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FINDING MASS DISTRIBUTION FROM THE EXTERIOR POTENTIAL

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ABSTRACT. Let $D \subset \mathbb{R}^3$ be a bounded domain. Consider the equation $\int_D \frac{m(y)}{4\pi|x-y|}dy = f(x),\ x \in D'$, where $D' = \mathbb{R}^3 \setminus D$. We give a method for calculating $m(y)$ from the knowledge of $f(x)$ in a subdomain of D'.

1. INTRODUCTION

In this paper an idea from [3] (see also monograph [7]) is used in a multidimensional problem. In [2] results on numerical solution of operator equations are presented.

Let

$$Am := \int_D \frac{m(y)}{4\pi|x-y|}dy = f(x),\ x \in D', \tag{1.1}$$

where $D' = \mathbb{R}^3 \setminus D$. Let D_1 be a finite subdomain in D'.

Assume that equation (1.1) *has a solution in* $C(D)$. For this, it is necessary that $f(x)$ be a harmonic function inD' and sufficient that it be continuous up to the boundary S of D. Indeed, $Am = f$ is harmonic inD', and for m to be continuous up to S it is suffiecent that f be continuous in $\mathbb{R}^3$.

We assume S to be a smooth, closed and connected surface.

The operator A acts from $C(D)$ into $C(D_1)$. Let

$$Bm := \int_{D_1} dx \frac{1}{4\pi|\xi - x|} \int_D \frac{1}{4\pi|x-y|} m(y)dy := \int_D B(\xi, y)m(y)dy = \int_{D_1} dx \frac{1}{4\pi|\xi - x|} f(x) := h(\xi), \tag{1.2}$$

where

$$Bm = \int_D B(\xi, y)m(y)dy = h(\xi), \tag{1.3}$$

and

$$B(\xi, y) = \frac{1}{16\pi^2} \int_D \frac{1}{|\xi - y||y-x|}dy. \tag{1.4}$$

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The kernel $B(\xi, y) = B(y, \xi)$ is symmetric, the operator $B : X \to X$, where $X := L^2(D)$, is self-adjoint and compact. Thus, the spectrum of B consists of eigenvalues $\lambda_n \to 0$ as $n \to \infty$.

2. PROOFS

Lemma 1. The operator B is non-negative in X, $(Bm, m) \geq 0$.

Proof of Lemma 1. One has:

$$(Bm, m) = \frac{1}{16\pi^2} \int_D \frac{1}{|\xi - y||y - x|} m(x)m(\xi)dxd\xi dy = \frac{1}{16\pi^2} \Big(\int_D \frac{1}{|y - x|} m(x)dx \Big)^2 \geq 0. \tag{2.1}$$

Lemma 1 is proved. □

Equation (1.3) is obviously equivalent to the equation

$$m = (I - B)m + h(\xi). \tag{2.2}$$

Lemma 2. Equation (1.1) **is equivalent to equation** (1.3)**.**

Proof of Lemma 2. From equation (1.1) one derives equation (1.3) by multiplying equation (1.1) by $\frac{1}{4\pi|\xi - x|}$ and integrating over D_1. Denote this operation as $A^\star$. Then $B = A^\star A$. Let us derive from (1.3) equation (1.1).

Let $\|m\|^2 := \int_D |m(y)|^2 dy$, and $(m, p) := \int_D m(y)\overline{p(y)}dy$, where the overline denotes complex conjugate; one can consider the space of real-valued functions and then the overline can be dropped. One has

$$\|Am - f\|^2 = (Am, Am) + (f, f) - (Am, f) - (f, Am) = 0.$$

Here we have used the equation $Am = f$, which was assumed solvable.

Lemma 2 is proved. □

Let us choose the domain D_1 sufficiently far from D. Then the norm of the operator B will be less than 1,

$$\|B\| < 1, \quad B \geq 0. \tag{2.3}$$

Then

$$0 < (I - B) \leq 1. \tag{2.4}$$

Consider the iterative process

$$m_{n+1} = (I - B)m_n + h(\xi), \quad m_0 = p_0, \tag{2.5}$$

where p_0 is an arbitrary element in $L^2(D)$ or $C(D)$. If there is a subsequence of m_n which converges in $L^2(D)$, $\lim_{j\to\infty} m_{n_j} = m$, then its limit m solves eqution (2.2). This equation is equivalent to $Bm = h$, and this equation is equivalent to equation (1.1). Therefore we have an iterative process for solving equation (1.1).

Does a convergent subsequence of m_n exist? The kernel $B(\xi, y)$ is positive. We have proved that. The operator B is self-adjoint, compact, and maps non-negative functions into non-negative ones: it maps a cone of non-negative functions into itself. The operator $I - B$ is self-adjoint, and if the domain D_1 is sufficiently far from D, then the norm of B in either $L^2(D)$ or $C(D)$ is sufficiently small, it less than 1, so the operator $I - B$ is non-negative, the inequalities (2.4) hold.

Subtract from (2.2) equation (2.5) and get

$$q_{n+1} = (I - B)q_n, \quad q_n := m - m_n. \tag{2.6}$$

Since $T := I - B$ is self-adjoint and $0 \le T \le 1$, it has a spectral representation:

$$T = \int_0^1 s dE(s), \quad \|T\| \le 1, \tag{2.7}$$

where $E(s)$ is the resolution of the identity of the self-adjoint operator T.

From (2.6) it follows that

$$q_{n+1} = T^{n+1} q_0, \quad q_n := m - m_n. \tag{2.8}$$

Choose q_0 orthogonal to the set $H_1 := \{\phi : B\phi = 0$. If the equation (1.3), or the equivalent to (1.3) equation (2.2), is solvable, then h must be orthogonal to H_1. Let us choose q_0 orthogonal to H_1. Let $\epsilon > 0$ be an arbitrary small fixed number. One can find $\eta > 0$ sufficiently small, so that

$$\|T^{n+1} q_0\| < \| \int_{1-\eta}^1 s^{n+1} dE(s) q_0\| < \epsilon, \tag{2.9}$$

and a sufficiently large n so that

$$\| \int_0^{1-\eta} s^{n+1} dE(s) q_0\| \le \eta^{n+1} \|q_0\| < \epsilon. \tag{2.10}$$

From (2.9) and (2.10) it follows that

$$\lim_{n\to\infty} \|q_n\| = 0. \tag{2.11}$$

We have proved

Theorem 1. If equation (1.1) **has a solution, then the iterative process** (2.5) **converges to a solution to equation** (1.1)**.**

Let us prove that the iterative process (2.5) *allows one to construct a stable approximation to the solution of equation* (1.1). Suppose that h_δ is given in place of h, and equation (2.2) is not necessarily solvable, but $\|h - h_\delta\| < \delta$. One has:

$$\|m - m_{n,\delta} \le \|m - m_n\| + \|m_n - m_{n,\delta}\|. \tag{2.12}$$

We have proved that

$$\lim_{n\to\infty} \|m - m_n\| = 0. \tag{2.13}$$

Let us choose $\delta(n)$ so that

$$\lim_{n\to\infty} [n\delta(n)] = 0. \tag{2.14}$$

If (2.14) holds, then

$$\lim_{n\to\infty} \|m_n - m_{n,\delta}\| = 0. \tag{2.15}$$

Indeed,

$$\|m_n - m_{n,\delta}\| \le \|(I-B)(m_{n-1} - m_{n-1,\delta})\| + \delta \le \|T\|^n \delta + n\delta \le (n+1)\delta \to 0 \text{ if } n \to \infty, \tag{2.16}$$

where we have used the inequality $\|I - B\| \le 1$, that we have established above.

An iterative method for solving linear Fredholm equations of the first kind was published earlier in [1]. This method for solving equation (1.3) is of the form

$$m_{n+1} = m_n + \lambda(h - Bm_n), \quad m_0 \in L^2(D), \ 0 < \lambda\lambda_1 < 1, \tag{2.17}$$

where λ_1 is the largest eigenvalue of the self-adjoint compact operator B. Let $m_n = m + u_n$, where m is a solution to equation (1.3), which is assumed to exist. Let us prove that $\|u_n\| \to 0$ as $n \to \infty$. From (2.17) one derives

$$u_{n+1} = u_n - \lambda B u_n. \tag{2.18}$$

Let $B\phi_j = \lambda_j \phi_j$. The set of ϕ_j is complete in $L^2(D)$ since B is self-adjoint. Let $b_{n,j};= (u_n, \phi_j)$. Equation (2.18) implies:

$$b_{n+1,j} = b_{n,j}(1 - \lambda\lambda_j) = b_{0,j}(1 - \lambda\lambda_1)^n, \tag{2.19}$$

where we have used the inequalities $0 \leq \lambda\lambda - j < \lambda\lambda_1 < 1$. One has $\sum_{j=1}^{\infty} |b_{0,j}|^2 = \|u_0\|^2 < c$, where $c = const$. Also, $\sum_{j=1}^{\infty} |b_{n,j}|^2 \leq \sum_{j=1}^{\infty} |b_{0,j}|^2(1 - \lambda\lambda_1)^n \leq c$ For an arbitrary small $\epsilon > 0$ choose k such that $\sum_{j=k}^{\infty} |b_{n,j}|^2 < \epsilon$. For a fixed k, choose n sufficiently large so that $\sum_{j=1}^{k} |b_{n,j}|^2 < \epsilon$. This is possible since $(1 - \lambda\lambda_1) < 1$. As a result, one gets $\|u_n\|^2 < 2\epsilon$ if n is sufficiently large. Since $\epsilon > 0$ was arbitrary small, one gets $\|u_n\| \to 0$ as $n \to \infty$. We have proved

Theorem 2. Iterative process (2.17) **converges to a solution of equation** (1.3)**, if this equation has a solution.**

Several other papers were published in Ann. of Communications in Math., [4], [5], [6].

3. Conclusion

A method is given for the calculation of the solution to equation (1.1).

4. Conflict of interest

There is no conflict of interest.

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