ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 1 (2025), 103-115
DOI: 10.62072/acm.2025.080108
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

HIGHER DERIVATIVE DIRECT BLOCK METHODS FOR FOURTH-ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Direct Block Methods for solving fourth-order Initial Value Problems (IVPs) are presented. The derivation of the methods is achieved by applying the technique of interpolation and collocation to a power series polynomial, which is considered an approximate solution to the problems. Higher derivative terms are introduced to improve the order of accuracy of the methods. Details of the block methods are presented, showing that the methods are zero stable, consistent and convergent. Some scalar and vector problems of IVPs are presented to illustrate the accuracy of the proposed approach, providing a comprehensive comparison with other methods in the literature.

1. INTRODUCTION

Considered in this article is the direct block methods for the solution of fourth order initial value problems of the form

$$y^{(4)} = f(x, y, y', y'', y'''), \quad y(\alpha) = y_0, y'(\alpha) = y_0', y''(\alpha) = y_0'', y'''(\alpha) = y_0''' \tag{1.1}$$

the fifth derivative of the solution to equation (1) is

$$y^{(5)} = f'(x, y, y', y'', y''') = f_t + f_y y' + f_{y'} y'' + f_{y''} y''' + f f_{y'''} \tag{1.2}$$

assuming that f in equation (1.1) is differentiable and continuous on the domain R.
Higher-order linear and nonlinear IVPs are the results of modeled problems arising from engineering and other sciences, to mention a few. The static deflection of a uniform beam or a cantilever beam often leads to fourth-order problems. There are many computational numerical methods available for solving this kind of problems in literatures. Not until recently, higher order ordinary differential equations were solved by reducing them to an equivalent system of first order ordinary differential equations, this process is opined to be too rigorous when compared with the direct methods. Several authors who studied and applied this approach to find the solution of higher order includes Jator [9], Jacob [8], Omar & Adeyeye [2] and others. According to Awoyemi [3], The un-economical nature in term of cost of implementation, increased computational burden and wastage of computer time and the increase in dimension of the resulting systems of equations to be solved are the setbacks associated with the reduction approach despite of the success of this approach.

2020 *Mathematics Subject Classification.* 06F35, 03G25.
Key words and phrases. Fourth-order; Higher-derivatives; Collocation & interpolation.
Received: January 30, 2025. Accepted: March 14, 2025. Published: March 31, 2025.
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Block method is an approach that have been recently used for solving higher order ODEs directly, this method has gain more attention. The block method was proposed first by Milne [11]. It advantages are; being more efficient in terms of cost implementation, accuracy, and time of execution over the predictor-corrector methods. Out of the several numerical methods in literatures, only few are specially designed to solve fourth order ordinary differential equations.
While Awoyemi [4] proposed a multi-derivative collocation method for the approximation of fourth order IVPs, Yap & Ismail [14] developed a Block Hybrid Collocation Method (BHCM) for the solution of fourth order IVPs, Abdelrahim & Omar [1] developed a four-step implicit block method with three generalized off-step points and applied it to solve fourth order IVPs directly. Kubye & Omar [10] developed a zero-stable block method for solving fourth order initial value problems of ordinary differential equations, Familua & Omole [5] proposed a five points mono hybrid linear multistep method for solving nth order ordinary differential equations using power series function. Recently, Olabode $et.al.$ [12] consider the existence results and a family of highly stable fifth derivative block methods for the direct solution of systems of beams equations. The article dealt with system of nonlinear fourth-order boundary value problems arising from structural mechanics.
This research focuses on the development of new block methods of the general form

$$\begin{aligned} A^0(t_n)Y_m &= A_0^{(i)}(t_n)Y_{m-i} + hA_1{}^{(i)}(t_n)Y'_{m-i} + h^2A_2{}^{(i)}(t_n)Y''_{m-i} + h^3A_3{}^{(i)}(t_n)Y'''_{m-i} \\ &\quad + h^4\sum_{i=0}^{k}B^{(i)}(t_n)F_{m-i} + h^5\sum_{i=0}^{k}C^{(i)}(t_n)F'_{m-i} \end{aligned} \tag{1.3}$$

which will be used to solve fourth order boundary value problems of ordinary differential equations directly.

2. Derivation of the Block Method

Assume the analytical solution $y(t)$ of (1.1) be approximated by power series polynomial $p(t)$ given as

$$y(t) = p(t) = \sum_{j=0}^{k} a_j t^j \tag{2.1}$$

where $a_j \in R$ are unknown coefficients and t is continuously differentiable. The derivatives of (2.1) yields the following

$$y'(t) = p^{'}(t) = \sum_{j=1}^{k} j a_j t^{j-1} \tag{2.2}$$

$$y''(t) = p^{''}(t) = \sum_{j=2}^{k} j(j-1) a_j t^{j-2} \tag{2.3}$$

$$y'''(t) = p^{'''}(t) = \sum_{j=3}^{k} j(j-1)(j-2) a_j t^{j-3} \tag{2.4}$$

$$y^{(4)}(t) = p^4(t) = \sum_{j=4}^{k} j(j-1)(j-2)(j-3) a_j t^{j-4} \tag{2.5}$$

and

$$y^{(5)}(t) = p^5(t) = \sum_{j=4}^{k} j(j-1)(j-2)(j-3)(j-4)a_j t^{j-5} \tag{2.6}$$

respectively.
Equating (1.1) and (2.5) gives the differential system

$$f(t, y, y', y'', y''') = \sum_{j=0}^{k} j(j-1)(j-2)(j-3)a_j t^{j-4} \tag{2.7}$$

Equations (2.1), (2.5) and (2.6) are the interpolation and collocation equations that shall be use in the derivation of the methods. The equations arising from it are expressed matrix form of equations written as

$$XA = B \tag{2.8}$$

but expressed as a linear system

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & \cdots & t_n^N \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & \cdots & Nt_n^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & \cdots & N(N-1)(N-2)t_n^{N-3} \\ 0 & 0 & 0 & 0 & 24 & 120t_n & \cdots & N(N-1)(N-2)(N-3)t_n^{N-4} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 120 & \cdots & N(N-1)(N-2)(N-3)(N-4)t_n^{N-5} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ \cdots \\ \cdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} y_n \\ y'_n \\ \cdots \\ y'''_n \\ f_n \\ \cdots \\ f'_{n+k} \end{bmatrix} \tag{2.9}$$

which shall be solved to obtain the unknown parameters.
The continuous schemes is recovered after the parameters are substituted into equation (2.1) and then evaluated to obtain the block methods.
For the one-step method (i.e $K = 1$), equation (2.9) becomes

$$\begin{bmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 \\ 0 & 0 & 2 & 6t_n & 12t_n^2 & 20t_n^3 & 30t_n^4 & 42t_n^5 \\ 0 & 0 & 0 & 6 & 24t_n & 60t_n^2 & 120t_n^3 & 210t_n^4 \\ 0 & 0 & 0 & 0 & 24 & 120t_n & 360t_n^2 & 840t_n^3 \\ 0 & 0 & 0 & 0 & 24 & 120t_n & 360t_n^2 & 840t_n^3 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720t_n & 2520t_n^2 \\ 0 & 0 & 0 & 0 & 0 & 120 & 720t_n & 2520t_n^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_n \\ y'_n \\ y''_n \\ y'''_n \\ f_n \\ f_{n+1} \\ f'_n \\ f''_{n+1} \end{bmatrix} \tag{2.10}$$

Solving equation (2.10) by Gaussian elimination method for a_j and substituting the resulting values into (2.1) yield the continuous scheme

$$\begin{aligned} y(t) \quad = \quad & \alpha_1(t)y_n + \alpha_2(t)y'_n + \alpha_3(t)y''_n + \alpha_4(t)y'''_n + h^4(\beta_0(t)f_n + \beta_1(t)f_{n+1}) \\ & + h^5(\omega_0 f'_n + \omega_1 f'_{n+1}) \end{aligned} \tag{2.11}$$

Evaluating the continuous scheme (2.11), it first to third derivatives at $t = 1$ gives the block methods as follows;

2.1. The One-Step Direct Block Method.

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4\left(\frac{f_n}{28} + \frac{f_{n+1}}{168}\right) + h^5\left(\frac{f'_n}{252} - \frac{f'_{n+1}}{630}\right) \quad (2.12)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3\left(\frac{16f_n + 4f_{n+1}}{120}\right) + h^4\left(\frac{2f'_n - f'_{n+1}}{120}\right) \quad (2.13)$$

$$y''_{n+1} = y''_n + hy'''_n + h^2\left(\frac{21f_n + 9f_{n+1}}{60}\right) + h^3\left(\frac{3f'_n - 2f'_{n+1}}{60}\right) \quad (2.14)$$

$$y'''_{n+1} = y'''_n + h\left(\frac{6f_n + 6f_{n+1}}{12}\right) + h^2\left(\frac{f'_n - f'_{n+1}}{12}\right) \quad (2.15)$$

Using the same procedure of the one-step direct block method as stated above and evaluating the continuous scheme at all grids points for the two-step and three-step block method, the direct block methods are presented below.

2.2. The Two-Step Direct Block Method.

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4\left(\frac{67f_n}{2016} + \frac{f_{n+1}}{144} + \frac{f_{n+2}}{672}\right) + h^5\left(\frac{37f'_n}{12096} - \frac{4f'_{n+1}}{945} - \frac{5f'_{n+2}}{12096}\right) \quad (2.16)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4\left(\frac{8f_n}{21} + \frac{16f_{n+1}}{63} + \frac{2f_{n+2}}{63}\right) + h^5\left(\frac{8f'_n}{189} - \frac{16f'_{n+1}}{189} - \frac{8f'_{n+2}}{945}\right) \quad (2.17)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3\left(\frac{817f_n + 256f_{n+1} + 47f_{n+2}}{6720}\right) + h^4\left(\frac{83f'_n - 140f'_{n+1} - 13f'_{n+2}}{6720}\right) \quad (2.18)$$

$$y'_{n+2} = y'_n + 2hy''_n + 2h^2y'''_n + h^3\left(\frac{68f_n + 64f_{n+1} + 8f_{n+2}}{105}\right) + h^4\left(\frac{8f'_n - 16f'_{n+1} - 2f'_{n+2}}{105}\right) \quad (2.19)$$

$$y''_{n+1} = y''_n + hy'''_n + h^2\left(\frac{520f_n + 280f_{n+1} + 40f_{n+2}}{1680}\right) + h^3\left(\frac{59f'_n - 128f'_{n+1} - 11f'_{n+2}}{1680}\right) \quad (2.20)$$

$$y''_{n+2} = y''_n + 2hy'''_n + h^2\left(\frac{79f_n + 112f_{n+1} + 19f_{n+2}}{105}\right) + h^3\left(\frac{10f'_n - 16f'_{n+1} - 4f'_{n+2}}{105}\right) \quad (2.21)$$

$$y'''_{n+1} = y'''_n + h\left(\frac{101f_n + 128f_{n+1} + 11f_{n+2}}{240}\right) + h^2\left(\frac{13f'_n - 40f'_{n+1} - 3f'_{n+2}}{240}\right) \quad (2.22)$$

$$y'''_{n+2} = y'''_n + h\left(\frac{7f_n + 16f_{n+1} + 7f_{n+2}}{15}\right) + h^2\left(\frac{f'_n - f'_{n+2}}{15}\right) \quad (2.23)$$

2.3. **The Three-Step Direct Block Method.**

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4\left(\frac{569426f_n + 78678f_{n+1} + 85266f_{n+2} + 15070f_{n+3}}{17962560}\right) + h^5\left(\frac{47055f'_n - 134568f'_{n+1} - 53487f'_{n+2} - 3732f'_{n+3}}{17962560}\right) \quad (2.24)$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4\left(\frac{99220f_n + 58536f_{n+1} + 25218f_{n+2} + 4136f_{n+3}}{280665}\right) + h^5\left(\frac{9672f'_n - 40068f'_{n+1} - 15120f'_{n+2} - 1020f'_{n+3}}{280665}\right) \quad (2.25)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2}h^2y''_n + \frac{9}{2}h^3y'''_n + h^4\left(\frac{33264f_n + 33534f_{n+1} + 14580f_{n+2} + 1782f_{n+3}}{24640}\right) + h^5\left(\frac{3429f'_n - 14580f'_{n+1} - 6561f'_{n+2} - 432f'_{n+3}}{24640}\right) \quad (2.26)$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3\left(\frac{62387f_n + 14418f_{n+1} + 11853f_{n+2} + 2062f_{n+3}}{544320}\right) + h^4\left(\frac{5637f'_n - 19386f'_{n+1} - 7371f'_{n+2} - 510f'_{n+3}}{544320}\right) \quad (2.27)$$

$$y'_{n+2} = y'_n + 2hy''_n + 2h^2y'''_n + h^3\left(\frac{5048f_n + 4428f_{n+1} + 1620f_{n+2} + 244f_{n+3}}{8505}\right) + h^4\left(\frac{516f'_n - 2268f'_{n+1} - 918f'_{n+2} - 60f'_{n+3}}{8505}\right) \quad (2.28)$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{9}{2}h^2y'''_n + h^3\left(\frac{3285f_n + 4374f_{n+1} + 2187f_{n+2} + 234f_{n+3}}{2240}\right) + h^4\left(\frac{351f'_n - 1458f'_{n+1} - 729f'_{n+2} - 54f'_{n+3}}{2240}\right) \quad (2.29)$$

$$y''_{n+1} = y''_n + hy'''_n + h^2\left(\frac{78076f_n + 35127f_{n+1} + 19548f_{n+2} + 3329f_{n+3}}{272160}\right) + h^3\left(\frac{7791f'_n - 33804f'_{n+1} - 12015f'_{n+2} - 822f'_{n+3}}{272160}\right) \quad (2.30)$$

$$y''_{n+2} = y''_n + 2hy'''_n + h^2\left(\frac{h^25731f_n + 7992f_{n+1} + 2943f_{n+2} + 344f_{n+3}}{8505}\right) + h^3\left(\frac{618f'_n - 2700f'_{n+1} - 1404f'_{n+2} - 84f'_{n+3}}{8505}\right) \quad (2.31)$$

$$y''_{n+3} = y''_n + 3hy'''_n + h^2\left(\frac{1206f_n + 2187f_{n+1} + 1458f_{n+2} + 189f_{n+3}}{1120}\right) + h^3\left(\frac{135f'_n - 486f'_{n+1} - 243f'_{n+2} - 36f'_{n+3}}{1120}\right) \quad (2.32)$$

$$y'''_{n+1} = y'''_n + h\left(\frac{34465f_n + 42255f_{n+1} + 12015f_{n+2} + 1985f_{n+3}}{90720}\right) + h^2\left(\frac{3849f'_n - 22977f'_{n+1} - 7263f'_{n+2} - 489f'_{n+3}}{90720}\right) \quad (2.33)$$

$$y'''_{n+2} = y'''_n + h\left(\frac{1115f_n + 2700f_{n+1} + 1755f_{n+2} + 100f_{n+3}}{2835}\right) + h^2\left(\frac{129f'_n - 432f'_{n+1} - 513f'_{n+2} - 24f'_{n+3}}{2835}\right) \quad (2.34)$$

$$y'''_{n+3} = y'''_n + h\left(\frac{465f_n + 1215f_{n+1} + 1215f_{n+2} + 465f_{n+3}}{1120}\right)$$
$$+ + h^2\left(\frac{h^2 57f'_n - 81f'_{n+1} + 81f'_{n+2} - 57f'_{n+3}}{1120}\right) \quad (2.35)$$

3. Analysis of the method

The derived methods are analysed based on order and error constant of the block methods, consistency, zero stability, convergence and domain of stability of the block methods.

3.1. **Local truncation error and order.** The block method is of the general form

$$y(t) = \sum_{i=0}^{3} \alpha_i y_n^i(t_n)h^i + h^4 \sum_{i=0}^{k} \beta_i(t_n) f_{n+i} + h^5 \sum_{i=0}^{k} \beta_i(t_n) f'_{n+i} \quad (3.1)$$

Assuming,

$$y_{n+v} \approx y(t_n + vh), f_{n+j} \equiv (t_n + jh, y(t_n + jh)), f'_{n+v} = \frac{df(t, y(t))}{dt}|_{y=y_{n+v}}^{t=t_{n+v}}$$

and $y(t_n)$ is continuously differentiable arbitrary function on $[a, b]$.
The Local Truncation Error (LTE) of the block methods is defined by the linear operator $L[y(t); h]$ such that

$$L[y(t); h] = y(t_n+ih) - \left(\alpha_1 hy'(t) - \alpha_2 h^2 y''(t) - \alpha_3 h^3 y'''(t) - h^4 \sum_{j=0}^{k} \beta_j(t) f_{n+j} - h^5 \sum_{j=0}^{k} \omega_j f'_{n+j}\right) \quad (3.2)$$

Applying the Taylor series expansion to the right hand side (RHS) of (3.2) at about point t, the method is of order $p = 2k + 2$

$$\begin{aligned} L.T.E.(t_n) &= -y(t) + y_n + \alpha_1 hy'(t) + \alpha_2 h^2 y''(t) + \alpha_3 h^3 y'''(t) + h^4 \sum_{j=0}^{k} \beta_j(t) f_{n+j} \\ &\quad + h^5 \sum_{j=0}^{k} \omega_j f'_{n+j} \\ &\leq C_{p+1} h^{p+1} y_n^{p+1}(t) + \bigcirc\left(h^{(p+1)}\right) \\ &= C_{2k+3} h^{2k+3} y_n^{2k+3}(t) + \bigcirc\left(h^{(2k+4)}\right) \end{aligned} \quad (3.3)$$

The block methods are of uniform order $p = 4$, $p = 6$ and $p = 8$ respectively and with the following error constants.

$$C_{p+4} = \left[\frac{1}{24192}, \frac{1}{5040}, \frac{1}{1440}, \frac{1}{720}\right]^T$$

$$C_{p+4} = \left[\frac{1}{259200}, \frac{1}{14175}, \frac{1}{56700}, \frac{2}{14175}, \frac{1}{17280}, \frac{1}{4725}, \frac{1}{9450}, \frac{1}{4725}\right]^T$$

and

$$C_{p+4} = \left[\frac{3359}{6706022400}, \frac{221}{26195400}, \frac{1053}{27596800}, \frac{89}{39916800}, \frac{1}{62370}, \frac{81}{1724800}, \frac{359}{50803200}, \frac{17}{793800}, \frac{27}{627200}, \frac{313}{25401600}, \frac{13}{793800}, \frac{9}{313600}\right]^T$$

respectively

3.2. **Zero-Stability and Convergence.** The zero stability of a numerical method is established if as $h \to 0$ the solutions remain bounded, it simply implies that the method provide no solutions that

grow unbounded with increase in steps numbers. Considering h $\to 0$ for the derived block methods, the method is expressed in matrix form as

$$A_0 Y_\mu = A_1 Y_{\mu-1}$$

A_0 is the identity matrix of order n, $A_0 = I_n$, and A_1 is a $n \times n$ matrix given by

$$X = \begin{bmatrix} t_{1,1} & t_{1,2} & \cdots & \cdots & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & \cdots & \cdots & \cdots & t_{2,n} \\ t_{3,1} & t_{3,2} & \cdots & \cdots & \cdots & t_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{n,1} & t_{n,2} & \cdots & \cdots & \cdots & t_{n,n} \end{bmatrix}$$

By the above definition, the methods are found to be zero stable after finding the characteristic polynomial of the matrix of the block methods which is as follows $(\lambda-1)^4 = 0, \quad (\lambda-1)^4 \lambda^4 = 0$ and $(\lambda-1)^4\lambda^6 = 0$ respectively.

3.3. Consistency.

Theorem: The convergence of the methods with respect to properties discussed in conjunction from the fundamental theorem of Dahlquist according to Henrici [7]) for multistep methods. Consistency and zero stability are the necessary and sufficient condition for a linear multistep method to be convergent. The above block methods are of order $p = 4, 6, 8$ respectively and they are also zero-stable. Therefore, the methods is said to be convergent.

3.4. **Linear stability.** The stability properties of the discrete fourth derivative block method as obtained from the continuous algorithm in (2.11) for output point values are determined. The region of absolutely stability of the block methods with application on the scalar test problem

$$y' = \lambda y, \quad y'' = \lambda^2 y, \quad y''' = \lambda^3 y, \quad y'^v = \lambda^4 y \tag{3.4}$$

produced the stability polynomial

$$\pi(w,z) \quad = \quad \det\left[\sum_{i=0}^{3} A^{(i)} + z^4 \sum_{i=0}^{k} B^{(i)} w^{k-i} + z^5 \sum_{i=0}^{k} C^{(i)} w^{k-i}\right] \tag{3.5}$$

The one-step block method has the stability polynomial

$$\pi(w,z) \quad = \quad \frac{wz^5}{630} - \frac{wz^4}{168} + w - \frac{z^5}{252} - \frac{z^4}{28} - \frac{z^3}{6} - \frac{z^2}{2} - z - 1 \tag{3.6}$$

and the determinant is now plotted to generate the stability region of the methods. $k = 1, 2$ are $A - Stable$ while $k = 3$ is $A(\alpha) - Stable$.

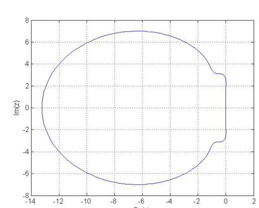

FIGURE 1. Stability Region for the One-Step Direct Block Method (1SDBM)

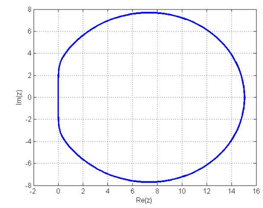

FIGURE 2. Stability Region for the Two-Step Direct Block Method(2SDBM)

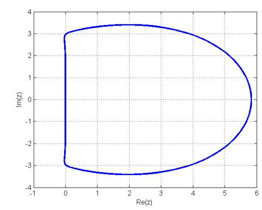

FIGURE 3. Stability Region for the Three-Step Direct Block Method(3SDBM)

4. NUMERICAL APPLICATION

The accuracy of the methods are ascertained by solving following fourth order initial value problems (IVPs).

ACRONYMS
1SDBM - One-Step Direct Block Method
2SDBM - Two-Step Direct Block Method
3SDBM - Three-Step Direct Block Method
E1SDBM - Error in One-Step Direct Block Method
E2SDBM - Error in Two-Step Direct Block Method
E3SDBM - Error in Three-Step Direct Block Method

Problem 1: This problem model the sinusoidal wave of frequency Ω as it passes along a ship or offshore structure.

$$y^{(4)} + 3y'' + y(2 + \epsilon Cos(\Omega t)) = 0$$

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0; \quad y'''(0) = 0; \quad h = \frac{1}{320}$$

The theoretical solution is known

$$y(t) = 2Cos(t) - Cos(t\sqrt{2})$$

Problem 2: An inhomogeneous nonlinear problem

$$y^{(4)} - y^2 = Cos^2(t) + Sin(t) - 1$$

$$y'(0) = 1, \quad y''(0) = 0; \quad y'''(0) = -1; \quad h = 0.1$$

The exact solution is given by $y(t) = Sin(t)$ The problem is integrated in the interval $[0, 8]$

Problem 3: Consider the nonlinear IVP

$$y^{(4)} - (y')^2 = -4t^2 + e^t(1 - 4t + t^2) + yy'''$$

$$y(0) = 1, y'(0) = 1, \quad y''(0) = 3; \quad y'''(0) = 1; \quad h = \frac{0.1}{32}$$

whose exact solution is $y(t) = t^2 + e^t$

Problem 4:

$$y^{(4))} = -y'' \quad 0 \le t \le \frac{\pi}{2}$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = -\frac{1.1}{72 - 50\pi}, \quad y''(0) = \frac{1}{144 - 100\pi}; \quad y'''(0) = \frac{1.2}{144 - 100\pi}; \quad h = 0.01$$

Exact Solution

$$y(t) = \frac{-t1.2\sin(t) - \cos(t) + 1}{144 - 100\pi}$$

Problem 5: Consider the linear system below, solved by erudite scholar [13]

$$\begin{aligned} y^4 &= e^{3t}u \quad y(0) = 1, y'(0) = -1, y''(0) = 1, y'''(0) = -1, \\ z^4 &= 16e^{-t}y \quad z(0) = 1, z'(0) = -2, z''(0) = 4, z'''(0) = -8, \\ w^4 &= 81e^{-t}z \quad w(0) = 1, w'(0) = -3, w''(0) = 9, w'''(0) = -27, \\ u^4 &= 256e^{-t}yw \quad wu(0) = 1, u'(0) = -4, u''(0) = 16, u'''(0) = -64, \end{aligned}$$

Exact Solution

$$y = e^{-t}, \qquad z = e^{-2t}, \qquad w = e^{-3t}, \qquad u = e^{-4t}$$

The problem is integrated in the interval $[0, 3]$

TABLE 1. Absolute Error Comparison for Problem 1

h	$y - exact$	$E2SBM$ $p = 6$	Familua $et.al.$ [5] $p = 8$
0.003125	0.9999999999920527	0	6.685763E-13
0.006250	0.9999999998728438	0	1.458489E-11
0.00937	0.9999999993562755	1.11022302462515651E-16	1.082968E-10
0.012500	0.9999999979655266	0	3.917803E10
0.015625	0.9999999950330675	0	1.025145E-09
0.018750	0.9999999897006795	1.11022302462515652E-16	2.217319E-09
0.021875	0.9999999809194795	0	4.226068E09
0.025000	0.9999999674499512	0	7.358019E-09
0.028125	0.9999999478619811	1.11022302462515653E-16	1.196868E-08
0.031250	0.999999920534901	1.11022302462515654E-16	1.846249E-08

TABLE 2. Absolute Error Comparison for Problem 2

h	$y - exact$	$E1SBM$ $p=4$	$E2SBM$ $p=6$	$E3SBM$ $p=8$
0.1	0.09983341664682815	1.9290125052862095E-14	1.3877787807814457E-17	1.3877787807814457E-17
0.2	0.19866933079506122	4.581057755359552E-13	0	2.7755575615628914E-17
0.3	0.29552020666133955	3.119782210347921E-12	5.5511151231257834E-17	5.5511151231257843E-17
0.4	0.3894183423086505	1.2561951479028721E-11	1.66533453693773481E-16	5.5511151231257833E-17
0.5	0.479425538604203	3.7438385724897216E-11	5.5511151231257832E-16	5.5511151231257834E-17
0.6	0.5646424733950354	9.180267657171726E-11	1.2212453270876722E-15	0
0.7	0.644217687237691	1.963581519603963E-10	2.55351295663786E-15	1.1102230246251565E-17
0.8	0.7173560908995228	3.796469805195102E-10	4.6629367034256575E-15	1.11022302462515652E-16
0.9	0.7833269096274834	6.7916883228491544E-10	8.326672684688674E-15	1.11022302462515651E-16
1.0	0.8414709848078965	1.142425487543619E-09	1.3877787807814457E-14	3.3306690738754696E-16

TABLE 3. Absolute Error Comparison for Problem 3

h	$E1SBM$ $p=4$	$E2SBM$ $p=6$	Kuboye $et.al.$ [10] $p=7$	Familua $et.al.$ [5] $p=7$
0.103125	0	2.220446049250313E-16	1.7878077E-10	9.02145880E-10
0.206250	2.220446049250313E-16	2.220446049250313E-16	1.1337991E-08	1.216821428E-09
0.306250	2.220446049250313E-16	8.8817841970012522E-16	1.1962766E-07	1.21681228E-09
0.406250	0	1.5543122344752192E-15	6.4013462E-07	1.713796095E-09
0.506250	2.220446049250313E-16	3.1086244689504383E-15	2.3491017E-06	1.481970916E-08
0.603125	8.8817841970012522E-16	4.884981308350689E-15	6.5726253E-06	3.058338503E-08
0.703125	1.33226762955018780E-15	7.549516567451064E-15	1.6100258E-05	4.941858156E-08
0.803125	3.1086244689504383E-15	1.1102230246251565E-14	3.5007632E-05	7.128679089E-08
0.903125	5.329070518200751E-15	1.554312234475219E-14	6.8384810E-05	1.058773080E-08

TABLE 4. Absolute Error Comparison for Problem 4

h	$E3SBM$ $p=8$	Gebremedhin $et.al.$ [6] $p=12$
0.1	2.168404344971009E-19	6.5052E-19
0.2	0	1.3010 E-19
0.3	8.673617379884035E-19	4.3368 E-19
0.4	0	1.6348 E-18
0.5	8.673617379884035E-19	1.8431E-18
0.6	8.673617379884035E-19	1.9753 E-18
0.7	8.673617379884035E-19	1.8347E-18
0.8	1.734723475976807E-18	1.7346 E-18
0.9	0	2.3214 E-18
1.0	0	3.4695 E-18

TABLE 5. Absolute Error Comparison for Problem 5

h	$y - exact$	$y - E3SBM$ $p = 8$	Omole $et.al.$ [13] (2020) $p = 10$
0.1	0.9048374180359595	1.1102230246251565E-17	0
0.2	0.8187307530779818	1.1102230246251565E-17	1.110223E-16
0.3	0.7408182206817179	0	1.110223E-16
0.4	0.6703200460356393	0	0
0.5	0.6065306597126334	1.11022302462515651E-17	1.110223E-16
0.6	0.5488116360940265	1.11022302462515652E-17	0
0.7	0.4965853037914095	1.11022302462515652E-16	1.110223E-16
0.8	0.44932896411722156	1.11022302462515651E-16	3.330669E-16
0.9	0.4065696597405991	1.11022302462515654E-16	3.885781E-16
1.0	0.36787944117144233	1.11022302462515653E-16	5.551115E-16

TABLE 6. Absolute Error Comparison for Problem 5

h	$z - exact$	$z - E3SBM$ $p = 8$	Omole $et.al.$ [13] (2020) $p = 10$
0.1	0.8187307530779818	0	3.885781E-15
0.2	0.6703200460356393	1.1102230246251565E-16	5.662137E-14
0.3	0.5488116360940264	0	2.287059E-13
0.4	0.44932896411722156	5.5511151231257830E-17	5.896394E-13
0.5	0.36787944117144233	1.11022302462515650E-16	1.208700E-12
0.6	0.301194211912202	5.5511151231257830E-17	2.155331E-12
0.7	0.24659696394160643	1.38777878078144570E-16	1.110223E-12
0.8	0.20189651799465538	1.66533453693773480E-16	5.309170E-12
0.9	0.16529888822158653	2.2204460492503130E-16	7.655349E-12
1.0	0.1353352832366127	2.77555756156289140E-16	1.060740E-11

TABLE 7. Absolute Error Comparison for Problem 5

h	$w - exact$	$w - E3SBM$ $p = 8$	Omole $et.al.$ [13] (2020) $p = 10$
0.1	0.7408182206817179	0	1.112999E-12
0.2	0.5488116360940264	0	1.596079E-11
0.3	0.40656965974059905	5.5511151231257830E-17	6.451839E-11
0.4	0.301194211912202	5.5511151231257830E-17	1.662694E-10
0.5	0.22313016014842982	0	3.408267E-10
0.6	0.1652988882215865	5.5511151231257830E-17	6.077662E-10
0.7	0.1224564282529819	5.5511151231257830E-17	9.866660E-10
0.8	0.09071795328941247	1.11022302462515605E-16	1.497135E-09
0.9	0.06720551273974976	1.3877787807814457E-16	2.158671E-09
1.0	0.049787068367863944	1.8041124150158794E-16	2.991185E-09

TABLE 8. Absolute Error Comparison for Problem 5

h	$u - exact$	$u - E3SBM$ $p = 8$	Omole $et.al.$ [13] (2020) $p = 10$
0.1	0.6703200460356393	1.1102230246251565E-16	5.409717E-11
0.2	0.44932896411722156	5.551115123125783E-17	7.753621E-10
0.3	0.301194211912202	5.551115123125783E-17	3.134138E-09
0.4	0.20189651799465538	2.7755575615628914E-17	8.077239E-09
0.5	0.1353352832366127	2.7755575615628914E-17	1.655765E-08
0.6	0.09071795328941247	1.1102230246251565E-16	2.952660E-08
0.7	0.06081006262521795	1.1102230246251565E-16	4.793539E-08
0.8	0.04076220397836621	1.249000902703301E-16	7.273677E-08
0.9	0.02732372244729256	1.3530843112619095E-16	1.048785E-07
1.0	0.01831563888873418	1.3183898417423734E-16	1.453268E-07

5. Conclusions and Discussions

The obtained data when the block methods are applied to problem 1 - 5 are presented in Tables 1-8. The new method is capable to to solve fourth-order scalar and vector problems. The numerical results are compared with the results of some existing method proposed by Familua and Omole [5], Kuboye and Omar [10], Gebremedhin and Jena [6], and Omole and Ukpebor [13].

The newly derived Block methods (BM) is applied to solve fourth-order IVPs in ordinary differential equations directly. The flexibility of the method and the easy to derive it shown it can be applied to solve diverse kinds of fourth-order IVPs which can be seen in the numerical examples solved. The application of these methods to real life problems have shown that it is suitable. It shows a high level of accuracy when the numerical results is compared to the exact solution and a very good performance in comparison with existing methods in the cited literature. Hence, it might be considered for solving fourth-order IVPs of ODEs.

6. Acknowledgements

The authors would like to thank the anonymous reviewers for their careful reading of the manuscript.

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