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REVISITING AN EXISTING INTEGRAL INEQUALITY

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ABSTRACT. In this article, we revisit a well-known result from the literature that can be considered a variant of the Hardy integral inequality. First, we present a counterexample to demonstrate the invalidity of the current formulation. We then revise the result by identifying and addressing a gap in the original proof. Finally, as an additional contribution, we derive a new integral inequality.

1. INTRODUCTION

Integral inequalities provide bounds on integrals, making them among the most valuable tools in mathematical analysis. In particular, they are essential for controlling error terms in key approximation results. Their usefulness extends across numerous disciplines, including physics, engineering, and economics. The fundamentals of integral inequalities can be found in [6, 2, 10, 1, 12].

An important result on this subject is attributed to Godfrey Harold Hardy (see [5]). It can be expressed as follows: For any $p > 1$ and any function $f : (0, +\infty) \mapsto (0, +\infty)$, by setting

$$F(x) := \int_0^x f(t)dt,$$

we have

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(x)dx, \tag{1.1}$$

provided that the integrals exist. In this general framework, the constant in the factor cannot be improved; it is optimal. This inequality is known as the Hardy integral inequality. Norman Levinson extended it in [7] to a more general integration domain as follows: For any $(a, b) \in (0, +\infty)^2 \cup \{+\infty\}^2$ with $a < b$ and any function $f : (a, b) \mapsto (0, +\infty)$, by setting

$$F(x) := \int_a^x f(t)dt,$$

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we have

$$\int_a^b \frac{1}{x^p} F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx, \tag{1.2}$$

provided that the integrals exist. This inequality is known as the Levinson integral inequality. In recent decades, various parameter and functional schemes have been used to extend the Hardy and Levinson integral inequalities, leading to numerous variants. See the results in [9, 8, 11, 3, 4].

In particular, nine theorems are proved in [9]. They are notable for their attempt to combine standard, monotonicity, submultiplicativity, and convexity properties in order to derive different forms of the Hardy integral inequality. In terms of bibliometrics, the reference [9] has been cited over thirty-seven times, demonstrating its significant impact and level of interest.

This article builds upon the referenced work. More precisely, it revisits a specific theorem established in [9], namely [9, Theorem 2.6]. First, we present an example that challenges the mathematical validity of the theorem. We then propose a correction to the proof. This theoretical gap is formalized by an additional integral term that is missing from the upper bound described in [9, Theorem 2.6]. Finally, we provide a new alternative result based on a similar approach. This result employs the Levinson integral inequality as an intermediate step.

The rest of the article is organized as follows: Section 2 recalls [9, Theorem 2.6] and presents a counterexample. Section 3 is devoted to a correction of this result. The new integral inequality is established in Section 4. The article concludes in Section 5.

2. A counterexample to an existing theorem

The theorem below is equivalent to [9, Theorem 2.6, page 522], with only minor changes to the presentation.

Theorem 2.1. [9, Theorem 2.6]. *Let $p \geq 2$, $f : (0, +\infty) \mapsto (0, +\infty)$ be a function and*

$$F(x) := \int_0^x f(t)dt,$$

provided that the integral exists.

Then the following inequality holds:

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx \leq \int_0^{+\infty} \frac{1}{x} f^{p-1}(x)F(x)dx,$$

still provided that the integrals exist.

The proof of this theorem can be found in [9, pages 522 and 523]. It is based on a suitable integral decomposition, multiple exchange of the order of integration, and a well-parameterized Hölder integral inequality. In particular, for the case $p = 2$, this inequality becomes

$$\int_0^{+\infty} \frac{1}{x^2} F^2(x)dx \leq \int_0^{+\infty} \frac{1}{x} f(x)F(x)dx.$$

The proposition below gives an elegant counterexample to this inequality (for the case $p = 2$).

Proposition 2.2. *We consider the framework of Theorem 2.1 with* $p = 2$ *and, for any* $t \in (0, +\infty)$,

$$f(t) := \frac{1}{t^2} e^{-1/t}.$$

Then we have

$$\int_0^{+\infty} \frac{1}{x^2} F^2(x) dx > \int_0^{+\infty} \frac{1}{x} f(x) F(x) dx.$$

This contradicts the inequality in Theorem 2.1.

Proof. The proof consists in finding all the integrals involved and comparing the ones given. First, we determine F as follows:

$$F(x) = \int_0^x f(t) dt = \int_0^x \frac{1}{t^2} e^{-1/t} dt = \left[e^{-1/t} \right]_{t \to 0}^{t = x} = e^{-1/x}.$$

We therefore have

$$\int_0^{+\infty} \frac{1}{x^2} F^2(x) dx = \int_0^{+\infty} \frac{1}{x^2} e^{-2/x} dx = \left[\frac{1}{2} e^{-2/x} \right]_{x \to 0}^{x \to +\infty} = \frac{1}{2}.$$

On the other hand, by an integration by parts, we obtain

$$\int_0^{+\infty} \frac{1}{x} f(x) F(x) dx = \int_0^{+\infty} \frac{1}{x} \times \frac{1}{x^2} e^{-1/x} \times e^{-1/x} dx = \int_0^{+\infty} \frac{1}{x^3} e^{-2/x} dx$$
$$= \left[\frac{1}{2x} e^{-2/x} \right]_{x \to 0}^{x \to +\infty} + \frac{1}{2} \int_0^{+\infty} \frac{1}{x^2} e^{-2/x} dx = 0 + \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

We conclude that

$$\int_0^{+\infty} \frac{1}{x^2} F^2(x) dx = \frac{1}{2} > \frac{1}{4} = \int_0^{+\infty} \frac{1}{x} f(x) F(x) dx,$$

which indeed contradicts Theorem 2.1 for $p = 2$. This ends the proof. □

Proposition 2.2 thus reveals a gap in the proof of [9, Theorem 2.6], at least for the case $p = 2$. In fact, this gap goes beyond the specific case of $p = 2$. Let us illustrate this claim with a short analysis. Considering the function f in Proposition 2.2, i.e., $f(t) = (1/t^2) e^{-1/t}$ for any $t \in (0, +\infty)$, we set

$$h(p) := \int_0^{+\infty} \frac{1}{x^p} F^p(x) dx - \int_0^{+\infty} \frac{1}{x} f^{p-1}(x) F(x) dx.$$

If we make the change of variables $y = p/x$ and leave out the details, we can see that

$$h(p) = p^{1-2p} \left[p^p \Gamma(p-1) - p \Gamma(2p-2) \right], \tag{2.1}$$

where $\Gamma(x)$ is the standard gamma function, i.e., $\Gamma(x) := \int_0^{+\infty} t^{x-1} e^{-t} dt$, for any $x \in (0, +\infty)$. To examine the sign of h, we perform a graphical analysis in Figure 1. More precisely, we plot the curve of $h(p)$ for $p \in [2, 12]$.

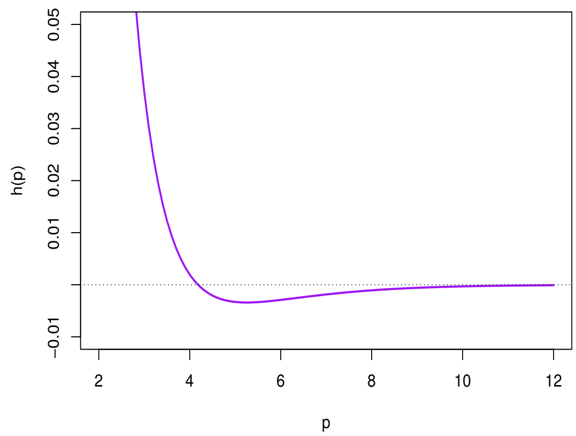

FIGURE 1. Curve of the function $h(p)$ given by Equation (2.1) for $p \in [2, 12]$

The purple curve shows that $h(p) > 0$ for $p \in [2, 4.18]$, contradicting [9, Theorem 2.6] for this range of values. This confirms that the gap in [9, Theorem 2.6] is deeper than just for the single value $p = 2$.

3. REVISION OF THE THEOREM

In the theorem below, we revisit [9, Theorem 2.6], showing that a term is missing in the upper bound.

Theorem 3.1. *Let $p \geq 2$, $f : (0, +\infty) \mapsto (0, +\infty)$ be a function,*

$$F(x) := \int_0^x f(t)dt$$

and

$$G(x; p) := \int_0^x f^{p-1}(t)dt,$$

provided that the integrals exist.

Then the following inequality holds:

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx \leq \int_0^{+\infty} \frac{1}{x} f(x)G(x; p)dx + \int_0^{+\infty} \frac{1}{x} f^{p-1}(x)F(x)dx,$$

still provided that the integrals exist. The inequality is an equality for the case $p = 2$, and we get

$$\int_0^{+\infty} \frac{1}{x^2} F^2(x)dx = 2\int_0^{+\infty} \frac{1}{x} f(x)F(x)dx.$$

Proof. Using the definition of F and changing the order of integration (which is possible because of the positivity of the functions involved, according to the Fubini-Tonelli integral theorem), we have

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx = \int_0^{+\infty} \frac{1}{x^p} F^{p-1}(x)F(x)dx \\
&= \int_0^{+\infty} \frac{1}{x^p} F^{p-1}(x) \int_0^x f(t)dtdx = \int_0^{+\infty} \int_0^x \frac{1}{x^p} F^{p-1}(x)f(t)dtdx \\
&= \int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^p} F^{p-1}(x)f(t)dxdt \\
&= \int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^p} \left[\int_0^x f(u)du\right]^{p-1} f(t)dxdt. \qquad (3.1)
\end{aligned}$$

For the case $p > 2$, applying the Hölder integral inequality with the parameter $p-1 > 1$ and again a change of the order of integration, we get

$$\begin{aligned}
&\int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^p} \left[\int_0^x f(u)du\right]^{p-1} f(t)dxdt \\
&\leq \int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^p} \left[\int_0^x f^{p-1}(u)du\right] \left(\int_0^x du\right)^{p-2} f(t)dxdt \\
&= \int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^2} \left[\int_0^x f^{p-1}(u)du\right] f(t)dxdt \\
&= \int_0^{+\infty} \int_t^{+\infty} \int_0^x \frac{1}{x^2} f^{p-1}(u)f(t)dudxdt \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_{\max(u,t)}^{+\infty} \frac{1}{x^2} f^{p-1}(u)f(t)dxdudt \\
&= \int_0^{+\infty} \int_0^{+\infty} \left[\int_{\max(u,t)}^{+\infty} \frac{1}{x^2} dx\right] f^{p-1}(u)f(t)dudt \\
&= \int_0^{+\infty} \int_0^{+\infty} \frac{1}{\max(u,t)} f^{p-1}(u)f(t)dudt. \qquad (3.2)
\end{aligned}$$

Note that, for the case $p = 2$, the inequality in the second line becomes an equality (there is no need to use the Hölder integral inequality).

In [9, Proof of Theorem 2.6], the term $\max(u,t)$ was erroneously reduced to u, with $u \in (t, +\infty)$, which explains the gap noticed in Proposition 2.2. Taking this maximum term into account, we continue the proof by applying the Chasles relation, changing the order of integration again and using the definition of G, as follows:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\max(u,t)}f^{p-1}(u)f(t)dudt\\
&=\int_0^{+\infty}\int_0^{t}\frac{1}{t}f^{p-1}(u)f(t)dudt+\int_0^{+\infty}\int_t^{+\infty}\frac{1}{u}f^{p-1}(u)f(t)dudt\\
&=\int_0^{+\infty}\frac{1}{t}f(t)\left[\int_0^{t}f^{p-1}(u)du\right]dt+\int_0^{+\infty}\int_0^{u}\frac{1}{u}f^{p-1}(u)f(t)dtdu\\
&=\int_0^{+\infty}\frac{1}{t}f(t)G(t;p)dt+\int_0^{+\infty}\frac{1}{u}f^{p-1}(u)\left[\int_0^{u}f(t)dt\right]du\\
&=\int_0^{+\infty}\frac{1}{t}f(t)G(t;p)dt+\int_0^{+\infty}\frac{1}{u}f^{p-1}(u)F(u)du. \qquad (3.3)
\end{aligned}$$

Putting Equations (3.1), (3.2) and (3.3) together, and uniformizing the notation, we get

$$\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\le\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx+\int_0^{+\infty}\frac{1}{x}f^{p-1}(x)F(x)dx.$$

For the case $p=2$, as indicated earlier, this inequality becomes an equality, and we have

$$\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx=\int_0^{+\infty}\frac{1}{x}f^{p-1}(x)F(x)dx=\int_0^{+\infty}\frac{1}{x}f(x)F(x)dx.$$

We therefore have

$$\int_0^{+\infty}\frac{1}{x^2}F^2(x)dx=2\int_0^{+\infty}\frac{1}{x}f(x)F(x)dx.$$

This concludes the proof.

To reinforce our findings, an alternative proof for the case $p>2$ is provided below. This proof is of mathematical interest because it uses several intermediate results, namely [6, Item 9.8.7, page 242], the Hölder integral inequality and the Young inequality for products.

Alternative proof for the case $p>2$. Based on [6, Item 9.8.7, page 242], we obtain

$$\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\le\frac{p}{p-1}\int_0^{+\infty}\frac{1}{x^{p-1}}F^{p-1}(x)f(x)dx.$$

This can be rewritten as follows:

$$\begin{aligned}
&\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\\
&\le\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x^{p-1}}F^{p-1}(x)f(x)dx+\int_0^{+\infty}\frac{1}{x^{p-1}}F^{p-1}(x)f(x)dx. \qquad (3.4)
\end{aligned}$$

Let us majorize the two main terms of this upper bound in turn. Applying the Hölder integral inequality with the parameter $p-1>1$, we get

$$F^{p-1}(x)=\left[\int_0^x f(u)du\right]^{p-1}\le\left[\int_0^x f^{p-1}(u)du\right]\left(\int_0^x du\right)^{p-2}=G(x;p)x^{p-2}.$$

The first main term can thus be majorized as follows:

$$\begin{aligned}
&\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x^{p-1}}F^{p-1}(x)f(x)dx\le\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x^{p-1}}G(x;p)x^{p-2}f(x)dx\\
&=\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx. \qquad (3.5)
\end{aligned}$$

For the second main term, a different strategy is used. The following product decomposition holds:

$$\frac{1}{x^{p-1}}F^{p-1}(x)f(x) = \left[\frac{1}{x}f^{p-1}(x)F(x)\right]^{1/(p-1)}\left[\frac{1}{x^p}F^p(x)\right]^{1-1/(p-1)}.$$

The Young inequality for products applied with the parameter $p-1>1$ gives

$$\begin{aligned}&\left[\frac{1}{x}f^{p-1}(x)F(x)\right]^{1/(p-1)}\left[\frac{1}{x^p}F^p(x)\right]^{1-1/(p-1)}\\&\leq \frac{1}{p-1}\left[\frac{1}{x}f^{p-1}(x)F(x)\right]+\left(1-\frac{1}{p-1}\right)\left[\frac{1}{x^p}F^p(x)\right].\end{aligned}$$

Combining the two equations above and integrating both sides, the second main term can be majorized as follows:

$$\begin{aligned}&\int_0^{+\infty}\frac{1}{x^{p-1}}F^{p-1}(x)f(x)dx\\&\leq \frac{1}{p-1}\int_0^{+\infty}\frac{1}{x}f^{p-1}(x)F(x)dx+\left(1-\frac{1}{p-1}\right)\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx.\end{aligned}\tag{3.6}$$

Putting Equations (3.4), (3.5) and (3.6) together, we obtain

$$\begin{aligned}&\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\\&\leq \frac{1}{p-1}\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx+\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x}f^{p-1}(x)F(x)dx\\&+\left(1-\frac{1}{p-1}\right)\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\\&\leq \frac{1}{p-1}\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx+\frac{1}{p-1}\int_0^{+\infty}\frac{1}{x}f^{p-1}(x)F(x)dx,\end{aligned}$$

so

$$\int_0^{+\infty}\frac{1}{x^p}F^p(x)dx\leq\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx+\int_0^{+\infty}\frac{1}{x}f^{p-1}(x)F(x)dx.$$

The desired inequality is established, ending this alternative proof. □

Compared to [9, Theorem 2.6], Theorem 3.1 considers the following additional term in the upper bound:

$$M(p):=\int_0^{+\infty}\frac{1}{x}f(x)G(x;p)dx.$$

This addition therefore fills a theoretical gap.

Considering the function f in Proposition 2.2, i.e., $f(t)=(1/t^2)e^{-1/t}$ for any $t\in(0,+\infty)$, after some integral developments, omitting the details, we get

$$M(p)=\frac{1}{2(2p-3)}\left[(p-1)^{2(1-p)}-3p^{2(1-p)}\right]\Gamma(2p-1).\tag{3.7}$$

For the case $p = 2$ as in Proposition 2.2, we have $M(2) = 1/4$ and it is no longer a counterexample of Theorem 3.1, since we have

$$\int_0^{+\infty} \frac{1}{x^2} F^2(x)dx = \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = M(2) + \int_0^{+\infty} \frac{1}{x} f(x)F(x)dx.$$

The former missing term $M(2)$ is as important as the other term in the upper bound in this particular example. This illustrates its determinant role in the inequality obtained.

In order to analyze more deeply this example, with a general value for p, let us consider the following function:

$$j(p) := \int_0^{+\infty} \frac{1}{x} f(x)G(x;p)dx + \int_0^{+\infty} \frac{1}{x} f^{p-1}(x)F(x)dx - \int_0^{+\infty} \frac{1}{x^p} F^p(x)dx.$$

Then, based on Equations (2.1) and (3.7), we can express it as

$$\begin{aligned} j(p) &= M(p) - h(p) \\ &= \frac{1}{2(2p-3)} \left[(p-1)^{2(1-p)} - 3p^{2(1-p)} \right] \Gamma(2p-1) \\ &\quad - p^{1-2p} \left[p^p \Gamma(p-1) - p\Gamma(2p-2) \right]. \end{aligned} \tag{3.8}$$

Theorem 3.1 ensures that, for any $p \geq 2$, we have $j(p) \geq 0$. This is illustrated in Figures 2 and 3 with the curve of $j(p)$ for $p \in [2, 13]$ for small values and $p \in [2, 50]$ for a visual of large values of p, respectively.

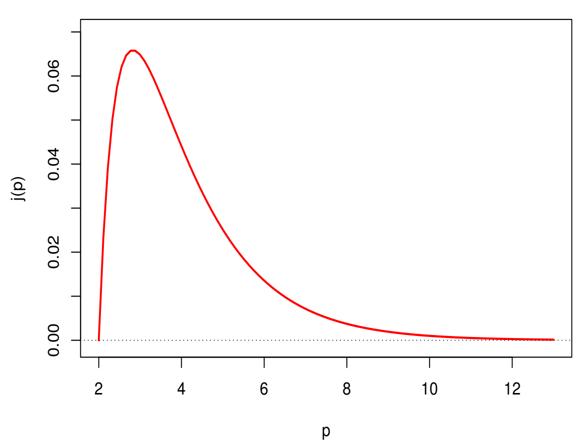

FIGURE 2. Curve of the function $j(p)$ given by Equation (3.8) for $p \in [2, 13]$

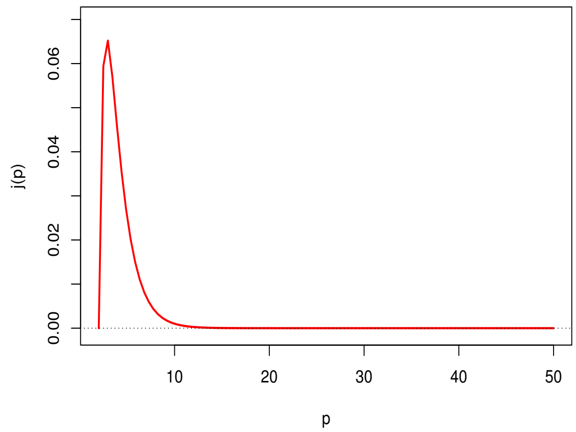

FIGURE 3. Curve of the function $j(p)$ given by Equation (3.8) for $p \in [2, 50]$

In any case, the red curve shows that, as expected, $j(p) \geq 0$ for $p \geq 2$. We also note that the inequality is sharp, with approximately 0.065 as the maximum error amplitude. We have thus proved the following inequality on the gamma function, which may be of independent interest: For any $p \geq 2$ we have $j(p) \geq 0$, i.e.,

$$\frac{1}{2(2p-3)}\left[(p-1)^{2(1-p)} - 3p^{2(1-p)}\right]\Gamma(2p-1) - p^{1-2p}\left[p^p\Gamma(p-1) - p\Gamma(2p-2)\right] \geq 0.$$

We leave it to future work to investigate this further.

4. A NEW INTEGRAL INEQUALITY

The result below gives an upper bound for $\int_0^{+\infty}(1/x^p)F^p(x)dx$ in a similar way to Theorem 3.1, considering only one integral term close to that suggested in [9, Theorem 2.6], but with a constant factor different from 1. It is also only valid for $p > 2$ (the case $p = 2$ is excluded).

Theorem 4.1. *Let $p > 2$, $f : (0, +\infty) \mapsto (0, +\infty)$ be a function,*

$$F(x) := \int_0^x f(t)dt$$

and

$$H(x) := \int_0^x \frac{1}{t} f(t)dt,$$

provided that the integrals exist.

Then we have

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx \leq \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{+\infty} f^{p-1}(x)H(x)dx,$$

still provided that the integrals exist.

Proof. Using the definition of F and changing the order of integration, we have

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx = \int_0^{+\infty} \frac{1}{x^p} F^{p-1}(x)F(x)dx$$
$$= \int_0^{+\infty} \frac{1}{x^p} F^{p-1}(x) \int_0^x f(t)dtdx = \int_0^{+\infty} \int_0^x \frac{1}{x^p} F^{p-1}(x)f(t)dtdx$$
$$= \int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^p} F^{p-1}(x)f(t)dxdt. \tag{4.1}$$

By the simple bound $1/x \leq 1/t$ for $x \in (t, +\infty)$, we get

$$\int_0^{+\infty} \int_t^{+\infty} \frac{1}{x^p} F^{p-1}(x)f(t)dxdt \leq \int_0^{+\infty} \left[\int_t^{+\infty} \frac{1}{x^{p-1}} F^{p-1}(x)dx\right] \frac{1}{t} f(t)dt. \tag{4.2}$$

Now, by applying the Levinson integral inequality on the interval $(t, +\infty)$ (see Equation (1.2) with $a = t$ and $b \to +\infty$) with the parameter $p - 1 > 1$, doing an exchange of order of integration and using the definition of H, we obtain

$$\int_0^{+\infty} \left[\int_t^{+\infty} \frac{1}{x^{p-1}} F^{p-1}(x)dx\right] \frac{1}{t} f(t)dt$$
$$\leq \int_0^{+\infty} \left[\left(\frac{p-1}{p-2}\right)^{p-1} \int_t^{+\infty} f^{p-1}(x)dx\right] \frac{1}{t} f(t)dt$$
$$= \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{+\infty} \int_t^{+\infty} f^{p-1}(x)\frac{1}{t} f(t)dxdt$$
$$= \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{+\infty} \int_0^x f^{p-1}(x)\frac{1}{t} f(t)dtdx$$
$$= \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{+\infty} f^{p-1}(x) \left[\int_0^x \frac{1}{t} f(t)dt\right] dx$$
$$= \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{+\infty} f^{p-1}(x)H(x)dx. \tag{4.3}$$

Putting Equations (4.1), (4.2) and (4.3) together, we have

$$\int_0^{+\infty} \frac{1}{x^p} F^p(x)dx \leq \left(\frac{p-1}{p-2}\right)^{p-1} \int_0^{+\infty} f^{p-1}(x)H(x)dx.$$

This concludes the proof. □

To the best of our knowledge, this integral inequality is a novel addition to the literature. In a sense, it supplements the theorems in [9], which are similar in nature.

5. Conclusion

The reference [9] contains many important and influential theorems in the field of integral inequalities. Most of these are valuable variants of the well-known Hardy integral inequality. However, in this article, we have identified a gap in one of these theorems (i.e., [9, Theorem 2.6]), and found a counterexample to the current formulation. Following analysis, we revised the statement and provided a correct proof, thereby closing the gap. Additionally, in the spirit of [9], a new integral inequality has been established.

Possible avenues for future research include generalizing these results by introducing a tuning parameter or an intermediate function, or extending them to the multidimensional case. Such generalizations would make them more adaptable to a broader range of integral analyses across various applications.

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