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A PROOF OF THE DIMENSION THEOREM FOR VECTOR SPACES

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ABSTRACT. We provide a proof of the theorem that, assuming Zorn's lemma, two Hamel bases of a vector space have equal cardinality.

The purpose of this work is to present an original and self-contained, apart from the Cantor-Schöder-Bernstein theorem, exposition of the fact that two bases of a vector space have the same cardinality, assuming Zorn's lemma. A similar result, albeit following a different approach, appeared in [3]. In the following, "almost all" means all except finitely many.

Recall that in a (left) vector space V over a field F, given $X \subseteq V$, a relation

$$\sum_{v \in X} a_v v = 0$$

where almost all (but not all) $a_v \in F$ are equal to zero is called a relation of linear dependence among the $\{v\}_{v \in X}$ and the v's are said linear dependent. If a relation like above implies that $a_v = 0$ for all $v \in X$, then the v's are said linearly independent or that X is a linearly independent set. Note that the empty set, $\emptyset$, is vacuously linear independent. Elements of a set $W \subseteq V$ are said to generate (are generators of) V if any $v \in V$ can be written as a linear combination

$$v = \sum_{w \in W} b_w w$$

with $b_w \in F$ almost all zero. We will also say that v lies in the (linear) span of W. Without further reminding, we will henceforth suppose that all such sums involve only a *finite* number of nonzero terms.

By definition, a basis of V is a set of independent vectors which also generate V. We want to show that a basis always exists, but we have to rely on the axiom of choice, in the form of Zorn's lemma.

Recall that a *partial order* on a set S is a binary relation $\preceq$ such that, for any $s, s', s'' \in S$, the following are true: $s \preceq s$ (reflexivity), $s \preceq s'$ and $s' \preceq s$ implies $s = s'$ (antisymmetry), $s \preceq s'$ and $s' \preceq s''$ implies $s \preceq s''$ (transitivity). The order is called *total* if given

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$s, s' \in S$ either $s \preceq s'$ or $s' \preceq s$ is true. A *maximal element* of S is $\sigma \in S$ such that $s \in S$ and $\sigma \preceq s$ implies $\sigma = s$.

Axiom (Zorn's Lemma)**.** Let S be a partially ordered set. If any chain (i.e., a subset of *totally* ordered elements of S) admits an upper bound in S, then S has a maximal element.

Theorem 1. *Let $V \neq 0$ be a vector space over a field F. Then V admits a basis.*

Proof. Let

$$S = \{\mathcal{I} \subseteq V : \mathcal{I} \text{ consists of linearly independent vectors}\} \ .$$

S is a nonempty partially ordered set (by inclusion). We show that any chain in S is upper bounded. Let $\{\mathcal{I}_i\}_{i \in I}$ be an arbitrary chain. Define

$$\mathcal{I} = \bigcup_{i \in I} \mathcal{I}_i \ .$$

If $\mathcal{I}$ contains linearly dependent vectors, say $v_1, \dots v_n$, with $v_k \in \mathcal{I}_{i_k}$. Since in any totally ordered set any finite subset contains a maximum element, there exists a $\bar{k}$ such that $\mathcal{I}_{i_k} \subseteq \mathcal{I}_{i_{\bar{k}}}$ for all $k = 1, \dots, n$. But then $\mathcal{I}_{i_{\bar{k}}}$ would contain the linearly dependent vectors $v_1, \dots v_n$, absurd. Therefore $\mathcal{I}$ is an upper bound of the chain.

By Zorn's lemma, S contains a maximal element $\mathcal{I}_{\mathfrak{m}} \neq \emptyset$ since $V \neq 0$. We claim that this maximal element is a set of generators of V (i.e. a basis, which is a set of linearly independent vectors that generate V). Indeed, if $v \in \mathcal{I}_{\mathfrak{m}}$, then v is generated by the elements of $\mathcal{I}_{\mathfrak{m}}$. On the other hand, if $v \in V - \mathcal{I}_{\mathfrak{m}}$, then $\mathcal{I}_{\mathfrak{m}} \cup \{v\}$ cannot be a set of linearly independent vectors, by the maximality of $\mathcal{I}_{\mathfrak{m}}$. Therefore there exists coefficients $a, a_u \in F$ almost all zero such that

$$\sum_{u \in \mathcal{I}_{\mathfrak{m}}} a_u u + av = 0 \iff v = \sum_{u \in \mathcal{I}_{\mathfrak{m}}} (-a_u/a)u$$

since $a \neq 0$ (otherwise we would get a relation of linear dependence among elements of $\mathcal{I}_{\mathfrak{m}}$). Again v is generated by elements of $\mathcal{I}_{\mathfrak{m}}$.

□

Next we show the familiar (in finite dimensional vector spaces) fact that two bases have equal cardinality (i.e. they are in bijection with one another). This can be generalized to arbitrary vector spaces by another use of Zorn's lemma. We will also need the Cantor-Schröder-Bernstein Theorem [1, Theorem 4 p.134].

Theorem 2 (Cantor-Schröder-Bernstein)**.** *Given any two sets A and B, if there exists an injection from A to B and an injection from B to A, then there is a bijection between the two sets.*

We will now show the following fact, which implies that two bases have equal cardinality, since a basis is both a set of linearly independent vectors and a set of generators.

Theorem 3. *Let $\mathcal{I}$ and $\mathcal{G}$ be respectively a linearly independent set and a set of generators of a vector space $V \neq 0$ over F. There exists an* injection $f : \mathcal{I} \hookrightarrow \mathcal{G}$ *with the additional property that $f(\mathcal{I})$ is a linearly independent set.*

Proof. Let us define

$$S = \{(\mathcal{I}', f') : \mathcal{I} \cap \mathcal{G} \subseteq \mathcal{I}' \subseteq \mathcal{I}, f' : \mathcal{I}' \hookrightarrow \mathcal{G} \text{ is an injection with } f'(x) = x \\ (\forall x \in \mathcal{I} \cap \mathcal{G}) \text{ and } \big(\mathcal{I} - \mathcal{I}'\big) \cup f'(\mathcal{I}') \text{ is a linearly independent set}\} \ .$$

S is nonempty, since, if $\mathcal{I} \cap \mathcal{G} = \emptyset$, then $(\emptyset, f') \in S$, where $f' \colon \emptyset \to \mathcal{G}$ is the empty function. If $\mathcal{I} \cap \mathcal{G} \neq \emptyset$, then $(\mathcal{I} \cap \mathcal{G}, f') \in S$, where $f' \colon \mathcal{I} \cap \mathcal{G} \to \mathcal{G}$ is the inclusion map. This set is partially ordered by the following order relation, where $(\mathcal{I}', f') \preceq (\mathcal{I}'', f'')$ if by definition $\mathcal{I}' \subseteq \mathcal{I}''$ and $f''\big|_{\mathcal{I}'} = f'$.

Lemma 4. *Let $(\mathcal{I}', f') \in S$. For any $v \in \mathcal{I} - \mathcal{I}'$, there exists $w \in \mathcal{G} - f'(\mathcal{I}')$ such that $\big(\mathcal{I} - (\mathcal{I}' \cup \{v\})\big) \cup (f'(\mathcal{I}') \cup \{w\})$ is a linearly independent set.*

This means that if we define $\mathcal{I}'' = \mathcal{I}' \cup \{v\}$ and f'' by extending f' with $f''(v) = w$, then $(\mathcal{I}'', f'') \in S$ and $(\mathcal{I}', f') \preceq (\mathcal{I}'', f'')$.

Proof of Lemma 4. The lemma is shown by contradiction. Note initially that $f'(\mathcal{I}') \neq \mathcal{G}$ as long as $\mathcal{I} - \mathcal{I}' \neq \emptyset$. Indeed if $\big(\mathcal{I} - \mathcal{I}'\big) \cup f'(\mathcal{I}') = \big(\mathcal{I} - \mathcal{I}'\big) \cup \mathcal{G}$ then any $v \in \mathcal{I} - \mathcal{I}'$ would be a linear combination of elements of $\mathcal{G}$ and therefore $\big(\mathcal{I} - \mathcal{I}'\big) \cup f'(\mathcal{I}')$ would not be a linearly independent set.

Choose an arbitrary $v \in \mathcal{I} - \mathcal{I}'$ and suppose that any $w \in \mathcal{G} - f'(\mathcal{I}')$ lies in the span of $\big(\mathcal{I} - (\mathcal{I}' \cup \{v\})\big) \cup f'(\mathcal{I}')$. Note that

$$v = \sum_{w \in \mathcal{G}} a_w w = \sum_{w \in \mathcal{G} - f'(\mathcal{I}')} a_w w \; + \sum_{w \in f'(\mathcal{I}')} a_w w \ ,$$

since $\mathcal{G}$ generates V. Let $X = \{w \in \mathcal{G} - f'(\mathcal{I}') \colon a_w \neq 0\}$. By assumption, any $w \in X$ lies in the span of $\big(\mathcal{I} - (\mathcal{I}' \cup \{v\})\big) \cup f'(\mathcal{I}')$. Any linear combination of elements of X is also in the span of $\big(\mathcal{I} - (\mathcal{I}' \cup \{v\})\big) \cup f'(\mathcal{I}')$ and so is also v, by the above equation. Therefore we obtain

$$v = \sum_{w \in (\mathcal{I} - (\mathcal{I}' \cup \{v\}))} b_w w \; + \sum_{w \in f'(\mathcal{I}')} b_w w \ .$$

Note that v on the left-hand side of the previous equation cannot equal any w on the right-hand side: it is clear for the first sum, which excludes v explicitly, and also in the second sum $w \in \mathcal{G}$ but $v \in \mathcal{I} - \mathcal{I}'$ implies $v \notin \mathcal{G}$. Therefore the equation gives rise to a relation of linear dependence in $\big(\mathcal{I} - \mathcal{I}'\big) \cup f'(\mathcal{I}')$, contradicting the definition of S and the lemma is proved. □

We now show that every chain in S is upper bounded by an element of S. Start from an arbitrary chain $\{(\mathcal{I}_i, f_i)\}_{i \in I} \subseteq S$. Define $(\mathcal{I}', f')$ by

$$\mathcal{I}' = \bigcup_{i \in I} \mathcal{I}_i, \quad f'\big|_{\mathcal{I}_i} = f_i \ .$$

Note that if $(\mathcal{I}_i, f_i) \preceq (\mathcal{I}_j, f_j)$, then for all $v \in \mathcal{I}_i \subseteq \mathcal{I}_j$ we have $f'(v) = f_i(v) = f_j(v)$ which shows that f' is well defined. We remark in what follows that $(\mathcal{I}', f') \in S$:

(1) $\mathcal{I} \cap \mathcal{G} \subseteq \mathcal{I}' \subseteq \mathcal{I}$: this is clear.
(2) f' is the identity on $\mathcal{I} \cap \mathcal{G}$ (clear) and f' is injective: if $v_1, v_2 \in \mathcal{I}'$ with $v_1 \neq v_2$, then there exists $i \in I$ such that $v_1, v_2 \in \mathcal{I}_i$. But then $f'(v_1) = f_i(v_1) \neq f_i(v_2) = f'(v_2)$, since f_i is injective.
(3) $\big(\mathcal{I} - \mathcal{I}'\big) \cup f'(\mathcal{I}')$ is a set of linearly independent vectors: otherwise $v_1, \ldots v_n$, say, are linearly dependent. Since

$$f'(\mathcal{I}') = \bigcup_{i \in I} f'(\mathcal{I}_i)$$

and $\{f'(\mathcal{I}_i)\}_{i\in I}$ is a chain because $\mathcal{I}_i \subseteq \mathcal{I}_j \Rightarrow f'(\mathcal{I}_i) \subseteq f'(\mathcal{I}_j)$, there exists $i \in I$ for which

$$v_1, \dots, v_n \in \big(\mathcal{I} - \mathcal{I}'\big) \cup f'(\mathcal{I}_i) \subseteq \big(\mathcal{I} - \mathcal{I}_i\big) \cup f'(\mathcal{I}_i) = \big(\mathcal{I} - \mathcal{I}_i\big) \cup f_i(\mathcal{I}_i)$$

but this leads to a contradiction, since this last set consists of linearly independent vectors.

Finally, note that $(\mathcal{I}_i, f_i) \preceq (\mathcal{I}', f')$ for all $i \in I$ by construction, so $(\mathcal{I}', f') \in S$ is an upper bound of our chain.

Invoking Zorn's lemma, S has a maximal element, say $(\mathcal{I}_\mathfrak{m}, f)$. We claim that in fact, $\mathcal{I}_\mathfrak{m} = \mathcal{I}$. Since otherwise Lemma 4 applied to $(\mathcal{I}', f') = (\mathcal{I}_\mathfrak{m}, f)$, would violate maximality. The theorem is thus proved. □

Remark. *The difficulty in the previous theorem consisted in defining an appropriate condition on a set S. Otherwise, one cannot show that a maximal injection is defined on all of $\mathcal{I}$; in fact, it could well result in a bijection from a strict subset of $\mathcal{I}$ to $\mathcal{G}$. This is a natural difficulty, arising from the fact that an infinite set can always be put in bijection with a strict subset.*

Remark. *The same argument as in Theorem 1 allows to prove the existence of Hilbert bases in Hilbert spaces (i.e., orthonormal sets which span a dense subspace). However, Zorn's lemma is not needed to show that all Hilbert bases of a Hilbert space have equal cardinality* [2, Theorem IV.4.14].

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