



## ON JERUSALEM SQUARE FRACTAL, ITS CONSTRUCTION AND PROPERTIES

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**ABSTRACT.** This short communication proposes a novel Iterated Function System (IFS) to construct the Jerusalem square fractal, a two-dimensional projection of the Jerusalem cube, characterized by its cross-like self-similar structure. We provide a step-by-step construction, a proof of its fractal nature, and Python code for visualization. Properties such as self-similarity, fractal dimension, and connectivity are analyzed with formal proofs. The IFS offers a new perspective on Jerusalem square fractal generation. Results demonstrate the fractal's intricate geometry, with applications in computer graphics and geometric modeling.

### 1. INTRODUCTION

Fractals are geometric objects exhibiting self-similarity across scales and have been extensively studied since Mandelbrot's seminal work [8]. Iterated Function Systems (IFSs), introduced by Hutchinson [6], provide a mathematical framework for generating fractals through iterative application of contractive mappings. The Jerusalem cube, a three-dimensional fractal with cross-shaped protrusions, inspires the Jerusalem square fractal, a 2D projection preserving its self-similar cross motif. While the Jerusalem cube has been explored [2], its 2D square variant is still not discussed much in the literature. This short article proposes a unique IFS to generate the Jerusalem square fractal, including its construction, visualization, and properties, contributing to fractal geometry and computer graphics.

### 2. PRELIMINARIES

**Definition 2.1.** Let  $(X, d)$  be a complete metric space. Let  $\mathcal{H}(X)$  be the set of non-empty compact subsets of  $X$ .  $\mathcal{H}(X)$  is called 'space of fractals'.

The space  $\mathcal{H}(X)$  is a metric space with respect to the Hausdorff metric  $h$  defined by

$$h(A, B) = \max \{d(A, B), d(B, A)\}$$

where,  $d(A, B) = \sup_{x \in A} \{ \inf_{y \in B} \{d(x, y)\} \}$  and  $A, B \in \mathcal{H}(X)$  [2].

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**Definition 2.2.** Let  $(X, d)$  be a complete metric space. An iterated function system (in short IFS) on  $X$  is a finite set of contraction mappings  $\phi_i : X \rightarrow X$ , having contractivity factors  $s_i$ , for  $i = 1, 2, \dots, n$ . The number  $s = \max_{1 \leq i \leq n} s_i$  is called ‘contractivity’ factor of the IFS.

**Theorem 2.1.** Let  $(X, d)$  be a complete metric space and  $\{X, \phi_i : i = 1, 2, \dots, \phi_n\}$  be an IFS with contractivity factor  $s$ . Then the transformation  $W : \mathcal{H}(X) \rightarrow \mathcal{H}(X)$  (called Hutchinson operator) defined by

$$W(B) = \bigcup_{i=1}^n \phi_i(B) \quad (2.1)$$

for all  $B \in \mathcal{H}(X)$  is a contraction mapping on  $\mathcal{H}(X, h(d))$  with contractivity factor  $s$ . Moreover, by the contraction mapping theorem its unique fixed point  $A \in \mathcal{H}(X)$ , called the ‘attractor’ of the IFS, satisfies

$$A = W(A) = \bigcup_{i=1}^n \phi_i(A). \quad (2.2)$$

The attractor  $A$  described in Theorem 2.1 is obtained by taking the limit of the sequence of forward iterates of  $W$  i.e.  $A = \lim_{n \rightarrow \infty} W^{\circ n}(K)$  for any  $K \in \mathcal{H}(X)$ .

We recall a classical result from the theory of iterated function systems by Hutchinson [6]

**Theorem 2.2** (Hutchinson’s Theorem (1981)). Let  $\{\phi_i\}_{i=1}^n$  be an IFS consisting of contraction mappings on a complete metric space  $(X, d)$ . If  $K \subset X$  is compact and connected, and  $\mathcal{F}(K) = \bigcup_i \phi_i(K)$  is connected, then the attractor  $A = \lim_{n \rightarrow \infty} \mathcal{F}^n(K)$  is also connected.

### 3. A NOVEL IFS FOR THE CONSTRUCTION OF JERUSALEM SQUARE FRACTAL

The Jerusalem square fractal is constructed within the unit square  $A_0 = [0, 1]^2$ . The IFS comprises eight affine transformations, reflecting the cube’s face projection, where each face forms a fractal resembling a Jerusalem cross [1]. The Jerusalem square’s face is described as a 2D fractal cross pattern where four corner squares of size  $k$  and four small square of size  $k^2$  (interlaced in between two corner squares on each side of  $A_0$ ) are retained, satisfying the equation  $k + k^2 + k = 1$  for the scaling factor. Thus, the scaling factor  $k = \sqrt{2} - 1$  satisfies  $k^2 + 2k - 1 = 0$ , ensuring the fractal fits within the previous iteration’s space [3]. This suggests 8 scaled (and translated) copies in the IFS (4 at scale  $k$ , 4 at scale  $k^2$ ).

Therefore, the desired IFS is given by  $\{\phi_i(x)\}$ ,  $1 \leq i \leq 8$ , where each  $\phi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is an affine transformation of the form:

$$\phi_i(\mathbf{x}) = A_i \mathbf{x} + \mathbf{b}_i, \quad x \in \mathbb{R}^2 \quad (3.1)$$

where  $A_i = kI_2$  (scaling by  $k$ ) for  $1 \leq i \leq 4$ ,  $A_i = k^2I_2$  for  $5 \leq i \leq 8$  (scaling by  $k^2$ ) and  $\mathbf{b}_i$  are translation vectors. The explicit transformations are

$$\begin{aligned}\phi_1(\mathbf{x}) &= k\mathbf{x}, & \phi_2(\mathbf{x}) &= k\mathbf{x} + \begin{bmatrix} 1-k \\ 0 \end{bmatrix}, \\ \phi_3(\mathbf{x}) &= k\mathbf{x} + \begin{bmatrix} 0 \\ 1-k \end{bmatrix}, & \phi_4(\mathbf{x}) &= k\mathbf{x} + \begin{bmatrix} 1-k \\ 1-k \end{bmatrix}, \\ \phi_5(\mathbf{x}) &= k^2\mathbf{x} + \begin{bmatrix} \frac{1}{2} - \frac{k^2}{2} \\ 0 \end{bmatrix}, & \phi_6(\mathbf{x}) &= k^2\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{2} - \frac{k^2}{2} \end{bmatrix}, \\ \phi_7(\mathbf{x}) &= k^2\mathbf{x} + \begin{bmatrix} \frac{1}{2} - \frac{k^2}{2} \\ 1-k^2 \end{bmatrix}, & \phi_8(\mathbf{x}) &= k^2\mathbf{x} + \begin{bmatrix} 1-k^2 \\ \frac{1}{2} - \frac{k^2}{2} \end{bmatrix},\end{aligned}$$

Here,  $k = \sqrt{2} - 1 \approx 0.414213562$ , and  $k^2 = 3 - 2\sqrt{2} \approx 0.171572175$ . Starting with the initial set  $A_0$  (Figure 1(a)), these mappings place eight scaled squares in a cross-like pattern, as shown in Figure 1(b) and the corresponding Jerusalem square fractal is shown in Figure 1(c) (obtained with the Python code given in Appendix A using deterministic algorithm). A fractal enthusiast can construct this attractor using IFS construction kit [11] (an easy to use GUI program to render high resolution images of fractals) without going into intricacy of knowing Python language.

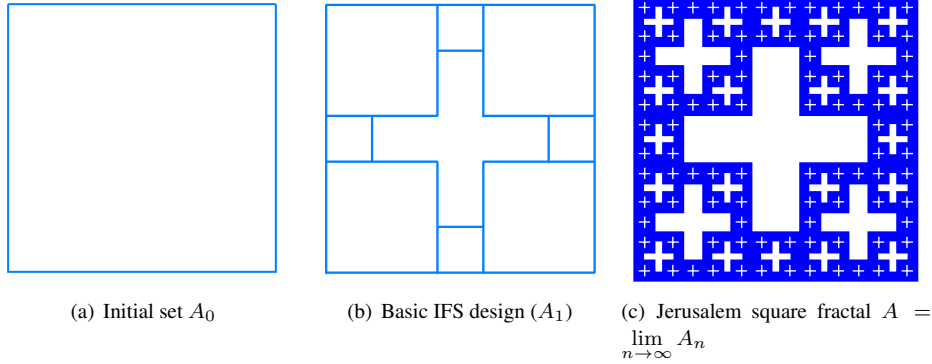


FIGURE 1. (a) Initial set, (b) basic IFS design, and (c) the Jerusalem square fractal.

### Step-by-Step Construction:

- (1) Start with the unit square  $A_0 = [0, 1]^2$ .
- (2) Apply IFS to  $A_0$ , yielding  $A_1 = \bigcup_{i=1}^8 \phi_i(A_0)$ , having four large squares of side length  $k$  and four small squares of side length  $k^2$ .
- (3) For  $n \geq 1$ , compute  $A_{n+1} = \bigcup_{i=1}^8 \phi_i(A_n)$ . The Hutchinson operator  $H(S) = \bigcup_{i=1}^8 \phi_i(S)$  ensures convergence to the fractal attractor  $A$ .

(4) The Jerusalem square fractal is the limit  $A = \lim_{n \rightarrow \infty} A_n$ , a compact set in  $\mathbb{R}^2$ .

**Theorem 3.1.** *The attractor of the IFS in (3.1) is the Jerusalem square and a self-similar fractal.*

*Proof.* Each  $\phi_i$  is contractive with scaling factor  $k = (\sqrt{2} - 1) < 1$  or  $k^2 < 1$ . By the contraction mapping theorem [2], there exists a unique non-empty compact set  $A$  such that  $H(A) = A$  and the self-similarity equation is given by

$$A = \bigcup_{i=1}^8 \phi_i(A).$$

The mappings are similitudes (being combinations of scaling and translation), confirming  $A$  as a self-similar fractal.  $\square$

#### 4. PROPERTIES OF THE JERUSALEM SQUARE

**Theorem 4.1.** *The Jerusalem square fractal is self-similar.*

*Proof.* The Jerusalem square fractal exhibits strict self-similarity, because each  $\phi_i(A)$  is a scaled copy of  $A$ . This is evident from the IFS definition and visualization in the iterative construction (Figure 1).  $\square$

**Theorem 4.2** (Similarity dimension). *The similarity (Hausdorff) dimension of the Jerusalem square fractal is 1.786440.*

*Proof.* For a self-similar fractal defined by an IFS with  $N_i$  copies at scaling factor  $r_i$ , the similarity dimension  $D$  satisfies the Moran equation [9, 4]

$$\sum_{i=1}^8 r_i^D = 1. \quad (4.1)$$

The Jerusalem square has 4 copies scaled by  $r_i = k = \sqrt{2} - 1$ ,  $1 \leq i \leq 4$  and 4 copies scaled by  $r_j = k^2 = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$ ,  $5 \leq j \leq 8$ . Thus, the similarity dimension  $D$  is the solution of non-linear equation

$$4k^D + 4k^{2D} = 1. \quad (4.2)$$

Taking  $k^D = u$ , we can write this equation as  $4u^2 + 4u - 1 = 0$  having roots  $u = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$ . Neglecting the negative root (which gives  $D < 0$ , not possible) and substituting back for  $k$ , we get

$$k^D = -\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{k}{2}. \quad (4.3)$$

Solving for  $D$  (see Appendix-B for a Python code), the similarity dimension of the Jerusalem square (face of the Jerusalem cube) is

$$D \approx 1.786440.$$

The non-integer value of the fractal dimension reflects the fractal's complexity, driven by the irrational scaling factor and the cross pattern. Since the mappings are non-overlapping (squares touch only at boundaries), the Hausdorff dimension equals the similarity dimension [5].  $\square$

**Theorem 4.3.** *The Jerusalem square fractal is connected.*

*Proof.* Let  $A$  denotes the attractor of the IFS in equation (3.1). For any compact and connected set  $K \subset \mathbb{R}^2$ , let  $\mathcal{F}$  denote the Hutchinson operator  $\mathcal{F}(K) = \bigcup_{i=1}^8 \phi_i(K)$ , and let  $A$  be the unique non-empty compact set such that

$$A = \mathcal{F}(A) = \bigcup_{i=1}^8 \phi_i(A).$$

Appealing to Theorem 2.2, it is enough to show that  $\mathcal{F}(K)$  is connected.

Choose  $K = A_0 = [0, 1]^2$ . Then  $K$  is compact and connected. Now, each map  $\phi_i$  is a similarity and hence continuous. The images  $\phi_i(K)$  are all compact squares placed at the corners and midpoints of the unit square. The geometry of the mappings is such that

- (1) The first 4 mappings ( $\phi_1$  through  $\phi_4$ ) place smaller squares in the 4 corners of  $K$ .
- (2) The next 4 mappings ( $\phi_5$  through  $\phi_8$ ) place even smaller squares at the midpoints of the sides of  $K$ .
- (3) The images  $\phi_i(K)$  are either adjacent or overlapping at edges or vertices.

Hence, the set  $\mathcal{F}(K) = \bigcup_{i=1}^8 \phi_i(K)$  is connected. Since  $\mathcal{F}(K)$  is connected, Hutchinson's theorem implies that the attractor  $A = \lim_{n \rightarrow \infty} \mathcal{F}^n(K)$  is connected.  $\square$

## 5. CONCLUSIONS AND DISCUSSIONS

The Jerusalem square fractal exhibits a fractal dimension of approximately 1.786440, indicating a complexity between a line and a plane, suitable for modeling intricate patterns. The Python visualization confirms the cross-like structure, with increasing detail at higher iterations. The connected nature ensures its applicability in network modeling. Unlike the Sierpinski carpet [2], the Jerusalem square has a denser central region, offering unique aesthetic and structural properties for architectural design [7] and has also been used in fractal shaped antenna designs due to its compact and intricate structure [10].

This brief communication presented a novel IFS for constructing the Jerusalem square fractal, a 2D fractal inspired by the Jerusalem cube. The step-by-step construction, supported by a proof of its fractal nature, and Python visualization demonstrate its feasibility. We also established the properties such as fractal dimension and connectivity for the Jerusalem fractal. Future work could explore 3D extensions or applications in fractal antenna generation and texture synthesis.

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