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# ON NORMALIZER AND CYCLIC SOFT MULTIGROUPS

J. A. OTUWE*, I. A. ONYEOZILI*, M. M. MOGBONJU* AND D. A. OLUYORI †

ABSTRACT. The theory of soft multigroup is an extension of soft group theory. The study of soft multigroups drawn from soft group and soft multisets is an awakening research domain in non classical group. This paper establishes and investigates some properties of normalizer of a soft group in the framework of soft multigroup. Also, the notion of cyclic soft multigroup is proposed and some of its properties are examined.

## 1. INTRODUCTION

Multiset theory as a notion is an important generalization of classical set theory which has emerged by violating a basic property of classical set that an element can belong to a set only once. Multiset is otherwise known as heap, list, sample, bag, fire sets and weighted set etc. Molodtsov in [1] initiated a novel idea of soft set theory which is completely a new approach for modeling vagueness and uncertainty. Soft set theory has a very rich potential for applications in several directions few of which has been indicated in [1]. Some authors have also generalized the concept of multisets in the setting of soft set [16] to form soft multisets [15, 24]. The term multigroup was first mentioned by Dresher and Ore (1938) and defined as an algebraic system that satisfied all the axioms of group except that the multiplication is multi-valued. This definition is neither in conformity with the notion of multisets nor in alignment with other non-classical groups (as in fuzzy group, intuitionistic fuzzy group, soft group, fuzzy soft group, soft multigroup, fuzzy multigroup and intuitionistic fuzzy multigroup). Nazmul et al [4] proposed the concept of multigroup as an algebraic structure of multisets that extended the concept of group. The idea is consistent with other standard groups in [2, 3, 4, 22, 23] etc. The idea of multigroup by Nazmul et al in [4], is well accepted due to the fact that it agrees with other non-classical groups and defined over multiset. (see [19, 20]) for details of multigroup. Again, the theory of group is one of the most important algebraic structures in modern mathematics. (Aktas and Cagman, 2007) introduced the notion of soft groups as a parameterized family of subgroups following the pattern and properties derived by Molodstsov's in (1999). Several

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*Corresponding author.

authors have introduced the idea of group theory in fuzzy sets, soft set, fuzzy soft set, soft group, fuzzy multigroup among others. see [16, 1, 17, 24]. It's natural to introduce the properties of group structures in soft multiset frame works and examine some of their basic properties.

## 2. Preliminaries

In this section, we present some existing definitions that are useful in the subsequent sections

**Definition 2.1.** [1] (**Soft Set**). A pair $(F, A)$ is called a soft set over $X$ where $F$ is a mapping given by $F : A \to P(X)$ and $A \subseteq E$.

**Definition 2.2.** [1] (**Union of Soft Sets**). Let $(F, A)$ and $(G, B)$ be two soft sets over $X$. Then their union is a soft set $(H, C)$ over $X$ where $C = A \cup B$ and for all $\alpha \in C$,

$$\mathrm{H}(\alpha) = \begin{cases} F(\alpha) \text{ if } \alpha \in A/B \\ G(\alpha) \text{ if } \alpha \in B/A \\ F(\alpha) \cup G(\alpha) \text{ if } \alpha \in A \cap B. \end{cases}$$

This relationship is written as $(F, A) \,\widetilde{\cup}\, (G, B) = (H, C)$.

**Definition 2.3.** [1] (**Intersection of Soft Set**). Let $(F, A)$ and $(G, B)$ be two soft sets over $X$. Then their intersection is a soft set $(H, C)$ over $X$ where $C = A \cap B$ and for all $\alpha \in C$, $H(\alpha) = F(\alpha) \cap G(\alpha)$. This relationship is written as $(F, A) \ \wedge \ (G, B) = (H, C)$.

**Definition 2.4.** [15] (**Soft Subset**). Let $(F, A)$ and $(H, B)$ be two soft sets over a common universe $U$, then we said that $(H, B)$ is a soft subset of $(F, A)$ if

i $B \subseteq A$ and
ii $H(e) \subseteq F(e) \; \forall \; e \; \in \; B$

We write $(H, B) \,\widetilde{\subset}\, (F, A)$, $(H, B)$ is said to be a soft super set of $(F, A)$, if $(F, A)$ is a subset of $(H, B)$ we denote it by $(H, B) \,\widetilde{\supset}\, (F, A)$.

**Definition 2.5.** [15] (**Equal Soft Set**). Two soft sets $(F, A)$ and $(H, B)$ over a common universe $U$ are said to be soft equal if $(F, A)$ is a soft subset of $(H, B)$ and $(H, B)$ is a soft subset of $(F, A)$.

**Definition 2.6.** [15] (**Absolute Soft Set**). A soft set $(F, A)$ over $U$ is said to be absolute soft denoted by $\tilde{A}$ if for all $e \; \in \; A, \; f(e) = U$.

**Definition 2.7.** [15] (**Soft Group**). Let $G$ be a group and $E$ be a set of parameters. For $A \subseteq E$, the pair $(F, A)$ is called a soft group over $G$ if and only if $F(\alpha) \leq G$ for all $\alpha \in A$ where $F$ is a mapping of $A$ into the set of subset of $G$.

**Definition 2.8.** [15] (**Soft Subgroup**). Let $(F, A)$ and $(H, K)$ be two soft group over $G$. Then $(H, K)$ is a soft subgroup of $(F, A)$ written as $(H, K) \,\overline{<}\, (A, A)$ if

i. $K \subseteq A$,
ii. $H(x) \leq F(x) \; \forall \; x \; \in \; K$

**Definition 2.9.** [9] (**Multiset**). Let $X$ be a set. A multiset $A$ is characterized by a count function

$$C_A(x) : X \to \mathbb{N}.$$

Such that for $x \in Dom(A)$ implies $A(x) = C_A(x) > 0$, where $C_A(x)$ denotes the number of times an object $x$ occurs in $A$. Whenever $C_A(x) = 0$ implies $x \notin Dom(A)$ the set of all multiset is denoted by $MS(X)$. The root of a multiset $A$, denoted by $A$, is defined as $A = \{x \in A : A(x) > 0\}$.

**Definition 2.10.** [9] (**Submultiset**). A multiset $A$ is called a submultiset or a msubset of a multiset $B$ denoted by $A \subseteq B$, if $C_A(x) \leq C_B(x)$, for all $x$.

$A$ is a proper submultiset of $B$ $(A \subset A)$ if $C_A(x) \leq C_B(x)$ for all $x$ and there exists at least one $x$ such that $C_A(x) < C_B(x)$.

**Definition 2.11.** [24] (**Equal Multisets**). Two multisets $M_1$ and $M_2$ are equal $(M_1 = M_2)$ if $(M_1 \subseteq M_2)$ and $(M_2 = M_1)$.

**Definition 2.12.** [24] (**Intersection of Multisets**). The intersection of two multisets $M_1$ and $M_2$ drawn from a set $X$ is an mset $M$ denoted by $M = M_1 \cap M_2$ such that for all $x \in X, C_M(x) = min\{C_{M_1}(x), C_{M_2}(x)\}$

**Definition 2.13.** [24] (**Union of Multisets**). The union of two multisets $M_1$ and $M_2$ drawn from a set $X$ is an mset $M$ denoted by $M = M_1 \cap M_2$ such that for all $x \in X$, $C_M(x) = max\{C_{M_1(x)}, C_{M_2(x)}\}$.

**Definition 2.14.** [24] (**Sum of Multisets**). The addition or sum of two multisets $M_1$ and $M_2$ drawn from a set $X$ results in a new multiset $M = M_1 \bigoplus M_2$ such that for all $x \in X$, $C_M(x) = C_{M_1}(x) + C_{M_2}(x)$.

**Definition 2.15.** [5] (**Soft Multiset**). Let $U$ be a universal multiset and $E$ be a set of parameters. Then an ordered pair $(F, E)$ is called a soft multiset where $F$ is a mapping given by $F : A \to PW(U)$.

**Definition 2.16.** [5] (**Equal Soft Multisets**). $(F, A)$ and $(G, B)$ over a common universe $U$ are said to be equal if $(F, A)$ is a soft multi subset of $(G, B)$ and $(G, B)$ is a soft multi subset of $(F, A)$.

**Definition 2.17.** [5] (**Intersection of Soft Multisets**). The intersection of $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft multiset $(H, C)$ where $C = A \cap B$ and for all $e \in C$,

$$\text{H(e)} = \begin{cases} F(e) \text{ if } e \in A - B \\ G(e) \text{ if } e \in B - A \\ F(e) \cup G(e) \text{ if } e \in A \cap B. \end{cases}$$

**Definition 2.18.** [5] (**Union of Soft Multiset**). The union of $(F, A)$ and $(G, B)$ over a common universe $U$ is the soft multiset $(H, C)$ where $C = A \cup B$ for all $e \in C$,

$$\text{H(e)} = \begin{cases} F(e) \text{ if } e \in A - B \\ G(e) \text{ if } e \in B - A \\ F(e) \cup G(e) \text{ if } e \in A \cap B. \end{cases}$$

**Definition 2.19.** [18] (**Multigroup**). Let $X$ be a group. A multiset $G$ over $X$ is said to be a multigroup over $X$ if the count function $G$ or $C_G$ satisfies the following two conditions:

i. $C_G(xy) \geq [C_G(x) \bigwedge C_G(y)],\ \forall\, x,\ y \in\ X$,
ii. $C_G(x^{-1}) \geq C_G(x),\ \forall\, x \in\ X$,

i.e., a multiset $G$ is called a multigroup over $X$ if $C_G(xy^{-1}) \geq [C_G(x) \wedge C_G(y)],\ \forall\, x,\ y \in X$. The set of all multigroups defined over $X$ is denoted by $MG(X)$.

**Definition 2.20.** [18] (**Intersection and Union of Multigroup**). Let $\{A_i\}_{i \in I}, I = 1, 2, \cdots, n$ be an arbitrary family of multigroups of a group $X$. Then

$$C_{\cap A_i}(x) = \bigwedge C_{A_i}(x)\ x \in X.$$
$$C_{\cup A_i}(x) = \bigvee C_{A_i}(x)\ x \in X.$$

**Definition 2.21.** [18] (**Abelian Multigroup**). Let $A \in MG(X)$. Then $A$ is said to be abelian or commutative if for all $x,\ y \in X,\ C_A(xy) = C_A(yx)$.

**Definition 2.22.** [18] (**Submultigroup**). Let $A \in MG(X)$. A submultiset $B$ of $A$ is called a submultigroup of $A$ denoted by $B \subseteq A$ if $B$ is a multigroup. A submultigroup $B$ of $A$ is a proper submultigroup denoted by $B \subset A$, if $B \subseteq A$ and $A \neq B$.

**Definition 2.23.** [2] (**Normal Submultigroup**). Let $A,\ B \in MG(X)$ such that $A \subseteq B$. Then $A$ is called a normal submultigroup of $B$ if

$$C_A(xyx^{-1}) \geq C_A(y)\ \forall\ x,\ y \in X.$$

In general, $C_A(xyx^{-1}) = C_A(y)\ \forall\ x,\ y \in X$.

**Definition 2.24.** [18] (**Soft Multigroup**). Let $X$ be a group, $M$ be an mgroup over $X$ and $A \subseteq E$ be a set of parameters. A soft Mset $(F, A)$ drawn from $M$ is said to be a soft multigroup (shortly soft Mgroup) over $M$ if and only if $F(\alpha)$ is a submultigroup of $M$, for all $\alpha$ in $A$.

**Definition 2.25.** [18] (**Soft Submultigroup**). Let $(F_1, A_1)$ and $(F_2, A_2)$ be two soft multigroups over $M$. Then $(F_1, A_1)$ is said to be a soft submgroup of $(F_2, A_2)$ denoted by $(F_1, A_1)\tilde{\leq}(F_2, A_2)$ if $A_1 \subseteq A_2$ and $F_1(\alpha)$ is a submultigroup of $F_2(\alpha), \forall\ \alpha \in A_1$.

**Definition 2.26.** [18] (**Soft Abelian Multigroup**). A soft multigroup $(F, A)$ over a multigroup $M$ of a group $X$ is called a soft abelian multigroup if $F(\alpha)$ is an abelian submgroup of $M, \forall\ \alpha \in A$.

**Definition 2.27.** [18]. Let $(F, A)$ be a soft multigroup over $M$. Then

(i) $(F, A)$ is said to be an identity soft multigroup over $M$ if $F(\alpha) = [C_M(e)]_e\ \forall\ \alpha \in A$, where $e$ is the identity element of $X$.
(ii) $(F, A)$ is said to be an absolute soft multigroup over $M$ if $F(\alpha) = M,\ \forall\ \alpha \in A$.

## 3. Concept of Normalizer and Cyclic Soft Multigroup

In this section the concept of normalizer of a soft multigroup and cyclic soft multigroups is studied and some of its properties is elaborated.

**Definition 3.1. Normalizer of a Soft Submultigroup** Let $(F, A)$ and $(G, B)$ be a soft multigroups over $X$ such that $(F, A)$ is a soft submultigroup of $(G, B)$. Then define the normalizer of $(F, A)$ in $(G, B)$ denoted as $N(F, A)$ as the set given by

$$N(F, A) = \{\alpha \in A, c \in M | C_{F,(\alpha)}(cd) = C_{F(\alpha)}(dc)\ \forall\ d \in M\}.$$

We now note that, $N(F, A) = \{\alpha \in A, c \in M | C_{F,(\alpha)^c}(d) = C_{F(\alpha)}(d)\ \forall\ d \in M\}$.

**Example 3.2.** Let $X = \{\pm 1, \pm i, j, \pm k\}$ be quaternion group and $M = [\pm 1, \pm i, j, \pm k]_{6,5,4,4,5,5,4,4}$. Suppose $E = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \cdots\}$ and $A = \{\alpha_1, \alpha_2\}$ such that

$$F(\alpha_1) = [\pm 1, \pm i, j, \pm k]_{4,3,4,4,3,3,4,4}$$
$$F(\alpha_2) = [\pm 1, \pm i, j, \pm k]_{3,3,2,2,3,3,2,2}$$

Then from the table below,

| $*$ | $1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ | $k$ | $-k$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $-1$ | $i$ | $-i$ | $j$ | $-j$ | $k$ | $-k$ |
| $-1$ | $-1$ | $1$ | $-i$ | $i$ | $-j$ | $j$ | $-k$ | $k$ |
| $i$ | $i$ | $-i$ | $-1$ | $1$ | $k$ | $-k$ | $-j$ | $j$ |
| $-1$ | $-i$ | $i$ | $1$ | $-1$ | $-k$ | $k$ | $j$ | $-j$ |
| $j$ | $j$ | $-j$ | $-k$ | $k$ | $-1$ | $1$ | $i$ | $-i$ |
| $-j$ | $-j$ | $j$ | $k$ | $-k$ | $1$ | $-1$ | $-i$ | $i$ |
| $k$ | $k$ | $-k$ | $j$ | $-j$ | $-i$ | $i$ | $-1$ | $1$ |
| $-k$ | $-k$ | $k$ | $-j$ | $j$ | $i$ | $-i$ | $1$ | $-1$ |

TABLE 1. Multiplication table of the Quaternion Group, $Q_8$.

Therefore, $N(F, A) = \{\alpha([1, -1])\}$.

**Proposition 3.1.** *Let $(G, B)$ be a soft multigroup over $M$ and $(F, A)$ be a soft submultigroup of $(G, B)$. Then the following assertion holds;*

i. *$N(F, A)$ is a soft subgroup of $X$*
ii. *$(F, A)$ is a normal soft submultigroup of $(G, B)$ if and only if $N(F, A) = X$.*

*Proof.* i. Let $a,\ c \in N(F, A)$ then $\forall\ x \in X$ and $\forall\ \alpha \in A$, we have $C_{F(\alpha)^{ac}}(x)$ since $C_{F(\alpha)^a}(x) = C_{F(\alpha)}(a^{-1}xa) = C_{F(\alpha)}(x)$. Hence $a,\ c \in N(F, A)$. Also, let $a \in N(F, A)$ we now show that $a^{-1} \in N(F, A)$. Now for all $y \in X$, $C_{F(\alpha)}(ay) = C_{F(\alpha)}(ya)$ and so, $C_{F(\alpha)}((a)) = C_{F(\alpha)}((ya)^{-1})$. Thus for all $y \in X$, $C_{F(\alpha)}(y^{-1}a^{-1}) = C_{F(\alpha)}(a^{-1}y^{-1})$ and so $C_{F(\alpha)}(ya^{-1}) = C_{F(\alpha)}(ya^{-1})$ since $C_{F(\alpha)}(y) = C_{F(\alpha)}(y^{-1})$. Thus, $a^{-1} \in N(F, A)$. Hence, $N(F, A)$ is a soft subgroup of $X$.

ii. Let $(F, A)$ be a normal soft submultigroup of $(G, B)$ and $a \in X$, then we have $C_{F(\alpha)^a}(x) = C_{F(\alpha)}(a^{-1}xa) = C_{F(\alpha)}((a^{-1}x)a) = C_{F(\alpha)}(x)$. Thus $C_{F(\alpha)^a}(x) = C_{F(\alpha)}(x)$ and so, $a \in N(F, A)$. Therefore, $N(F, A) = X$.

Conversely, assume that $N(F, A) = X$, then let $x, y \in X$, to show that $(F, A)$ is normal, its sufficient to show that $\forall \alpha \in A$, $C_{F(\alpha)}(xy) = C_{F(\alpha)}(xyxx^{-1}) = C_{F(\alpha)}(x(yx)x^{-1}) = C_{F(\alpha)^{x^{-1}}}(yx) = C_{F(\alpha)}(yx)$. The last equality hold immediately since $N(F, A) = X$. And so $x^{-1} \in N(F, A)$. Hence, $C_{F(\alpha)^{x^{-1}}}(y) = C_{F(\alpha)}(y)$ (that is, $F(\alpha)^{x^{-1}} = F(\alpha)^x$). Therefore, $(F, A)$ be a normal soft submultigroup of $(G, B)$.

□

**Proposition 3.2.** *Let $(F, A)$ and $(G, B)$ be soft multigroups over an abelian multigroup $M \in SMG(X)$ such that $(F, A) \subseteq (G, B)$, then $N(F, A) \cap N(G, B) \subseteq N[(F, A) \cap (G, B)]$.*

*Proof.* Let $c \in N(F, A)$ and $c \in N(G, B) \Rightarrow c \in N(F, A) \cap N(G, B)$. Now, for $a, b \in X$, we have $C_{(F,A)\cap(G,B)}(aba^{-1}) = C_{(F,A)\cap(G,B)}(b)$.

Now,

$$\begin{aligned} C_{(F,A)\cap(G,B)}(aba^{-1}) &= C_{(F,A)}(aba^{-1}) \wedge C_{(G,B)}(aba^{-1}) \\ &= C_{(F,A)}(baa^{-1}) \wedge C_{(G,B)}(baa^{-1}) \\ &= C_{(F,A)}(b) \wedge C_{(G,B)}(b) \\ &= C_{(F,A)\cap(G,B)}(b) \end{aligned}$$

Therefore, $b \in N[(F,A) \cap (G,B)]$. Hence the result. □

**Proposition 3.3.** *Let $(F,A)$ and $(G,B)$ be two soft multigroups over $X$ such that $(F,A) \subseteq (G,B)$ and $\forall\ \alpha \in A$ and $\forall\ \beta \in B$. If $C_{F(\alpha)}(e) = C_{G(\beta)}(e)$, then $N(F,A) \cap N(G,B) = N[(F,A) \cap (G,B)]$.*

*Proof.* We recall from the definition of normalizer that

$$\begin{aligned} N(F,A) &= \{\alpha \in A, c \in M | C_{F(\alpha)}(cd) = C_{F(\alpha)}(dc) \forall d \in M\} \\ &= \{\alpha \in A, c \in M | C_{F(\alpha)}(cdc^{-1}d^{-1}) = C_{F(\alpha)}(e) \forall d \in M\} \end{aligned}$$

Now, let $\alpha \in N[(F,A) \cap (G,B)]$. Then from definition 4.1 for $b \in X$ we have

$$\begin{aligned} C_{F(\alpha)\cap G(\beta)}(aba^{-1}b^{-1}) &= C_{F(\alpha)}(aba^{-1}b^{-1}) \wedge C_{G(\beta)}(aba^{-1}b^{-1}) \\ &= C_{F(\alpha)}(e) \wedge C_{G(\alpha)}(e) \end{aligned}$$

This implies that $\alpha \in N(F,A)$ and $a \in N(G,B)$. Thus $a \in N[(F,A) \cap (G,B)]$ since $C_{F(\alpha)}(aba^{-1}b^{-1}) = C_{F(\alpha)}(e) \Rightarrow C_{F(\alpha)}(ab) = C_{F(\alpha)}(ba)$.
Similarly, in the case of $(G,B)$ from the fact that $C_{F(\alpha)}(e) = C_{G(\beta)}(e)$. Hence the result. □

**Proposition 3.4.** *Let $(F,A)$ and $(G,B)$ be soft multigroup over $X$. Then $N(F,A) \cap N(G,B) \subseteq N[(F,A) \diamond (G,B)]$.*

*Proof.* Let $\alpha \in [N(F,A) \cap N(G,B)]$, that is $a \in N(F,A)$ and $a \in N(F,A)$ and $a \in N(G,B)$. Now for all $b \in X$, we have

$$\begin{aligned} C_{(F,A)\diamond(G,B)}(a) &= \vee_{a=xy} \left(C_{F(\alpha)}(x) \wedge C_{G(\beta)}(y)\right) \forall x,y \in X,\ for\ every\ \alpha\ \in\ A\ and\ \beta\ \in\ B \\ &= \vee_{a=xy} \left(C_{F(\alpha)}(b^{-1}xb) \wedge C_{G(\beta)}(b^{-1}yb)\right)\ \forall\ x,\ y\ \in\ X \\ &\leq V_{b^{-1}ab=cd}(C_{F(\alpha)}(c) \wedge C_{G(\beta)}(d))\ \forall\ c,\ d\ \in\ X \\ &= C_{(F,A)\diamond(G,B)}(b^{-1}ab) \\ &\Rightarrow C_{(F,A)\diamond(G,B)}(a) \leq C_{(F,A)\diamond(G,B)}(b^{-1}ab). \end{aligned}$$

The inequality hold since from the fact that $a = xy$ which implies that $b^{-1}xyb = cd \Rightarrow xy = bcdb^{-1} = (bcb^{-1})(bdb^{-1})$.

Since $x = bcb^{-1}$ and $y = bdb^{-1}$ it implies that $b^{-1}xb = c$ and $b^{-1}yb = d$.
Also, $C_{(F,A)\diamond(G,B)}(b^{-1}ab) \leq C_{(F,A)\diamond(G,B)}(b^{-1}(b^{-1}ab)b) = C_{(F,A)\diamond(G,B)}(a)$
and so, $C_{(F,A)\diamond(G,B)}(a) \geq C_{(F,A)\diamond(G,B)}(b^{-1}ab)$.
Thus, $C_{(F,A)\diamond(G,B)}(a) = C_{(F,A)\diamond(G,B)}(b^{-1}ab)$.
Therefore, $a \in N[(F,A) \diamond (G,B)]$ and so $N(F,A) \cap N(G,B) \subseteq N[(F,A) \diamond (G,B)]$. □

**Definition 3.3. Cyclic Soft Multigroup** Let $X = \langle a \rangle$ be a group generated by $a$. If $(F,A) = \{\alpha([a^n]C_{F(\alpha)}(x))/n \in \mathbb{N},\ \forall\ \alpha\ \in\ A\}$ is a multigroup. Then $(F,A)$ is called a *cyclic soft multigroup* generated by $[a]C_{F(\alpha)}(x)$ depicted by $\langle [a]C_{F(\alpha)}(x) \rangle\ \forall\ \alpha\ \in\ A$. The element $a$ is then called the generator of $(F,A)$ otherwise, a non-generator of $(F,A)$.

**Example 3.4.** Let $\langle i \rangle$ that is $X = \{\pm 1, \pm i\}$ and $A = \{\alpha_1,\ \alpha_2\}$.

Suppose $F(\alpha_1) = [\pm 1, \pm i]_{6,5,4,4}$ and $F(\alpha_2) = [\pm 1, \pm i]_{4,4,3,3}$.

Now using $\{[a^n]C_{F(\alpha)}(x)/n\ \in\ \mathbb{N}\}$. For $n = 1$,

$$F(\alpha_1) = \{i^4, -1^5, -i^4, 1^6\} = [\pm 1, \pm i]_{6,5,4,4}$$

Similarly, for $F(\alpha_2) = \{i^3, -1^4, -i^3, 1^4\} = [\pm 1, \pm i]_{4,4,3,3}$.

Therefore, $(F, A)$ is a cyclic soft multigroup.

**Proposition 3.5.** *Let $(F, A)^i$ and $(F, A)^j$ be cyclic soft multigroups over $M$ and $i \leq j$, then $(F, A)^i \cup (F, A)^j$ is also a cyclic soft multigroup over $M$ for any $i, j \in \mathbb{Z}^+$.*

*Proof.* We proof this by considering the count function and without lost of generality $i \leq j$. Since $(F, A)^i \subseteq (F, A)^j$ then we have

$$\begin{aligned} C_{F^i(\alpha)\cup F^j(\alpha)}(a^n a^m) &= C_{F^i(\alpha)}(a^n a^m) \vee C_{F^j(\alpha)}(a^n a^m) \\ &= C_{F^j(\alpha)}(a^n a^m) \geq C_{F^j(\alpha)}(a^n) \wedge C_{F^j(\alpha)}(a^n a^m) \\ &= C_{F^i(\alpha)\cup F^j(\alpha)}(a^n) \wedge C_{F^i(\alpha)\cup F^j(\alpha)}(a^m) \\ and\ C_{F^i(\alpha)\cup F^j(\alpha)}(a^{-n}) &= C_{F^i(\alpha)}(a^{-n}) \vee C_{F^j(\alpha)}(a^{-n}) \\ &= C_{F^i(\alpha)}(a^n) \vee C_{F^j(\alpha)}(a^n) \\ &= C_{F^i(\alpha)\cup F^j(\alpha)}(a^n)\ \forall\ F^i(\alpha)\ \in\ A and F^j(\alpha)\ \in\ B. \end{aligned}$$

Hence the result ☐

**Proposition 3.6.** *Let (F,A) be a cyclic soft multigroup over M. Then $(F, A) \subseteq (F, A)^2 \subseteq (F, A)^3 \subseteq \cdots \subseteq (F, A)^n \subseteq \cdots \subseteq \varepsilon$.*

*Proof.* Since $\forall\ \alpha \in A, x \in M$, we have $C_{F(\alpha)}(x) \in D$. Hence, $C_{F(\alpha)}(x) \leq \left(C_{F(\alpha)^2}(x)\right)^2$, $C_{F(\alpha)}(x^2) \leq \left(C_{F(\alpha)^2}(x^2)\right)^2, \cdots C_{F(\alpha)}(x^n) \leq \left(C_{F(\alpha)^2}(x^n)\right)^2$. By definition 2.25, we have $(F, A) \subseteq (F, A)^2$. By generalizing it for $i, j \in \mathbb{Z}^+$ with $i \leq j$ we obtain $\left(C_{F(\alpha)^i}(x)\right)^i \leq \left(C_{F(\alpha)^j}(x)\right)^j$,
$\left(C_{F(\alpha)^i}(x^2)\right)^i \leq \left(C_{F(\alpha)^j}(x^2)\right)^j, \cdots \left(C_{F(\alpha)^i}(x^n)\right)^i \leq \left(C_{F(\alpha)^j}(x^n)\right)^j$. So $(F, A)^i \subseteq (F, A)^j$ for any $i, j \in \mathbb{Z}^+$ with $i \leq j$, which indicates that $(F, A) \subseteq (F, A)^2 \subseteq (F, A)^3 \subseteq \cdots \subseteq (F, A)^n$ ☐

**Proposition 3.7.** *Every frattini soft submultigroup of a cyclic soft multigroup is abelian.*

*Proof.* Let $(F, A)$ be a cyclic soft multigroup over $M$ and $\phi(F, A)$ be the Frattini soft submultigroup of $(F, A)$. Now since $(F, A)$ is cyclic, we have $(F, A) = \{\alpha([a^n]C_{F(\alpha)}(x))/n \in \mathbb{N},\ \forall\ \alpha \in A\}$. From the fact that $\phi(F, A)$ is the Frattini soft submultigroup of $(F, A)$, then there exist an element $c \in M$ such that for every $a, b \in M$, we have

$$\{\alpha([nc]C_{\phi(F,A)}(a))\} = C_{\phi(F,A)}(nc) = C_{\phi(F,A)}(a)$$

and

$$\{\alpha([mc]C_{\phi(F,A)}(a))\} = C_{\phi(F,A)}(mc) = C_{\phi(F,A)}(b)\ for\ every\ n,\ m\ \in\ \mathbb{N}\ and\ \forall\ \alpha\ \in\ A.$$

Now it follows that

$$\begin{aligned} C_{\phi(F,A)}(a+b)) &= C_{\phi(F,A)}(nc+mc) \\ &= C_{\phi(F,A)}c(n+m) = C_{\phi(F,A)}c(m+n) \\ &= C_{\phi(F,A)}(mc+nc) = C_{\phi(F,A)}(b+a). \end{aligned}$$

Therefore, $\phi(F, A)$ is abelian. □

## 4. Conclusions and/or Discussions

The notion of normalizer of a soft multigroup is proposed and some of its related results are outlined. Also, we have explicated the idea of cyclic soft multigroup and a number of related results were obtained. Notwithstanding, more properties of normalizer and cyclic soft multigroups could still be exploited either in the framework of soft multigroup or fuzzy soft multigroup and the idea of cyclic soft multigroup could encourage the establishment of generator and non-generator of a cyclic soft multigroups. Finally the concept of isomorphism of soft multigroups, factor soft multigroups and solvable soft multigroups remain challenging in the frame work of soft multigroups.

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## References

[1] D. Molodtsov. Soft set theory - first results, An International Journal of Computer and Mathematics with Application, 37 (1999), 9 - 31.

[2] P. A. Ejegwa, A. M. Ibrahim. Normal submultigroups and comultisets of a multigroup. Quasigroups and Related Systems, 25(2017), p 231-244.

[3] A. M. Ibrahim, P. A. Ejegwa. Characteristic submultigroups of multigroups. Gulf Journal of Mathematics, 5(4), (2017), p 1 - 8.

[4] S. K. Nazmul, P. Majumdar and S. K. Samantha. On multisets and Multigroups. Annals of fuzzy mathematics and informatics, 6(3), (2003), p 643-656.

[5] I. I. Ahmed. Some properties of Soft Multiset. Science World Journal, 13(4), (2018) p 44 - 46.

[6] K. V. Babitha, J. J. Sunil. On soft multisets. Annal of Fuzzy Mathematics and Informatics, 5(1), (2013) p 35 - 44.

[7] P. Majumdar, S. K. Samanta. Soft Sets and Soft theory. Journal of Mathematics and Computer science, 2(6), (2012), p 1700-1711.

[8] P. K. Maji, R. Biswas, and A. Roy. Soft Set Theory. Comput. Math. App., 45 (2003), p 555-562.

[9] S. Miyamoto. Generalization of Multisets. In: In M. Iniguichi S. Tsumito eds. Rough Set Theory and Granular Computing, Springer, (1987), p 59-68.

[10] W. D. Blizzard. Multiset Theory, Notre Dame Journal of Formal Logic, 30(1989), p 36 - 66.

[11] N. J. Wildberger. A new look at multisets, School of Mathematics, UNSW Sydney, 2052, (2003).

[12] D. Singh, A. M. Ibrahim, T. Yohanna and J. N. Singh. An overview of the Applications of Multisets. Novi Sad Journal of Math., 37(2), (2007), p 73-92.

[13] A. Syropoulous. Mathematics of Multisets. Springer-Verlag Berlin, Hiedelberg, (2001), p. 347 - 358.

[14] S. Alkhazdeh, A. R. Salleh, and N. Hassan. Soft Multiset Theory. Computer and Mathematics with Applications, 45(2003), p 555-562.

[15] H. Aktas, N. Cagman. Soft Sets and Soft Groups. Information Sciences, an International Journal of Mathematics, 177(2007), p 2726-2735.

[16] R. Biswas. Intuitionistic fuzzy subgroups. Mathematical Forum, 10(1989), p 37-46.

[17] S. K. Nazmul, S. K. Samantha. Fuzzy Soft Group. Journal of Fuzzy Mathematics, 19(1), (2011), p 101-114.

[18] S. K. Nazmul, S. K. Samanta. On Soft Multigroups. Annals of Fuzzy Mathematics and Informatics, 10(2), (2015), p 1-14.

[19] A. Barlotti, K. Strambach. Multigroups and the Foundations of Geometry. Rendiconti Del Circolo Matematico Di Palermo, 40(1), (1991), p 5 - 68.

[20] M. Dresher, O. Ore. Theory of Multigroups, American Journal of Mathematics, 60(1934), p 705 -733.
[21] W. Prenowitz. Projective Geometries as Multigroups. American Journal of Mathematics, 65(2002), p 235 - 256.
[22] T. K. Shinoj, J. J. Sunil. Intuitionistic Fuzzy Multigroups, Annal of Pure and Applied Mathematics, 9(1), (2015), p 133 - 143.
[23] N. E. Onoyima, A. I. Isah, Y. Tella. Some New Results and Operations on Soft Multigroups Theory, International Journal of Scientific Research in Mathematical and Statistical Sciences, 11(2), (2024), p 32 - 40.
[24] R. R. Yager. On the Theory of Bags, International J. of Gen. Systems, 13(1), (1989), p 23 - 37.
[25] L. A. Zadeh. Fuzzy Sets, Inform. Control, 8(1965), p 338 - 353.

J. A. OTUWE*
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABUJA, NIGERIA.
*ORCID: 0009-0009-6693-6825*
*Email address*: talk2josephotuwe2016@gmail.com

I. A. ONYEOZILI*
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABUJA, NIGERIA.
*ORCID: 0009-0007-1664-2875*
*Email address*: ijeozili@gmail.com

M. M. MOGBONJU*
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABUJA, NIGERIA.
*ORCID: 0000-0003-1545-2678*
*Email address*: morufu.mogbonju@uniabuja.edu.ng

D. A. OLUYORI†
DEPARTMENT OF APPLIED MATHEMATICS, FEDERAL UNIVERSITY OF TECHNOLOGY, BABURA, JIGAWA STATE, NIGERIA.
*ORCID: 0000-0002-8348-6717*
*Email address*: daoluyori.mth@futb.edu.ng