 0$, and $y = 0$ in (AB) , we get $((x * 0) * (0 * 0)) * (x * 0) = 0$. From here, with respect to (1) and (M), we get (Re).

Since our statement (2) differs from statement (2) in [6], Proposition 2.5, we state its proof. If we put $z = 0$ in (AB), we get $((x * y) * (0 * y))(x * 0) = 0$. From here, with respect to (L) and (M), we get $(x * y) * x = 0$. $\square$

We determine the relation $\preccurlyeq$ in an AB-algebra $\mathfrak{A} =: (A, *, 0)$ by the following way

$$(\forall x, y \in A)(x \preccurlyeq y \iff x * y = 0).$$

The induced relation $\preccurlyeq$ on the AB-algebra $\mathfrak{A} =: (A, *, 0)$, has the following properties:

Proposition 3.3. *Let* $\mathfrak{A} =: (A, *, 0)$ *be an AB-algebra. Then:*

(3) $(\forall x \in A)(0 \preccurlyeq x)$.

(4) $(\forall x \in A)(x \preccurlyeq x)$.

(5) $(\forall x \in A)(x \preccurlyeq 0 \implies x = 0)$.

(6) $(\forall x, y \in A)(x * y \preccurlyeq x)$.

(7) $(\forall x, y, z \in A)((x \preccurlyeq y \wedge y \preccurlyeq z) \implies x \preccurlyeq z)$.

(8) $(\forall x, y, z \in A)(x \preccurlyeq y \implies x * z \preccurlyeq y * z)$.

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \implies z * y \preccurlyeq z * x)$.

Proof. Since statements (3), (4) and (6) are obvious, we will prove the statement (5) and and others.

(5): Let $x \in A$ be such that $x \preccurlyeq 0$. This means $x * 0 = 0$. Since $x * 0 = x$ according to (M), we get $x = 0$.

(7): Let $x, y, z \in A$ be such that $x \preccurlyeq y$ and $y \preccurlyeq z$. This means that $x * y = 0$ and $y * z = 0$ If we put $z = y$ and $y = z$ in (AB), we get $((x * z) * (y * z)) * (x * y) = 0$. From here, taking into account the hypothesis, we get $((x * z) * 0) * 0 = 0$ from where it follows that $x * z = 0$ due to (M) . This proves that $x \preccurlyeq z$ which means that the induced relation $\preccurlyeq$ on A is a quasi-order on A.

(8): Let $x, y, z \in A$ be such that $x \preccurlyeq y$ Then $x * y = 0$. If we put $z = y$ and $y = z$ in (AB), we get $((x * z) * (y * z)) * (x * y) = 0$. Thus $(x * z) * (y * z) \preccurlyeq x * y = 0$. From here it follows that $(x * z) * (y * z) = 0$ according to (5). Hence, $x * z \preccurlyeq y * z$.

(9): Let $x, y, z \in A$ be arbitrary elements such that $x \preccurlyeq y$. This means $x * y = 0$. If we put $x = z$ and $z = x$ in (AB), we get $((z*y)*(x*y))*(x*z) = 0$. Thus, $(z*y)*(z*x) = 0$ if we use the hypothesis and taking into account (M), So, $z * y \preccurlyeq z * x$. $\square$

In the Previous Proposition it was shown that the induced relation $\preccurlyeq$ is a quasi-order (reflexive and transitive relation) on the set A and that it is right compatible and left anti-compatible with the operation in $\mathfrak{A}$ in accordance with the just proved statements (8) and (9). On the other hand, it is generally known that the relation

$$\equiv =: \preccurlyeq \cap \preccurlyeq^{-1}$$

is an equivalence on the set A. Therefore, $\equiv_{\preccurlyeq}$ is a congruence on $\mathfrak{A}$ since it is compatible with the operation in $\mathfrak{A}$. Therefore, in every AB-algebra $\mathfrak{A} =: (A, *, 0)$, the following formula

$$(\forall x, y \in A)((x \preccurlyeq y \wedge y \preccurlyeq x) \implies x \equiv y)$$

is valid.

Example 3.1. Let $A = \{0, a, b, c\}$ be set and let the operation $*$ be determined by the following table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
c	c	b	b	0

Then $\mathfrak{A} =: (A, *, 0)$ is a AB-algebra ([6], Example 2.2).

Based on the previous example, we can make the following observations:

Observation 4. In this case, we have $a * (b * c) = a * 0 = a$ and $b * (a * c) = b * 0 = b$, which shows that formula

(a) $(\forall x, y, z \in A)(x * (y * z) = y * (x * z))$

in [6], Proposition 2.7 is not a valid formula.

Observation 5. In the general case, formula

(b) $(\forall x, y \in A)((y * x) * (0 * x) = y)$

in [6], Proposition 2.7 is also not valid because, for example, we have $(b * a) * (0 * a) = a * 0 = a \neq b$.

Observation 6. In addition to the above, the formula

(2) $(\forall x, y \in A)((x * (x * y)) * y = 0.)$

in [7], Remark 2.3 is also not valid in the general case because, for example, we have $(c * (c * a)) * a = (c * b) * a = b * a = a \neq 0$.

3.2. **Direct product of AB-algebras.** In what follows, we deal with the creation of the direct product of AB-algebras. Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of AB-algebras. If on the set

$$\prod_{i\in I} A_i =: \{f : I \longrightarrow \cup_{i\in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i\in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) *_i g(i)),$$

we created the structure $(\prod_{i\in I} A_i, \odot, f_0)$, where f_0 was chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i).$$

Before we start working with direct products of AB-algebras, we say that the operation determined in this way is well-defined. If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

Theorem 3.4. *The direct product of any family of AB-algebras, determined as above, is an AB-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of AB-algebra:

Let $f, g, h \in \prod_{i\in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(M) $(f \odot f_0)(i) = f(i) *_i f_0(i) = f(i) *_i 0_i = f(i)$.

(L) $(f_0 \odot f)(i) = f_0(i) *_i f(i) = 0_i \odot f(i) = 0_i$.

(AB) Considering that

$$(((f \odot g) \odot (g \odot h))(h \odot f))(i) = ((f(i) *_i g(i))(h(i) *_i g(i))) *_i (f(i) *_i h(i)) = f_0(i),$$

we have that (AB) is a valid formula for the observed structure.

Therefore, the structure $(\prod_{i\in I} A_i, \odot, f_0)$ is an AB-algebra. $\square$

3.3. **Sub-algebras and ideals.** The concept of sub-algebras in AB-algebras is introduced by the standard way:

Definition 3.2. ([6], Definition 2.8) A nonempty subset S of an AB-algebra $\mathfrak{A} =: (A, *, 0)$ is called a sub-algebra in $\mathfrak{A}$ if it holds

(S1) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x * y \in S)$.

We denote the family of all sub-algebras of the AB-algebra $\mathfrak{A}$ by $\mathfrak{S}(A)$.

Let us show that the sub-algebra S in an AB-algebra $\mathfrak{A}$ also satisfies the condition

(S0) $0 \in S$.

Indeed, since S is not empty, there exists at least some $x \in A$ such that $x \in S$. From here, it follows immediately that $0 = x * x \in S$ according to (S1) and (Re).

Definition 3.3. ([6], Definition 2.9) A nonempty subset J of an AB-algebra $\mathfrak{A} =: (A, *, 0)$ is called an ideal in $\mathfrak{A}$ if it satisfies the following conditions:

(J0) $0 \in J$.

(J1) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge y \in J) \implies x * z \in J)$.

We denote the family of all ideals of the AB-algebra $\mathfrak{A}$ by $\mathfrak{J}(A)$.

Proposition 3.5 ([6], Proposition 2.11)**.** *Every ideal in an AB-algebra $\mathfrak{A}$ is a sub-algebra in $\mathfrak{A}$. So, $\mathfrak{J}(A) \subseteq \mathfrak{S}(A)$.*

Proof. Since our proof of this proposition differs somewhat from the proof in [6], we demonstrate it here.

Let J be an ideal in $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in J$ and $y \in J$. Then $J \ni x = x * 0 = x * (y * y)$ and $y \in J$. Thus $x * y \in J$ by (J1). $\square$

Also, we have:

Proposition 3.6. *For every ideal J in an AB-algebra $\mathfrak{A}$ holds*

(J2) $(\forall x, y \in A)((x * y \in J \wedge y \in J) \implies x \in J)$.

Proof. If we put $z = 0$ in (J1), respecting (M), we get (J2). $\square$

Corollary 3.7. *For every ideal J in an AB-algebra $\mathfrak{A}$ holds*

(J3) $(\forall x, y \in A)((x \preccurlyeq y \wedge y \in J) \implies x \in J)$.

Proof. Let J be an ideal in an AB-algebra $\mathfrak{A}$ and let $x.y \in A$ be such that $x \preccurlyeq y$ and $y \in J$. Then $x * y = 0 \in J$ and $y \in J$ in accordance with (J0). Thus $x \in J$ by (J2). $\square$

As a consequence of the claim (6) and Corollary 3.7, we have:

Corollary 3.8. *For every ideal J in an AB-algebra $\mathfrak{A}$ holds*

(J4) $(\forall x, y \in A)((x \in J \implies x * y \in J)$.

Proof. Since, for arbitrary $x, y \in A$, according to (6), $x * y \preccurlyeq x$ holds, we have $x * y \in J$ in accordance with Corollary 3.7. $\square$

Let us emphasize that the statements (J3) and (J4) are consequences of the statement (J2). This means that if a nonempty subset J in an AB-algebra $\mathfrak{A}$ satisfies the conditions (J0) and (J2), then it also satisfies the conditions (J3) and (J4). Let us now show that the converse of Proposition 3.6 also holds.

Proposition 3.9. *If a nonempty subset J of an AB-algebra $\mathfrak{A}$ satisfies the conditions* (J0) *and* (J2)*, then J is an ideal in $\mathfrak{A}$.*

Proof. Let J be a nonempty subset in an AB-algebra $\mathfrak{A}$ that satisfies the conditions (J0) and (J2) and let $x, y, z \in A$ be such that $x * (y * z) \in J$ and $y \in J$. Then, with respect to (J4), we have $x * (y * z) \in J$ and $y * z \in J$. Thus, $x \in J$ according to (J2). Hence, $x * z \in J$ according to (J4). Therefore, J is an ideal in $\mathfrak{A}$. $\square$

Example 3.4. Let the AB-algebra $\mathfrak{A}$ be as in Example 2.2.

Subsets $S_0 =: \{0\}$, $S_1 =: \{0, a\}$, $S_2 =: \{0, b\}$, $S_3 =: \{0, c\}$, $S_4 =: \{0, a, b\}$ and $S_6 =: \{0, b, c\}$ are sub-algebras in $\mathfrak{A}$. The subset $S_5 =: \{0, a, c\}$ is not a sub-algebra in $\mathfrak{A}$ since, for example, we have $c * a = b \notin S_5$.

Sub-algebra S_0 is an ideal in $\mathfrak{A}$. The other sub-algebras mentioned above are not ideals in $\mathfrak{A}$. $\square$

The previous example shows that, in the general case, $\mathfrak{J}(A) \subsetneq \mathfrak{S}(A)$ holds.

In the following theorem, we show that:

Theorem 3.10. *The families $\mathfrak{S}(A)$ and $\mathfrak{J}(A)$ of every AB-algebra $\mathfrak{A} =: (A, *, 0)$ form complete lattices.*

Proof. Let $\{S_i :\in I\}$ be a family of sub-algebras (ideals) in AB-algebra $\mathfrak{A}$.

Suppose that S_i is a sub-algebra in $\mathfrak{A}$ for every $i \in I$. Let $x, y \in A$ be such that $x \in \bigcap_{i \in I} S_i$ and $y \in \bigcap_{i \in I} S_i$. This means that $x \in S_i$ and $y \in S_i$ for each $i \in I$. Thus

$x * y \in S_i$ for each $i \in I$ since S_i is a sub-algebra in $\mathfrak{A}$. Hence $x * y \in \bigcap_{i \in I} S_i$. So, $\bigcap_{i \in I} S_i$ is a sub-algebra in $\mathfrak{A}$.

Suppose that S_i is an ideal in $\mathfrak{A}$ for every $i \in I$. Let $x, y \in A$ be such that $x * y \in \bigcap_{i \in I} S_i$ and $y \in \bigcap_{i \in I} S_i$. This means that $x * y \in S_i$ and $y \in S_i$ for each $i \in I$. Thus $x \in S_i$ for each $i \in I$ since S_i is an ideal in $\mathfrak{A}$. Hence $x \in \bigcap_{i \in I} S_i$. So, $\bigcap_{i \in I} S_i$ is an ideal in $\mathfrak{A}$.

Let $\mathcal{Z}$ be the family of all sub-algebras (ideals) in $\mathfrak{A}$ that contain $\bigcup_{i \in I} S_i$. Then $\cap \mathcal{Z}$ is a sub-algebra (an ideal) that contains $\bigcup_{i \in I} S_i$ as already shown in this proof. If we put $\sqcup_{i \in I} S_i = \bigcup_{i \in I} S_i$ and $\sqcap_{i \in I} S_i = \cap \mathcal{Z}$, then $(\mathfrak{S}(A), \sqcup, \sqcap)$ is a complete lattice. □

The previous theorem is a generalization of Theorem 2.13 in [6].

Example 3.5. Let $A = \{0, a, b, c, d\}$ be set and let the operation $*$ be determined by the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	b	a	0	0
d	d	d	d	d	0

Then $\mathfrak{A} =: (A, *, 0)$ is a AB-algebra ([2], Example 2.2).

Subsets $S_0 =: \{0\}$, $S_1 =: \{0, a\}$, $S_2 =: \{0, b\}$, $S_3 =: \{0, c\}$, $S_4 =: \{0, d\}$, $S_5 =: \{0, a, b\}$, $S_7 =: \{0, a, d\}$, $S_9 =: \{0, b, d\}$, $S_{10} =: \{0, c, d\}$, $S_{11} =: \{0, a, b, c\}$ and $S_{12} =: \{0, a, b, d\}$ are sub-algebras in $\mathfrak{A}$. However, the subsets $S_6 =: \{0, a, c\}$, $S_8 =: \{0, b, c\}$, $S_{13} =: \{0, a, c, d\}$ and $S_{14} =: \{0, b, c, d\}$ are not sub-algebras in A.

Sub-algebras S_0, S_1 and S_{11} are ideals in $\mathfrak{A}$. The remaining sub-algebras in $\mathfrak{A}$, mentioned above, are not ideals in $\mathfrak{A}$. For illustration, the sub-algebra $S_5 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c * b = b \in S_5$ and $b \in S_5$ but $c \notin S_5$. □

Remark. *The statement that the sub-algebra S_5 is not an ideal in the AB-algebra $\mathfrak{A}$ contradicts the statement about this sub-algebra reported in* [2]*, Example 2.8.*

Further on, we have:

Theorem 3.11. *Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of AB-algebras, K be a subset of I and let S_i be a sub-algebra (an ideal) in $(A_i, *_i, 0_i)$ for each $i \in K$. Then $\prod_{i \in I} T_i$, where $T_i = S_i$ for $i \in K$ and $T_j = A_j$ for $j \neq i \in K$, is a sub-algebra (an ideal) in the AB-algebra $\prod_{i \in I} A_i$.*

Proof. First, it is clear that $f_0 \in \prod_{i \in I} T_i$.

If $K = \emptyset$, then $\prod_{i \in I} T_i = \prod_{i \in I} A_i$, so $\prod_{i \in I} T_i$ is certainly a sub-algebra (an ideal) in $\prod_{i \in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Let $x, y \in \prod_{i \in I} A_i$ be such that $x \in \prod_{i \in I} T_i$ and $y \in \prod_{i \in I} T_i$. This means $x(i) \in S_i$ and $y(i) \in S_i$ for each $i \in K$. Then $(x \odot y)(i) = x(i) *_i y(i) \in S_i$ since J_i is an sub-algebra in $(A_i, *_i, 0_i)$ for each $i \in K$. Hence $x \odot y \in \prod_{i \in I} T_i$. As shown, $\prod_{i \in I} T_i$ is a sub-algebra in $\prod_{i \in I} A_i$.

Let $x, y \in \prod_{i \in I} A_i$ be such that $x \odot y \in \prod_{i \in I} T_i$ and $y \in \prod_{i \in I} T_i$. This means $(x \odot y)(i) = x(i) *_i y(i) \in S_i$ and $y(i) \in S_i$ for each $i \in K$. Then $x(i) \in S_i$ since S_i is an ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Hence $x \in \prod_{i \in I} T_i$. As shown, $\prod_{i \in I} T_i$ is an ideal in $\prod_{i \in I} A_i$. □

Example 3.6. Let the AB-algebra $\mathfrak{A} =: (A, *, 0)$ be as in Example 3.4. Therefore, $\mathfrak{A} \times \mathfrak{A} =: (A \times A, \odot, (0, 0))$ is an AB-algebra according to the Theorem 3.4 where the operation $\odot$ is determined as follows

$$(\forall x, y, u, v \in A)((x, y) \odot (u, v) =: (x * u, y * v)).$$

The subset $S_1 =: \{0, a\}$ is a sub-algebra (an ideal) in $\mathfrak{A}$, so the subsets $S_1 \times A$, $A \times S_1$ and $S_1 \times S_1$ are sub-algebras (ideals) in $\mathfrak{A} \times \mathfrak{A}$. $\square$

At the end of this subsection, let us give a criterion for recognizing ideals in AB-algebras.

Theorem 3.12. *Let $\mathfrak{A} =: (A, *, 0)$ be an AB-algebra. A subset J of A is an ideal in $\mathfrak{A}$ if and only if it holds*

(J5) $(\forall x, y, z \in A)((y \in J \wedge z \in J \wedge (x * y) * z = 0) \implies x \in J)$.

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $y, z \in J$ and $(x * y) * z = 0$. Then $x * y \preccurlyeq z \in J$. Thus $x * y \in J$ by (J3). Hence, $x \in J$ by (J1).

Conversely, let (J5) be valid and let $x, y \in A$ be such that $x * y \in J$ and $y \in J$. Then $(x * y) * (x * y) = 0$, $y \in J$ and $x * y \in J$ according to (Re). Thus $x \in J$ by (J5). Let us show that (J0) holds. Putting $x = 0$ in (J5), we get $0 \in J$ with respect (L). Therefore, J is an ideal in $\mathfrak{A}$. $\square$

3.4. **Ideals and congruences.** For an equivalence relation ρ on an AB-algebra $\mathfrak{A} =: (A, *, 0)$ we say that it is a left congruence on $\mathfrak{A}$ if it holds

$$(\forall x, y, z \in A)((x, y) \in \rho \implies (z * x, z * y) \in \rho).$$

The right congruence is determined analogously. The relation ρ is said to be a congruence on $\mathfrak{A}$ if it is a left and right congruence on $\mathfrak{A}$. By $\mathfrak{Q}(A)$ ($\mathfrak{Q}_L(A)$ respectively) we denote the family of all (left) congruences on the AB-algebra $\mathfrak{A} =: (A, *, 0)$.

In what follows, we need the following lemma:

Lemma 3.13. *Let J be an ideal in an AB-algebra $\mathfrak{A} = (A, *, 0)$. Then:*

(10) $(\forall x, y, z \in A)((x * y \in J \wedge y * z \in J) \implies x * z \in J)$.

(11) $(\forall x, y, z \in A)(x * y \in J \implies (x * z) * (y * z) \in J)$.

(12) $(\forall x, y, z \in A)(y * x \in J \implies (z * x) * (z * y) \in J)$.

Proof. (10): If we put $y = z$ and $z = y$ in (AB), we get $((x * z) * (y * z)) * (x * y) = 0 \in J$. From here, according to (J2), it follows $(x * z) * (y * z) \in J$ since $x * y \in J$. We can apply (J2) again, because $y * z \in J$. We get $x * z \in J$.

(11): If we put $z = y$ and $y = z$ in (AB), we get $((x * z) * (y * z)) * (x * y) = 0 \in J$. From here, according to (J2), it follows $(x * z) * (y * z) \in J$.

(12): If we put $x = z$, $y = x$ and $z = y$ in (AB), we get $((z * x) * (y * x)) * (z * y) = 0 \in J$. From here, according to (J1), $(z * x) * (z * y) \in J$ follows. $\square$

The concept of α-ideals in logical algebras (see, for example [4]) is determined by the conditions (J0) and (10). Therefore:

Proposition 3.14. *Every ideal in an AB-algebra $\mathfrak{A}$ is an α-ideal in $\mathfrak{A}$.*

Now we can prove the following theorem:

Theorem 3.15 ([6], Theorem 3.2, Lemma 3.2 and Corollary 3.4)**.** *Let J be an ideal in an AB-algebra $\mathfrak{A} =: (A, *, 0)$. Then the relation $\rho_J \subseteq A \times A$, defined by*

$$(\forall x, y \in A)((x, y) \in \rho_J \iff (x * y \in J \wedge y * x \in J)),$$

is an equality relation on A compatible with the operation in $\mathfrak{A}$.

Proof. Since our proof of this theorem differs significantly from the proofs presented in [6], Theorem 3.2, Lemma 3.2 and Corollary 3.4, we demonstrate it here.

Since it is obvious that the relation ρ_J is reflexive and symmetric, it remains to prove transitivity and its compatibility with the operation in $\mathfrak{A}$.

Let $x, y, z \in A$ such that $(x, y) \in \rho_J$ and $(y, z) \in \rho_J$. This means $x * y \in J$, $y * x \in J$, $y * z \in J$ and $z * y \in J$. Now, from $x * y \in J$ and $y * z \in J$ we get $x * z \in J$ according to (10). Assume that $z * y \in J$ and $y * x \in J$. If we put $x = z$ and $z = x$ in (10), we get $z * x \in J$. This shows that ρ_J is transitive.

Let $x, y, z \in A$ be such that $(x, y) \in \rho_J$. This means $x * y \in J$ and $y * x \in J$. If we use $x * y \in J$, (11) immediately gives $(x * z) * (y * z) \in J$. Let us take $y * x \in J$. If we put $x = y$ and $y = x$ in (11), we get $(y * z) * (x * z) \in J$. Therefore, $(x * z, y * z) \in \rho_J$, which means that ρ_J is a right congruence on $\mathfrak{A}$.

Let $x, y, z \in A$ be such that $(x, y) \in \rho_J$. This means $x * y \in J$ and $y * x \in J$. If we use $y * x \in J$, (12) immediately gives $(z * x) * (z * y) \in J$. Let us take $x * y \in J$. f we put $x = y$ and $y = x$ in (12), we get $(z * y) * (z * x) \in J$. Therefore, $(z * x, z * y) \in \rho_J$, which means that ρ_J is a left congruence on $\mathfrak{A}$. $\square$

The previous theorem illustrates the correspondence

$$\mathfrak{J}(A) \ni J \longmapsto \rho_J \in \mathfrak{Q}(A).$$

On the other hand, we have:

Theorem 3.16. *If ρ is a left congruence on an AB-algebra $\mathfrak{A}$, then the class $[0]_\rho =: \{x \in A : (x, 0) \in \rho\}$ is an ideal in $\mathfrak{A}$.*

Proof. Let ρ be a left congruence on an AB-algebra $\mathfrak{A} =: (A, *, 0)$ and let $x, y \in A$ be such that $x * y \in [0]_\rho$ and $y \in [0]_\rho$. This means $(x * y, 0) \in \rho$ and $(y, 0) \in \rho$. Then $(x * y, 0) \in \rho$ and $(x * y, x * 0) = (x * y, x) \in \rho$ since ρ is a left congruence on $\mathfrak{A}$. Hence, $(0, x) \in \rho$ due to the transitivity of the relation ρ. Thus, $x \in [0]_\rho$. Therefore, $[0]_\rho$ is an ideal in $\mathfrak{A}$ since $0 \in [0]_\rho$ by reflexivity of ρ. $\square$

The preceding theorem illustrates the correspondence

$$\mathfrak{Q}_L(A) \ni \rho \longmapsto [0]_\rho \in \mathfrak{J}(A).$$

3.5. **Strong ideals.** On the other hand, in the text [7], Definition 2.5, the concept of ideals in AB-algebras is introduced somewhat differently than in [6], Definition 2.9. Since this definition of an ideal, as done in the paper [7], is recognizable as a strong ideal (see, for example [1], Definition 3.1), we have:

Definition 3.7. A non-empty subset J of an AB-algebra $\mathfrak{A} =: (A, *, 0)$ is called a strong ideal in $\mathfrak{A}$ if it satisfies (J0) and the following condition:

(StJ) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \implies x * z \in J)$.

Proposition 3.17. *Any strong ideal in an AB-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. Putting $z = 0$ in (StJ), we obtain (J2). $\square$

Proposition 3.18. *In every AB-algebra* $\mathfrak{A} =: (A, *, 0)$*, the subset* $\{0\}$ *is a strong ideal in* $\mathfrak{A}$.

Proof. Let $x, y, z \in A$ be arbitrary elements such that $(x * y) * z = 0$ and $y = 0$. Then $x * z = (x * 0) * z = 0$ in accordance with (M). So, the subset $\{0\}$ is a strong ideal in $\mathfrak{A}$. □

Definition 3.8. ([2], Definition 3.1) Let $\mathfrak{A} =: (A, *, 0_A)$ and $\mathfrak{B} =: (B, \star, 0_B)$ be AB-algebras. A function $f : A \longrightarrow B$ is said to be an AB-homomorphism if holds

(f1) $(\forall x, y \in A)(f(x * y) = f(x) \star f(y))$.

We denote this AB-homomorphism by $f : \mathfrak{A} \longrightarrow \mathfrak{B}$.

Let's prove that it is also valid

(f0) $f(0_A) = 0_B$.

Indeed, for arbitrary $x \in A$, we have $f(0_A) = f(x * x) = f(x) \star f(x) = 0_B$.

Theorem 3.19. *Let* $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ *be an AB-homomorphism of AB-algebras. If* C *is a strong ideal of* $\mathfrak{B}$ *, then* $f^{-1}(C)$ *is a strong ideal in* $\mathfrak{A}$.

Proof. Since $f(0) = 0$, we have $0 \in f^{-1}(C)$.

Let $x, y, z \in A$ be such that $(x * y) * z \in f^{-1}(C)$ and $y \in f^{-1}(C)$. Then $(f(x) * f(y)) * f(z) = f((x * y) * z) \in C$ and $f(y) \in C$. Since C is a strong ideal in $\mathfrak{B}$, it follows from (StJ) that $f(x * z) = f(x) * f(z) \in C$. So that $x * z \in f^{-1}(C)$. Hence $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$. □

Corollary 3.20. *Let* $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ *be an AB-homomorphism of AB-algebras. Then* $Kerf := \{x \in A : f(x) = 0\}$ *is a strong ideal of* $\mathfrak{A}$.

Proof. Since the subset $\{0\}$ is a strong ideal in the AB-algebra $\mathfrak{B}$, by Proposition 3.18, we have that the kernel $Kerf = f^{-1}(\{0\})$ of the homomorphism f is a strong ideal in $\mathfrak{A}$ in accordance with the previous theorem. □

The claim of this Corollary is stronger than the claim exposed by Theorem 3.7 in [2].

Proposition 3.21. *Every strong ideal in an AB-algebra* $\mathfrak{A}$ *is a sub-algebra in* $\mathfrak{A}$.

Proof. Since every strong ideal in an AB-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$ (according to Proposition 3.17) and every ideal in $\mathfrak{A}$ is a sub-algebra in $\mathfrak{A}$ (according to Proposition 3.5), we conclude that every strong ideal in $\mathfrak{A}$ is a sub-algebra in $\mathfrak{A}$. □

Example 3.9. Let $\mathfrak{A}$ be an AB-algebra as in Example 3.5. Ideals $S_0 = \{0\}$, $S_1 = \{0, a\}$ and $S_{11} = \{0, a, b, c\}$ are strong ideals in $\mathfrak{A}$.

3.6. **Weak AB-algebras.** The concept of weak AB-algebras was introduced in [7] by A. T. Hameed and B. N. Abbas as follows:

Definition 3.10. ([7], Definition 3.1.1) An algebra $\mathfrak{A} =: (A, *, 0)$ of type $(2, 0)$ is a weak AB-algebra if and only if it satisfies the conditions (AB), (Re) and (M).

We denote this system of axioms by **wAB** and the corresponding algebraic structure as wAB-algebra.

Remark. *The requirement (Re) can be omitted in this determination, since it can be demonstrated from (AB) and (M), as already done in Proposition 3.2.*

It is obvious that every AB-algebra is a wAB-algebra and that, in the general case, the reverse need not be the case.

As is common in weakened logical algebras that are obtained by omitting the condition (L), so in the case of wAB-algebra $\mathfrak{A} =: (A, *, 0)$, the correspondence $\phi(x) =: 0 * x$ is introduced for arbitrary $x \in A$.

Observation 7. The statement [7], 3.1.4(2)

$$(\forall x, y \in A)(x \preccurlyeq y \implies \phi(x) = \phi(y))$$

cannot be proved without using the formula $(\forall x, y, z \in A)((x*y)*z = (x*z)*y)$, stated in [7], Lemma 2.1.7(4) since it is, in general, not a valid formula in the system **wAB**. For illustration, in Example 3.1, we have $(c * b) * a = b * a = a \neq 0 = b * b = (c * a) * b$.

The following observation follows from the previous observation and is related to the statement [7], Theorem 3.1.4(3)

$$(\forall x \in A)(\phi^3(x) = \phi(x)).$$

It is possible to prove:

Proposition 3.22. *Let* $\mathfrak{A} =: (A, *, 0)$ *be a wAB-algebra. Then:*

(13) $(\forall x \in A)(\phi^2(x) \preccurlyeq x)$.

(14) $(\forall x \in A)(\phi(x) \preccurlyeq \phi^3(x))$.

(15) $(\forall x, z \in A)(\phi(z * x) \preccurlyeq x * z)$.

(16) $(\forall x, z \in A)(\phi(x * z) \preccurlyeq \phi^2(z * x))$.

Proof. If we put $y = x$ and $z = 0$ in (AB), we get $((x * x) * (0 * x)) * (x * 0) = 0$ which gives $(0 * (0 * x)) * x = 0$ according to (Re) and (M). So $\phi^2(x) \preccurlyeq x$.

From the just proven (13), using (9), we obtain $\phi(x) \preccurlyeq \phi^3(x)$. Here it should be said that the formula (9) is a valid formula in the **wAB** system since its proof was demonstrated without using the axiom (L).

If we put $y = x$ in (AB), we get $((x * x) * (z * x)) * (x * z) = 0$. From here we get $(0 * (z * x)) * (x * z) = 0$ with respect to (Re), which gives $\phi(z * x) \preccurlyeq x * z$.

If we apply (9) to (15), we get (16). $\square$

As a consequence of (13) and (14), we have:

Corollary 3.23. *Let* $\mathfrak{A} =: (A, *, 0)$ *be a wAB-algebra. Then*

$$(\forall x \in A)(\phi^3(x) \equiv \phi(x)).$$

Proof. If we put $\phi(x)$ instead of x in (13), we get $\phi^3(x) \preccurlyeq \phi(x)$ which together with (14), gives $\phi^3(x) \equiv_{\preccurlyeq} \phi(x)$. $\square$

It should be noted that, in the general case, the condition (J1) is not equivalent to the condition (J2) in wAB-algebras. Additionally, if J is an ideal in a wAB-algebra $\mathfrak{A} =: (A, *, 0)$, then, in addition to (J2), also

$$(\forall y, z \in A)((\phi(y) * z \in J \wedge y \in J) \implies \phi(z) \in J)$$

holds. On the other hand, for a strong ideal in a wAB-algebra $\mathfrak{A}$, we have

$$(\forall y, z \in A)((\phi(y * z) \in J \wedge y \in J)) \implies \phi(z) \in J)$$

so, that's why, and the following formula

$$(\forall z \in A)(\phi^2(z) \in J \implies \phi(z) \in J)$$

is valid.

4. Final comments

The concept of AB-algebra was introduced in 2017 in the article [6] written by A. T. Hameed and B. N. Abbas. The axiomatization of this class of logical algebras, as well as the deduction of propositions from them, has not received sufficient attention from researchers. Thus, a number of stated propositions about AB-algebras still do not have satisfactory demonstrations of their proofs. Apart from the previous observations, the question about the correspondence

$$\mathfrak{Q}_L(A) \ni \varrho \longmapsto [0]_\varrho \longmapsto \rho_{[0]_\varrho} \in \mathfrak{Q}(A).$$

remains open. The existence of the aforementioned correspondence between $\mathfrak{Q}_L(A)$ and $\mathfrak{Q}(A)$ supports our expectation that it should be possible to prove that every left congruence on an AB-algebra is simultaneously a right congruence on that algebra.

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