ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 2 (2025), 303-316
DOI: 10.62072/acm.2025.080213
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

SOME RESULTS ON BICOMPLEX NUMBERS AND BICOMPLEX FUNCTIONS

MD. NASIRUZZAMAN*, MOHAMMAD AYMAN-MURSALEEN AND NADEEM RAO

ABSTRACT. In this paper, we obtain some results on infinite products in bi-complex space, and the exact order of simultaneous results with quantitative estimate for the bi-complex gamma functions and bi-complex Beta functions. Finally using the specific results of complex gamma operator and complex beta operator we introduce the bi-complex gamma functions and bi-complex beta operators and obtain some results.

1. INTRODUCTION

There exist several ways to generalize complex numbers to higher dimensions. The most well-known extension is given by the quaternions invented by Hamilton [4] which are mainly used to represent rotations in three-dimensional space. However, quaternions are not commutative in multiplication. Another extension was found at the end of the 19^{th} century by Corrado Segre who described special multidimensional algebras and he named their elements n-complex numbers [14]. These type of numbers are now commonly named multicomplex numbers which were studied in details by Price [12] and Fleury [3]. In the recent years there are many investigation on bi-complex modules are given by [9, 10, 11] and references their in. bi-complex number, just like the quaternion, is a generalization of complex number to four real dimensions introduced by Segre [14]. These two number systems differ because:

(i) Quaternions which form a division algebra, while bi-complex numbers do not, and
(ii) bi-complex numbers are commutative, whereas quaternions are not.

For such reasons, the bi-complex number system has been shown to be more attractive (compared to the quaternions).

$$f(z_0+hi_1+hi_2) := f(z_0)+h(i_1+i_2)f'(z_0)+h^2(i_1+i_2)^2\frac{f''(z_0)}{2!}+\cdots+h^n(i_1+i_2)^n\frac{f^{(n)}(z_0)}{n!}+O(h^{(n+1)}).$$

2020 *Mathematics Subject Classification.* 16D10, 30G35, 46C05, 46C50.

Key words and phrases. Bi-complex number; bi-complex holomorphic function; infinite product on bi-complex space; bi-complex Euler's gamma function; bi-complex Euler's beta operator.

Received: May 28, 2025. Accepted: June 21, 2025. Published: June 30, 2025.

*Corresponding author.

We describe how to define elementary functions such as algebra of series of a function on bi-complex space and Taylor series representation in case of holomorphic functions.

The aim of the present article is to obtain the results on infinite products of series of a function on bi-complex space of holomorphic functions; and obtain some results for bi-complex gamma functions and bi-complex beta operators in same space $\mathbb{C}^2$. In case of Gamma functions we define $\Gamma(z), z \in \mathbb{C}^2$, in terms of Eulerian integral as

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\, dz,\ Re(z) > 0.$$

The bi-complex Euler's Beta function $\mathcal{B}(\mu, \nu)$ is defined as

$$\mathcal{B}(\mu,\nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\, dx\ \mu, \nu \in \mathbb{C}^2, Re(\mu), Re(\nu) > 0,$$

so for all $n \in \mathbb{N}$ and $z \in \mathbb{C}^2$ satisfying $0 < Re(z) < 1$, and
the bi-complex Beta operator is defined as

$$\mathcal{C}_n(f,z) = \frac{1}{\mathcal{B}(nz, n(1-z))} \int_0^1 x^{nz-1}(1-x)^{n(1-z)-1} f(x)\, dx.$$

2. Preliminaries

2.1. **Bi-complex numbers.** We start with the following basic definition of bi-complex numbers.

Definition 2.1 (cf. [2], [1]). The set of the bi-complex numbers is defined as

$$\mathbb{C}^2 := \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}^1(i_1)\} \tag{2.1}$$

where i_1, i_2 are the imaginary units and governed by the rules

$$i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j \tag{2.2}$$

and so,

$$j^2 = 1, i_1 j = j i_1 = -i_2, i_2 j = j i_2 = -i_1 \tag{2.3}$$

Note that we define

$$\mathbb{C}^1(i_k) := \{x + y i_k \mid i_k^2 = -1 \text{ and } x, y \in \mathbb{R} \text{ for } k = 1, 2\} \tag{2.4}$$

where $\mathbb{C}^1$ is the set of complex numbers with the imaginary units i_k for $k = 1, 2$. Thus the bi-complex numbers are complex numbers with complex coefficients, which explain the name of bi-complex, and there is a deep similarities in properties of complex and bi-complex numbers.
With the addition and the multiplication of two bi-complex numbers defined in the obvious way, the set $\mathbb{C}^2$ makes up a commutative ring. In fact they are the particular case of the so called multicomplex numbers (denoted by $\mathbb{MC}$).
Clearly the bi-complex number.

$$\mathbb{C}^2 \cong \mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1) \tag{2.5}$$

are unique among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bi-complex numbers as

$$\mathbb{C}^2 := \{x_0 + x_1 i_1 + x_2 i_2 + x_3 i_1 i_2 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}. \tag{2.6}$$

We know the complex conjugation plays an important role for both algebraic and geometric properties of $\mathbb{C}^1$. So for bi-complex numbers there are three possibilities of conjugations. Let $z \in \mathbb{C}^2$ and $z_1, z_2 \in \mathbb{C}^1(i_1)$, such that $z := z_1 + z_2 i_2$, then we define the three conjugation as:

$$z^{\dagger_1} = (z_1 + z_2 i_2)^{\dagger_1} = \overline{z}_1 + \overline{z}_2 i_2 \tag{2.7}$$
$$z^{\dagger_2} = (z_1 + z_2 i_2)^{\dagger_2} = z_1 - z_2 i_2 \tag{2.8}$$
$$z^{\dagger_3} = (z_1 + z_2 i_2)^{\dagger_3} = \overline{z}_1 - \overline{z}_2 i_2. \tag{2.9}$$

These three kinds of conjugation have some of the standard properties of conjugations, such as

$$(z_1 + z_2)^{\dagger_k} = z_1^{\dagger_k} + z_2^{\dagger_k} \tag{2.10}$$
$$(z_1^{\dagger_k})^{\dagger_k} = z_1 \tag{2.11}$$
$$(z_1.z_2)^{\dagger_k} = z_1^{\dagger_k}.z_2^{\dagger_k}. \tag{2.12}$$

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. Thus the analogues of this, for bi-complex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}^1(i_1)$ and $z := z_1 + z_2 i_2 \in \mathbb{C}^2$, then we have:

$$\mid z \mid_{i_1}^2 = z.z^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}^1(i_1) \tag{2.13}$$
$$\mid z \mid_{i_2}^2 = z.z^{\dagger_1} = (\mid z_1 \mid^2 - \mid z_2 \mid^2) + 2Re(z_1\overline{z}_2)i_2 \in \mathbb{C}^1(i_2) \tag{2.14}$$
$$\mid z \mid_{j}^2 = z.z^{\dagger_3} = (\mid z_1 \mid^2 + \mid z_2 \mid^2) - 2Im(z_1\overline{z}_2)j \in \mathbb{D}, \tag{2.15}$$

where $\mathbb{D}$ is the subalgebra of hyperbolic numbers, and is defined as

$$\mathbb{D} := \{x + yj \mid j^2 = 1, x, y \in \mathbb{R}, \} \cong \mathrm{Cl}_{\mathbb{R}}(0, 1). \tag{2.16}$$

Note that for $z_1, z_2 \in \mathbb{C}^1(i_1)$ and $z := z_1 + z_2 i_2 \in \mathbb{C}^2$, we can define the usual (Euclidean in $\mathbb{R}^4$) norm of z as $\mid z \mid = \sqrt{\mid z_1 \mid^2 + \mid z_2 \mid^2} = \sqrt{Re(\mid z \mid_j^2)}$. It is easy to verifying that $z.\frac{z^{\dagger_2}}{|z|_{i_1}^2} = 1$. Hence the inverse of z is given by

$$z^{-1} = \frac{z^{\dagger_2}}{\mid z \mid_{i_1}^2}. \tag{2.17}$$

2.2. **Idempotent basis.** bi-complex algebra is considerably simplified by the introduction of two bi-complex numbers e_1 and e_2 defined as $e_1 = \frac{1+i_1 i_2}{2}$, $e_2 = \frac{1-i_1 i_2}{2}$ Infact e_1 and e_2 are hyperbolic numbers ($i_1 i_2 = i_2 i_1 = j$). They make up the so called idempotent basis of the bi-complex numbers, and one easily can check that

$$e_1^2 = e_1, e_2^2 = e_2, e_1 + e_2 = 1, e_1.e_2 = 0, e_k^{\dagger_3} = e_k \text{ (for } k = 1, 2). \tag{2.18}$$

Thus any bi-complex number can be written as

$$z = z_1 + z_2 i_2 = \alpha_1 e_1 + \alpha_2 e_2, \text{where } \alpha_1 = z_1 - z_2 i_1, \alpha_2 = z_1 + z_2 i_1. \tag{2.19}$$

Definition 2.2 (**cf. Definition 5** [13])**.** For any positive integer n and any $z \in \mathbb{C}^2$ we define

$$n^z = e^{z \ln(n)}. \tag{2.20}$$

As an immediate consequence of this definition it is clear that n^z is invertible for each $z \in \mathbb{C}^2$. Now we have

$$a^z = (exp)^{z \ln a} = (exp)^{(\alpha e_1 + \beta e_2) \ln a} = a^{\alpha} e_1 + a^{\beta} e_2. \tag{2.21}$$

Definition 2.3 (**cf.** [5])**.** Let A_1 and A_2 be complex spaces defined as $A_1 = \{z_1 - z_2 i_1 \mid z_1, z_2 \in \mathbb{C}^1\}$ and $A_2 = \{z_1 + z_2 i_1 \mid z_1, z_2 \in \mathbb{C}^1\}$. Then the Cartesian set determined by X_1 and X_2 in A_1 and A_2 respectively is defined as: $X_1 \times_e X_2 = z_1 + z_2 i_2 = \alpha_1 e_1 + \alpha_2 e_2$, where $\alpha_1 \in X_1, \alpha_2 \in X_2$. With the help of idempotent representation we define some functions : $h_1 : \mathbb{C}^2 \to A_1, h_2 : \mathbb{C}^2 \to A_2$ as follows

$$p_1(z_1+z_2i_2) = h_1[(z_1-z_2i_1)e_1+(z_1+z_2i_1)e_2] = (z_1-z_2i_1) \in A_1, \forall(z_1+z_2i_2) \in \mathbb{C}^2 \tag{2.22}$$

$$p_2(z_1+z_2i_2) = h_2[(z_1-z_2i_1)e_1+(z_1+z_2i_1)e_2] = (z_1+z_2i_1) \in A_2, \forall(z_1+z_2i_2) \in \mathbb{C}^2. \tag{2.23}$$

Theorem 2.1 (**cf. Theorem 2, cf.** [2]). *Let X_1 and X_2 be open sets in $\mathbb{C}^1(i_1)\}$. If $f_{e_1} : X_1 \to \mathbb{C}^1(i_1)$ and $f_{e_2} : X_2 \to \mathbb{C}^1(i_1)$ be holomorphic functions of $\mathbb{C}^1(i_1)$ on X_1 and X_2 respectively. Then the function $f : X_1 \times_e X_2 \to \mathbb{C}^2$ is defined as*

$$f(z_1+z_2i_2) = f_{e_1}(z_1-z_2i_1)e_1+f_{e_2}(z_1+z_2i_1)e_2, \forall z_1+z_2i_2 \in f : X_1 \times_e X_2 \tag{2.24}$$

which is $\mathbb{C}^2$ holomorphic on the open set $X_1 \times_e X_2$, and

$$f'(z_1+z_2i_2) = f'_{e_1}(z_1-z_2i_1)e_1+f'_{e_2}(z_1+z_2i_1)e_2, \forall z_1+z_2i_2 \in f : X_1 \times_e X_2. \tag{2.25}$$

Definition 2.4 (**cf. Definition 6** [7]). Let $z_n = \alpha_n e_1 + \beta_n e_2$ for $n \geq 1$. The sequence $\{z_n\}_{n\geq 1}$ is said to be convergent component-wise if the sequence of the complex numbers $\{\alpha_n\}$ and $\{\beta_n\}$ are convergent in the complex plane to complex numbers α_0 and β_0, respectively. In this case, we write $z_n \to z_0 = \alpha_0 e_1 + \beta_0 e_2$, and we say that z_n has limit z_0.

Theorem 2.2 (**cf. Theorem 12** [13]). *Let $\{z_n = z_{1,n} + z_{2,n}i_2\}_{n=1}^{\infty} \subseteq \mathbb{C}^2$ be invertible for each positive integer n. Then $\prod_{n=1}^{\infty} z_n$ converges if and only if both $(\prod_{n=1}^{\infty} z_{1,n} - z_{2,n}i_1)$ and $(\prod_{n=1}^{\infty} z_{1,n} + z_{2,n}i_1)$ converge. Further, when convergent, we obtain the identity.*

$$\prod_{n=1}^{\infty} z_n = (\prod_{n=1}^{\infty} z_{1,n} - z_{2,n}i_1)e_1 + (\prod_{n=1}^{\infty} z_{1,n} + z_{2,n}i_1)e_2.$$

2.3. **bi-complex holomorphic functions.** It is also possible to define differentiability of a function at a point of $\mathbb{C}^2$.

Definition 2.5 (**cf. Definition 1** [2]). Let U be an open set of $\mathbb{C}^2$ and $z_0 \in U$. Then, $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ is said to be $\mathbb{C}^2$- differentiable at z_0 with derivative equal to $f'(z_0) \in \mathbb{C}^2$ if

$$\lim_{\substack{z\to z_0 \\ z-z_0\ inv.}} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0). \tag{2.26}$$

We also say that the function f is bi-complex holomorphic ($\mathbb{C}^2$- holomorphic) on an open set U if and only if f is $\mathbb{C}^2$- differentiable at each point of U. Using $z = z_1 + z_2i_2$, a bi-complex number z can be seen as an element (z_1, z_2) of $\mathbb{C}^2$, so a function $f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$ of $\mathbb{C}^2$ can be seen as a mapping $f(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$ of $\mathbb{C}^2$.

Theorem 2.3 (**cf. Theorem 2** [6]). *Let $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ be a function, and $f(z_1 + z_2i_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$, where $z_1, z_2 \in \mathbb{C}^1$. Then the following are equivalent*

(i) *f is holomorphic in U.*

(ii) *f_1 and f_2 are holomorphic in z_1 and z_2 and satisfying the bi-complex Cauchy-Riemann equations:*

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \tag{2.27}$$

(iii) *f can be represented, near every point $z_0 \in U$, by a Taylor series.*

Theorem 2.4 (Weierstrass factorization theorem for complex number)**.** *Given an infinite sequences of complex numbers $a_0 = 0, a_1, \cdots, a_n, \cdots$ with no finite point of accumulation, the most general entire function having zeros at those points only (at zero $a_n, n \geq 1$, of multiplicity α being repeated α times in the sequence)is given by*

$$F(w) = e^{h(w)} w^m \prod_{n=1}^{\infty} E(\frac{w}{a_n}; k_n), \tag{2.28}$$

where $h(w)$ is an arbitrary entire function, $(m \geq 0)$, is the order of multiplicity $a_0 = 0$ and k_n are non negative integers such that the series

$$\sum_{n=1}^{\infty} \mid \frac{w}{a_n} \mid^{k_n+1} \tag{2.29}$$

converges for each finite value of w.

The Weierstrass factorization theorem for the complex numbers gives

$$F(w) = e^{h(w)} w \prod_{n=1}^{\infty} (1 + \frac{w}{n}) e^{-\frac{w}{n}} \tag{2.30}$$

as the most general entire function having the prescribed zeros. By taking $h(w) = \gamma w$ where the constant γ is chosen so that $F(1) = 1$, {as $w = 1$ }, a particular entire function $F_1(w)$ is given by

$$F_1(w) = \frac{1}{\Gamma(w)} = e^{\gamma w} w \prod_{n=1}^{\infty} (1 + \frac{w}{n}) e^{-\frac{w}{n}}. \tag{2.31}$$

We define $\Gamma(w)$ in terms of the Eulerian integral as

$$\Gamma(w) = \int_0^{\infty} \mathrm{e}^{-x} x^{w-1} \,\mathrm{d}x, \; Re(w) > 0. \tag{2.32}$$

Theorem 2.5 (cf. [5]**).** *Let $z = z_1 + z_2 i_2 = \alpha e_1 + \beta e_2, f(z) = f_1(\alpha) e_1 + f_2(\beta) e_2$. Then $f(z)$ is convergent in a domain D if and only if $f_1(\alpha)$ and $f_2(\beta)$ are convergent in domain $p_1 : D \to D_1$ and $p_2 : D \to D_2$ respectively, and we can write*

$$\int_D f(z) \mathrm{d}z = \int_{D_1} f(\alpha) \mathrm{d}\alpha e_1 + \int_{D_2} f(\beta) \mathrm{d}\beta e_2 \tag{2.33}$$

3. Convergence of sequence of bi-complex numbers

With the help of Theorem 2.4, we have the following result.

Theorem 3.1. *Let $z = z_1 + z_2 i_2 = \alpha e_1 + \beta e_2$ be any bi-complex number where as $\alpha = \{z_1 - z_2 i_1 \mid z_1, z_2 \in \mathbb{C}^1\}$ and $\beta = \{z_1 + z_2 i_1 \mid z_1, z_2 \in \mathbb{C}^1\}$, and let X_1 and X_2 be open sets in $\mathbb{C}^1(i_1)$. If $f_{e_1}, g_{e_1} : X_1 \to \mathbb{C}^1(i_1)$ and $f_{e_2}, g_{e_2} : X_2 \to \mathbb{C}^1(i_1)$ are holomorphic functions of $\mathbb{C}^1(i_1)\}$ on X_1 and X_2 respectively, then $\forall z_1 + z_2 i_2 \in \mathbb{C}^2$ we have $f, g : X_1 \times_e X_2 \to \mathbb{C}^2$ defined as*

$$f(z) = f_{e_1}(\alpha) e_1 + f_{e_2}(\beta) e_2$$
$$g(z) = g_{e_1}(\alpha) e_1 + g_{e_2}(\beta) e_2$$

are $\mathbb{C}^2$ holomorphic on the open set $X_1 \times_e X_2$ and

$$f'(z) = f'_{e_1}(\alpha) e_1 + f'_{e_2}(\beta) e_2$$
$$g'(z) = g'_{e_1}(\alpha) e_1 + g'_{e_2}(\beta) e_2.$$

Moreover we have the following

$$f(z) \pm g(z) = (f_{e_1}(\alpha) \pm g_{e_1}(\alpha))e_1 + (f_{e_2}(\beta) \pm g_{e_2}(\beta))e_2 \tag{3.1}$$

$$f'(z) \pm g'(z) = (f'_{e_1}(\alpha) \pm g'_{e_1}(\alpha))e_1 + (f'_{e_2}(\beta) \pm g'_{e_2}(\beta))e_2 \tag{3.2}$$

$$f(z).g(z) = f_{e_1}(\alpha)g_{e_1}(\alpha)e_1 + f_{e_2}(\beta)g_{e_2}(\beta)e_2 \tag{3.3}$$

$$\frac{f(z)}{g(z)} = \frac{f_{e_1}(\alpha)}{g_{e_1}(\alpha)}e_1 + \frac{f_{e_2}(\beta)}{g_{e_2}(\beta)}e_2. \tag{3.4}$$

Proof. Easy. □

If put $k = 2$ in Definition 3 of [6], we get the following.

Definition 3.1. Let $\mathbb{C}^2 := \{z = z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^1\}$ be a bi-complex number, and let $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ be a bi-complex holomorphic function in U. Then about a point z_0, the Taylor series can be expressed as :

$$f(z_0+hi_1+hi_2) := f(z_0)+h(i_1+i_2)f'(z_0)+h^2(i_1+i_2)^2\frac{f''(z_0)}{2!}+\cdots+h^n(i_1+i_2)^n\frac{f^{(n)}(z_0)}{n!}+O(h^{(n+1)}) \tag{3.5}$$

where f^n denotes the n^{th} order derivative, and

$$(i_1 + i_2)^k := \sum_{\substack{x_1,x_2 \\ x_1+x_2=k}} \frac{n!}{x_1!x_2!} i_1^{x_1} i_2^{x_2}. \tag{3.6}$$

Theorem 3.2. *Let $z = z_1 + z_2 i_2$ be any bi-complex number with $\mid z \mid< 1$ and $z = \{\alpha e_1 + \beta e_2 \mid \alpha, \beta \in \mathbb{C}^1\}$, where e_1, e_2 are idempotent basis. Then the series $\sum_{n=0}^{\infty} z^n$ is convergent to $\frac{1}{1-z}$, if $\sum_{n=0}^{\infty} \alpha^n$ converges to $\frac{1}{1-\alpha}$ and $\sum_{n=0}^{\infty} \beta^n$ converges to $\frac{1}{1-\beta}$, provided $\mid \alpha \mid, \mid \beta \mid< 1$.*

Proof. Simply from [8], we have $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ and $\sum_{n=0}^{\infty} \beta^n = \frac{1}{1-\beta}$.
Using Theorem 2.4 we have

$$\sum_{n=0}^{\infty} \alpha^n e_1 + \beta^n e_2 = \frac{1}{1-\alpha}e_1 + \frac{1}{1-\beta}e_2.$$

Hence

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

where if $\mid \alpha \mid, \mid \beta \mid< 1$, then we have $\mid z \mid< 1$. □

Corollary 3.3. *Let $z = z_1 + z_2 i_2$ be any bi-complex number with $\mid z \mid> 1$ and $\{z = \alpha e_1 + \beta e_2 \mid \alpha, \beta \in \mathbb{C}^1\}$, where e_1, e_2 are idempotent basis. Then the series $\sum_{n=1}^{\infty} z^{-n}$ is convergent to $\frac{1}{z-1}$, if $\sum_{n=1}^{\infty} \alpha^{-n}$ converges to $\frac{1}{\alpha-1}$ and $\sum_{n=1}^{\infty} \beta^{-n}$ converges to $\frac{1}{\beta-1}$, provided $\mid \alpha \mid, \mid \beta \mid> 1$.*

Proof. Simply $\sum_{n=1}^{\infty} \alpha^{-n} = \frac{1}{\alpha-1}$ and $\sum_{n=1}^{\infty} \beta^{-n} = \frac{1}{\beta-1}$ imply that

$$\sum_{n=1}^{\infty} \alpha^{-n} e_1 + \beta^{-n} e_2 = \frac{1}{\alpha-1}e_1 + \frac{1}{\beta-1}e_2.$$

Hence

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{z-1}.$$

□

Theorem 3.4. *Let $f : U \subset \mathbb{C}^2 \to \mathbb{C}^2$ be holomorphic in U, and $f(z) = \sum_{n=0}^{\infty} z^n$, with $\mathbb{C}^2 = \{z :\mid z \mid < 1\}$. Then at any point $z_0 \in \mathbb{C}^2$, we have*

$$f(z) = \frac{1}{1-z_0} + \frac{1}{(1-z_0)^2}(z-z_0) + \frac{1}{(1-z_0)^3}(z-z_0)^2 + \cdots + \frac{1}{(1-z_0)^{n+1}}(z-z_0)^n + O(n+1). \tag{3.7}$$

Proof. We have

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

which implies that

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}},\ n = 1, 2, \cdots.$$

Now put n^{th} derivative of f about a point z_0 in (3.5), we have

$$f(z_0+h(i_1+i_2)) = \frac{1}{1-z_0} + h(i_1+i_2)\frac{1}{(1-z_0)^2} + (h(i_1+i_2))^2\frac{1}{(1-z_0)^3} + \cdots + (h(i_1+i_2))^n\frac{1}{(1-z_0)^{n+1}} + O(h^{n+1}).$$

If $z_0 + h(i_1 + i_2) = z$, then we have

$$f(z) = \frac{1}{1-z_0} + (z-z_0)\frac{1}{(1-z_0)^2} + (z-z_0)^2\frac{1}{(1-z_0)^3} + \cdots + (z-z_0)^n\frac{1}{(1-z_0)^{n+1}} + O(n+1).$$

$\square$

Now we define the following:

Definition 3.2 (cf. Definition 3 [13]**).** For any sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}^2$, the infinite product $\prod_{n=1}^{\infty} a_n$ is said to be convergent provided only a finite number of terms in the sequence are non-invertible and the product formed by the invertible terms in the sequence tend to a finite limit.

Definition 3.3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in bi-complex space $\mathbb{C}^2$, where $a_n = \alpha_n e_1 + \beta_n e_2$, and $a_n \neq -1$ for all $n \in \mathbb{N}$. Then the infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ is said to be absolutely convergent iff the series $\sum_{n=1}^{\infty} \log(1 + a_n)$ is absolutely convergent, where $\sum_{n=1}^{\infty} \log(1 + a_n) = \sum_{n=1}^{\infty} \log(1 + \alpha_n)e_1 + \sum_{n=1}^{\infty} \log(1 + \beta_n)e_2$.

Theorem 3.5. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence in bi-complex space $\mathbb{C}^2$, where $z_n = \alpha_n e_1 + \beta_n e_2$, and $z_n \neq -1$ for all $n \in \mathbb{N}$. Then the infinite product $\prod_{n=1}^{\infty}(1 + z_n)$ is absolutely convergent iff the series $\sum_{n=1}^{\infty} \alpha_n$ and $\sum_{n=1}^{\infty} \beta_n$ are absolutely convergent, i.e., if $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.*

Proof. If $\mid \alpha \mid < 1, z \neq 0$, then we have

$$\log(1 + \alpha) = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} - \cdots. \tag{3.8}$$

Here

$$\mid \frac{\log(1+\alpha)}{\alpha} - 1 \mid \leq \frac{\mid \alpha \mid}{2} + \frac{\mid \alpha \mid^2}{3} + \frac{\mid \alpha \mid^3}{4} + \cdots \leq \frac{1}{2}\sum_{m=1}^{\infty} \mid \alpha \mid^m. \tag{3.9}$$

For all $n > \mathbb{N}$, such that $\alpha_n \neq 0$, let $\alpha = \alpha_n$ in (3.9), we obtain

$$\mid \frac{\log(1+\alpha_n)}{\alpha_n} - 1 \mid \leq \frac{1}{2}\sum_{m=1}^{\infty} \mid \alpha_n \mid^m \leq \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{2^m} = \frac{1}{2}.$$

So we have

$$\frac{1}{2} \leq \mid \frac{\log(1+\alpha_n)}{\alpha_n} \leq \frac{3}{2}$$

$$\frac{1}{2} \mid \alpha_n \mid \leq \mid \log(1+\alpha_n) \mid \leq \frac{3}{2} \mid \alpha_n \mid . \tag{3.10}$$

This inequality holds also for $\alpha_n = 0$. So that the equation (3.10) is valid for all $n > \mathbb{N}$. Hence,

$$\frac{1}{2} \sum_{n=N+1}^{\infty} \mid \alpha_n \mid \leq \sum_{n=N+1}^{\infty} \mid \log(1+\alpha_n) \mid \leq \frac{3}{2} \sum_{n=N+1}^{\infty} \mid \alpha_n \mid . \tag{3.11}$$

Similarly

$$\frac{1}{2} \sum_{n=N+1}^{\infty} \mid \beta_n \mid \leq \sum_{n=N+1}^{\infty} \mid \log(1+\beta_n) \mid \leq \frac{3}{2} \sum_{n=N+1}^{\infty} \mid \beta_n \mid . \tag{3.12}$$

Using Theorem 2.4 and equations (3.11) and (3.12), we have

$$\frac{1}{2} \sum_{n=N+1}^{\infty} (\mid \alpha_n \mid e_1 + \mid \beta_n \mid e_2) \leq \sum_{n=N+1}^{\infty} (\mid \log(1+\alpha_n) \mid e_1 + \mid \log(1+\beta_n) \mid e_2) \leq \frac{3}{2} \sum_{n=N+1}^{\infty} (\mid \alpha_n \mid e_1 + \mid \beta_n \mid e_2).$$

$$\frac{1}{2} \sum_{n=N+1}^{\infty} \mid z_n \mid \leq \sum_{n=N+1}^{\infty} \mid \log(1+z_n) \mid \leq \frac{3}{2} \sum_{n=N+1}^{\infty} \mid z_n \mid . \tag{3.13}$$

Therefore, if $\sum_{n=1}^{\infty} \mid \alpha_n \mid$and $\sum_{n=1}^{\infty} \mid \beta_n \mid$ converge then $\sum_{n=1}^{\infty} \mid \log(1+\alpha_n) \mid$and $\sum_{n=1}^{\infty} \mid \log(1+\beta_n) \mid$ converge. Hence if $\sum_{n=1}^{\infty} \mid z_n \mid$ converges then $\sum_{n=1}^{\infty} \mid \log(1+z_n) \mid$ converges and conversely. □

Example. Consider $\prod_{n=1}^{\infty}(1 + \frac{1}{(2+i_1+i_2+i_1i_2)^n})$.
We have

$$\sum_{n=1}^{\infty} \mid \frac{1}{(2+i_1+i_2+i_1i_2)^n} \mid = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{7})^n}.$$

Since the last series converges, the given product converges absolutely.

4. Euler's gamma function for bi-complex numbers

We prove the following theorems on Euler's Gamma functions for the bi-complex numbers.

Theorem 4.1. *Let $z = \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}^1(i_1)\} = \alpha e_1 + \beta e_2$. Then the Eulerian gamma integral for bi-complex number z is $\Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1}\, dz$, Re $(z) > 0$, if we have $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx$ and $\Gamma(\beta) = \int_0^{\infty} e^{-x} x^{\beta-1}\, dx$, provided Re $(\alpha) > 0$ and Re $(\beta) > 0$.*

Proof. $z_1 = x_1 + x_2 i_1, z_2 = x_2 + y_2 i_1$

$$\alpha = z_1 - z_2 i_1 = (x_1 + y_2) + (y_1 - x_2)i_1, \beta = z_1 + z_2 i_1 = (x_1 - y_2) + (y_1 + x_2)i_1$$

$$Re(\alpha) > 0, Re(\beta) > 0 \Rightarrow Re(z) > 0, \text{ where } z = x_1 + y_1 i_1 + x_2 i_2 + y_2 i_1 i_2$$

we have

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx,\ Re(\alpha) > 0$$

$$\Gamma(\beta) = \int_0^{\infty} e^{-x} x^{\beta-1}\, dx,\ Re(\beta) > 0$$

$$\Gamma(z) = \Gamma(\alpha)e_1 + \Gamma(\beta)e_2 \tag{4.1}$$

$$\begin{aligned}\Gamma(\alpha)e_1+\Gamma(\beta)e_2 &= \int_0^\infty e^{-x}x^{\alpha-1}\,dxe_1+\int_0^\infty e^{-x}x^{\beta-1}\,dxe_2\\ &= \int_0^\infty e^{-x}x^{(\alpha-1)e_1+(\beta-1)e_2}\,dx\\ &= \int_0^\infty e^{-x}x^{\alpha e_1+\beta e_2-1}\,dx\\ &= \int_0^\infty e^{-x}x^{z-1}\,dz=\Gamma(z),\ Re(z)>0.\end{aligned}$$

□

Theorem 4.2. *Let $z=z_1+z_2i_2=\alpha e_1+\beta e_2$. Then the Γ - function for bi-complex number z is*

$$\frac{1}{\Gamma(z)}=e^{\gamma z}z\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-\frac{z}{n}}. \tag{4.2}$$

If for any $\alpha,\beta\in\mathbb{C}^1(i_1)$ or $\mathbb{C}^1(i_2)$, we have

$$\frac{1}{\Gamma(\alpha)}=e^{\gamma\alpha}\alpha\prod_{n=1}^{\infty}(1+\frac{\alpha}{n})e^{-\frac{\alpha}{n}},\ \frac{1}{\Gamma(\beta)}=e^{\gamma\beta}\beta\prod_{n=1}^{\infty}(1+\frac{\beta}{n})e^{-\frac{\beta}{n}},$$

where the constant γ is Euler constant and is chosen so that $\Gamma(e_1+e_2)=e_1+e_2$.

Proof. We have

$$\begin{aligned}\Gamma(\alpha e_1+\beta e_2) &= \Gamma(\alpha)e_1+\Gamma(\beta)e_2\\ \frac{1}{\Gamma(\alpha e_1+\beta e_2)} &= \frac{1}{\Gamma(\alpha)}e_1+\frac{1}{\Gamma(\beta)}e_2\\ \frac{1}{\Gamma(\alpha)}e_1+\frac{1}{\Gamma(\beta)}e_2 &= e^{\gamma\alpha}\alpha\prod_{n=1}^{\infty}(1+\frac{\alpha}{n})e^{-\frac{\alpha}{n}}e_1+e^{\gamma\beta}\beta\prod_{n=1}^{\infty}(1+\frac{\beta}{n})e^{-\frac{\beta}{n}}e_2\\ &= (e^{\gamma\alpha}\alpha e_1+e^{\gamma\beta}\beta e_2)(\prod_{n=1}^{\infty}(1+\frac{\alpha}{n})e^{-\frac{\alpha}{n}}e_1+\prod_{n=1}^{\infty}(1+\frac{\beta}{n})e^{-\frac{\beta}{n}}e_2)\\ \Rightarrow\frac{1}{\Gamma(\alpha e_1+\beta e_2)} &= e^{\gamma(\alpha e_1+\beta e_2)}(\alpha e_1+\beta e_2)\prod_{n=1}^{\infty}(1+\frac{\alpha e_1+\beta e_2}{n})e^{-\frac{\alpha e_1+\beta e_2}{n}}\\ \Rightarrow\frac{1}{\Gamma(z)} &= e^{\gamma z}z\prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-\frac{z}{n}},\ \text{where } z\in\mathbb{C}^2.\end{aligned}$$

□

Lemma 4.3. *let $z_n=1+\frac{1}{2}+\cdots+\frac{1}{n}$. Then we have*

$$\lim_{n\to\infty}(z_n-\ln n)=\gamma,$$

where γ is Euler constant and chosen as $\Gamma(e_1+e_2)=e_1+e_2$.

Proof. Put $z=e_1+e_2$ in (4.2), we have

$$e_1+e_2=e^{\gamma}\prod_{n=1}^{\infty}(1+\frac{1}{n})e^{-\frac{1}{n}}.$$

Hence,

$$\begin{aligned}
-\gamma &= \sum_{n=1}^{\infty}(\ln\frac{n+1}{n}-\frac{1}{n}) \\
&= \lim_{n\to\infty}(\ln\frac{2}{1}-\frac{1}{1}+\ln\frac{3}{2}-\frac{1}{2}+\cdots+\ln\frac{n+1}{n}-\frac{1}{n}) \\
&= \lim_{n\to\infty}\{\ln(\frac{2}{1}.\frac{3}{2}.\cdots\frac{n}{n-1})+\ln\frac{n+1}{n}-(1+\frac{1}{2}+\cdots+\frac{1}{n})\} \\
&= \lim_{n\to\infty}(\ln n - z_n).
\end{aligned}$$

□

Theorem 4.4. *Let $z = z_1 + z_2 i_2 = \alpha e_1 + \beta e_2$. Then the Euler's function for bi-complex number z is*

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}. \tag{4.3}$$

Proof. Using Lemma 4.3 and equation (4.2), we get the result. □

Corollary 4.5. *Let $z = z_1 + z_2 i_2 = \alpha e_1 + \beta e_2$ be any bi-complex number. Then we have*

$$\Gamma(z+1) = z\Gamma(z), z \neq 0, -1, -2, \cdots, \tag{4.4}$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{4.5}$$

Theorem 4.6. *For any $\alpha, \beta \in \mathbb{C}^1(i_1)$ or $\mathbb{C}^1(i_2)$, if we have $\alpha\prod_{n=1}^{\infty}(1-\frac{\alpha^2}{n^2}) = \frac{\sin\pi\alpha}{\pi}$ and $\beta\prod_{n=1}^{\infty}(1-\frac{\beta^2}{n^2}) = \frac{\sin\pi\beta}{\pi}$. Then for a bi-complex number $z = z_1 + z_2 i_2$ we have*

$$z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2}) = \frac{\sin \pi z}{\pi}. \tag{4.6}$$

Proof. We have

$$\alpha\prod_{n=1}^{\infty}(1-\frac{\alpha^2 e_1}{n^2})e_1 = \frac{\sin\pi\alpha}{\pi}e_1$$

$$\beta\prod_{n=1}^{\infty}(1-\frac{\beta^2 e_2}{n^2})e_2 = \frac{\sin\pi\beta}{\pi}e_2.$$

On adding and using Theorem 3.1, we get

$$\begin{aligned}
\alpha\prod_{n=1}^{\infty}(1-\frac{\alpha^2}{n^2})e_1+\beta\prod_{n=1}^{\infty}(1-\frac{\beta^2}{n^2})e_2 &= \frac{\sin\pi\alpha}{\pi}e_1+\frac{\sin\pi\alpha}{\pi}e_2 \\
(\alpha e_1+\beta e_2)(\prod_{n=1}^{\infty}(1-\frac{\alpha^2}{n^2})e_1+\prod_{n=1}^{\infty}(1-\frac{\beta^2}{n^2})e_2) &= \frac{\sin\pi(\alpha e_1+\beta e_2)}{\pi} \\
(\alpha e_1+\beta e_2)\prod_{n=1}^{\infty}((1-\frac{\alpha^2}{n^2})e_1+(1-\frac{\beta^2}{n^2})e_2) &= \frac{\sin\pi(\alpha e_1+\beta e_2)}{\pi} \\
z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2}) &= \frac{\sin\pi z}{\pi},\ \text{where } z\in\mathbb{C}^2.
\end{aligned}$$

□

Corollary 4.7. *Let $z = z_1 + z_2 i_2 = \alpha e_1 + \beta e_2$ be any bi-complex number. Then we have*

$$\lim_{z \to -n} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}, n = 0, 1, 2, \cdots, \tag{4.7}$$

$$\lim_{n \to \infty} \frac{\Gamma(z+n)}{n^z \Gamma(z)} = 1, z \neq -n. \tag{4.8}$$

Theorem 4.8. *Let $z = z_1 + z_2 i_2 = \alpha e_1 + \beta e_2$ be any bi-complex number. Then there is just one function $F(z)$ in $\mathbb{C}^2 - \{0, -1, -2, \cdots\}$ which satisfies the following conditions*

(*i*) $F(1) = 1$;
(*ii*) $F((\alpha+1)e_1 + (\beta+1)e_2) = (\alpha e_1 + \beta e_2) F(\alpha e_1 + \beta e_2)$;
(*iii*) $\lim_{n \to \infty} \frac{F((\alpha+n)e_1 + (\beta+n)e_2)}{n^{(\alpha e_1 + \beta e_2)} F(n)} = 1$.

Proof. We have already seen that the Γ- function satisfies the above equations. Suppose that there exits a function $F(z)$ which also satisfies all three equations.
From condition (ii) it follows that

$$F((\alpha+n)e_1+(\beta+n)e_2) = (\alpha e_1+\beta e_2)((\alpha+1)e_1+(\beta+1)e_2)\cdots((\alpha+n-1)e_1+(\beta+n-1)e_2)F(\alpha e_1+\beta e_2). \tag{4.9}$$

Put $z = e_1 + e_2$ in (4.9) and using condition (i) we find
$F(n+1) = n!$ or $F(n) = (n-1)!, (n \geq 1)$.
Now taking condition (iii) into account, as well as from (4.3) and (4.9), we have

$$1 = \lim_{n \to \infty} \frac{F((\alpha+n)e_1 + (\beta+n)e_2)}{n^{\alpha e_1 + \beta e_2} F(n)}$$

$$= F(\alpha e_1+\beta e_2) \lim_{n \to \infty} \frac{(\alpha e_1+\beta e_2)((\alpha+1)e_1+(\beta+1)e_2)\cdots((\alpha+n-1)e_1+(\beta+n-1)e_2)F(\alpha e_1+\beta e_2)}{n! n^{(\alpha-1)e_1+(\beta-1)e_1}}$$

$$= \frac{F(\alpha e_1 + \beta e_2)}{\Gamma(\alpha e_1 + \beta e_2)}$$

□

5. Euler's beta operator for bi-complex numbers

We know that the Euler's beta function in complex plane is defined as

$$\mathcal{B}(\mu, \nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1}\, dx \;\; \mu, \nu \in \mathbb{C}^1, Re(\mu), Re(\nu) > 0. \tag{5.1}$$

Then for all $n \in \mathbb{N}$ and $z_1 \in \mathbb{C}^1(i_1)$ or $\mathbb{C}^1(i_2)$, the complex beta operator is

$$\mathcal{C}_n(f, z_1) = \frac{1}{\mathcal{B}(nz_1, n(1-z_1))} \int_0^1 x^{nz_1-1}(1-x)^{n(1-z_1)-1} f(x)\, \mathrm{d}x, \;\; 0 < Re(z_1) < 1. \tag{5.2}$$

We have following theorem in bi-complex space $\mathbb{C}^2$.

Theorem 5.1. *Let $z = \{z_1 + z_2 i_2 = \alpha e_1 + \beta e_2 \mid \alpha, \beta \in \mathbb{C}^1(i_1)$ or $\mathbb{C}^1(i_2)\}$. Then a beta operator for a bi-complex number z is*

$$\mathcal{B}(nz, n(1-z)) = \int_0^1 x^{nz-1}(1-x)^{n(1-z)-1}\, dx, \; \forall n \in \mathbb{N}, \; \textit{and } Re(z, 1-z) > 0, \tag{5.3}$$

where

$$\mathcal{B}(n\alpha, n(1-\alpha)) = \int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}\, dx$$

$$\mathcal{B}(n\beta, n(1-\beta)) = \int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}\,dx,$$

provided $\forall n \in \mathbb{N},$ *and* $Re(\alpha, 1-\alpha) > 0,\ Re(\beta, 1-\beta) > 0.$

Proof. We have

$$\mathcal{B}(nz, n(1-z)) = \mathcal{B}(n\alpha, n(1-\alpha))e_1 + \mathcal{B}(n\beta, n(1-\beta))e_2 \tag{5.4}$$

$$\begin{aligned}
\mathcal{B}(n\alpha, n(1-\alpha)) &= \int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}\,dx \\
\mathcal{B}(n\beta, n(1-\beta)) &= \int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}\,dx \\
\mathcal{B}(nz, n(1-z)) &= \int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}\,dxe_1 + \int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}\,dxe_2 \\
&= \int_0^1 (x^{n\alpha-1}e_1 + x^{n\beta-1}e_2)((1-x)^{n(1-\alpha)-1}e_1 + (1-x)^{n(1-\beta)-1}e_2)\,dx \\
&= \int_0^1 x^{(n\alpha-1)e_1+n\beta-1)e_2}(1-x)^{(n(1-\alpha)-1)e_1+n(1-\beta)-1)e_2}\,dx \\
&= \int_0^1 x^{n(\alpha e_1+\beta e_2)-1}(1-x)^{n(1-\alpha e_1+\beta e_2)-1}\,dx \\
&= \int_0^1 x^{n(\alpha e_1+\beta e_2)-1}(1-x)^{n(1-\alpha e_1+\beta e_2)-1}\,dx \\
&= \int_0^1 x^{nz-1}(1-x)^{n(1-z)-1}\,dx
\end{aligned}$$

for all $z \in \mathbb{C}^2$, and $Re(z, 1-z) > 0$.
whereas $Re(\alpha, 1-\alpha) > 0,\ Re(\beta, 1-\beta) > 0$ imply that $Re(z, 1-z) > 0$. □

Theorem 5.2. *Let* $z = \{z_1 + z_2 i_2 = \alpha e_1 + \beta e_2 \mid \alpha, \beta \in \mathbb{C}^1(i_1)$ *or* $\mathbb{C}^1(i_2)\}$. *Then for a bi-complex number* z

$$\mathcal{C}_n(f, z) = \frac{1}{\mathcal{B}(nz, n(1-z))}\int_0^1 x^{nz-1}(1-x)^{n(1-z)-1}f(x)\,\mathrm{d}x\ \forall n \in \mathbb{N},\ \textit{and}, 0 < Re(z) < 1, \tag{5.5}$$

if we have

$$\begin{aligned}
\mathcal{C}_n(f, \alpha) &= \frac{1}{\mathcal{B}(n\alpha, n(1-\alpha))}\int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}f(x)\,dx \\
\mathcal{C}_n(f, \beta) &= \frac{1}{\mathcal{B}(n\beta, n(1-\beta))}\int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}f(x)\,dx,
\end{aligned}$$

provided $\forall n \in \mathbb{N},$ *and* $0 < Re(\alpha) < 1, 0 < Re(\beta) < 1.$

Proof. We have

$$\mathcal{C}_n(f, \alpha) = \frac{1}{\mathcal{B}(n\alpha, n(1-\alpha))}\int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}f(x)\,dx\ \forall n \in \mathbb{N}, \alpha \in \mathbb{C}^1, 0 < Re(\alpha) < 1$$

$$\mathcal{C}_n(f, \beta) = \frac{1}{\mathcal{B}(n\beta, n(1-\beta))}\int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}f(x)\,dx\ \forall n \in \mathbb{N}, \beta \in \mathbb{C}^1, 0 < Re(\beta) < 1$$

$$\mathcal{C}_n(f, z) = \mathcal{C}_n(f, \alpha)e_1 + \mathcal{C}_n(f, \beta)e_2 \tag{5.6}$$

$$\mathcal{C}_n(f,\alpha)e_1+\mathcal{C}_n(f,\beta)e_2=\frac{1}{\mathcal{B}(n\alpha,n(1-\alpha))}\int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}f(x)\,dxe_1$$
$$+\frac{1}{\mathcal{B}(n\beta,n(1-\beta))}\int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}f(x)\,dxe_2$$
$$=(\frac{1}{\mathcal{B}(n\alpha,n(1-\alpha))}e_1+\frac{1}{\mathcal{B}(n\beta,n(1-\beta))}e_2)(\int_0^1 x^{n\alpha-1}(1-x)^{n(1-\alpha)-1}f(x)\,dxe_1$$
$$+\int_0^1 x^{n\beta-1}(1-x)^{n(1-\beta)-1}f(x)\,dxe_2)$$
$$=\frac{1}{\mathcal{B}(nz,n(1-z))}\int_0^1 x^{n(\alpha e_1+\beta e_2)-1}(1-x)^{n(1-\alpha e_1-\beta e_2)-1}f(x)\,dx$$
$$=\frac{1}{\mathcal{B}(nz,n(1-z))}\int_0^1 x^{nz-1}(1-x)^{n(1-z)-1}f(x)\,dx\;\forall n\in\mathbb{N},z\in\mathbb{C}^2,0<Re(z)<1$$

whereas $0<Re(\alpha)<1, 0<Re(\beta)<1\Rightarrow 0<Re(z)<1$.

□

Acknowledgment. The authors of the manuscript gratefully extend their appreciation to reviewers for their positive feedback and their relevant constructive comments.

Funding: Not applicable.

Availability of data and materials: Not applicable.

Competing interests: The author declares no competing interests.

REFERENCES

[1] D. Alpay, M. Elena Luna-Elizarrarás, M. Shapiro and D.C. Struppa Basics of Functional Analysis with bi-complex Scalars and bi-complex Schur Analysis, SpringerBriefs, 2014.
[2] K.S. Charak, D. Rochon and N. Sharma, Normal families of bi-complex holomorphic functions, arXiv:0806.4403v1 [math.CV] 26 June 2008.
[3] N. Fleury, M. R. Detraubenberg and R. M. Yamaleev, Commutative extended complex numbers and connected trigonometry, Jour. Math. Anal. Appl., 180 (1993), 431-457.
[4] W. R. Hamilton, On Quaternions; or On A New System of Imaginaries in Algebra, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, 15 (1844), 489-495.
[5] A. Kumar and P. Kumar, bi-complex version of laplace transform, International Journal of Engineering and Technology, 3(3) (2011), 225-232.
[6] G. Lantoine, R.P. Russell and T. Dargent, Using multicomplex variables for automatic computation of high-order derivatives, ACM Transactions on Mathematical Software, 38(3), Article 16, April 2012, 21 pages.
[7] M.E. Luna-Elizarrarás and M. Shapiro, bi-complex numbers and their elementary functions, CUBO A Mathematical Journal, 14(2) (2012), 61-80.
[8] J.H. Mathews and R.W. Howell, Complex Analysis for Mathematics and Engineering, Narosa Publication House, 2006.
[9] M. Mursaleen, M. Nasiruzzaman and H.M. Srivastava, Approximation by bi-complex beta operators in compact BC-disks, Math. Meth. Appl. Sci., 2016, 39 2916–2929.
[10] Md. Nasiruzzaman and M. Mursaleen, Some properties of fundamental formulation of tricomplex polynomials, Annals of Communications in Mathematics Volume 8, Number 1 (2025), 137–149 DOI: 10.62072/acm.2025.080111
[11] M. Nasiruzzaman and M. Mursaleen, The multicomplex numbers and their properties on some elementary functions Thai Journal of Mathematics Volume 21 Number 1 (2023) Pages 77–100.
[12] G.B. Price, An Introduction to Multicomplex Spaces and Functions, Marcel Dekker, 1991.

[13] F.L. Reid and R.A. Van Gorder, A multicomplex Riemann Zeta function, Adv. Appl. Clifford Algebras, 23 (2013), 237-251.

[14] C. Segre, The real representation of complex elements and hyperalgebraic entities (Italian), Mathematische Annalen, 40 (1892), 413-467.

MD. NASIRUZZAMAN
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF TABUK, PO BOX 4279, TABUK 71491, SAUDI ARABIA.
ORCID: 0000-0003-4823-0588
Email address: mfarooq@ut.edu.sa; nasir3489@gmail.com

MOHAMMAD AYMAN-MURSALEEN
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSTRAVA, MLÝNSKÁ 702/5, 702 00 MORAVSKÁ OSTRAVA, CZECHIA.
ORCID: 0000-0002-2566-3498
Email address: mohdaymanm@gmail.com, ayman.mursaleen@osu.cz

NADEEM RAO
DEPARTMENT OF MATHEMATICS, UNIVERSITY CENTER FOR RESEARCH AND DEVELOPMENT CHANDIGARH UNIVERSITY, MOHALI-140243, PUNJAB, INDIA.
ORCID: 0000-0002-5681-9563
Email address: nadeemrao1990@gmail.com