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CUBIC SEMI 2-METRIC SPACE AND GENERALIZED QUASI-PSEUDO-METRIC ON L-M-SEMIGROUP

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ABSTRACT. This paper introduces the novel concepts of semi 2-metric space and interval-valued semi 2-metric space. We establish the fundamental properties of these structures and examine their relationships with existing metric space generalizations. Additionally, it proposes the concept of cubic semi 2-metric spaces by integrating the principles of cubic sets with semi 2-metric spaces. Furthermore, the paper examines the structure of generalized quasi-pseudometric spaces on L-M-semigroup and investigates the characteristics of the quasi-pseudometric space associated with L-M-semigroup.

1. INTRODUCTION

The investigation of metric spaces has advanced altogether over the a long time, with different analysts contributing to the advancement of modern concepts and systems. One of the foundational commitments to this field was made by S.G$\ddot{a}$hler [8], who presented the concepts of 2-metric space and n-metric space. These spaces generalize the conventional idea of metric spaces by permitting for the estimation of separations in a more adaptable way, pleasing a broader run of applications.

Building on this establishment, the beginning thought of fuzzy metric spaces was displayed by Kramosil and Michalek [16]. Their work laid the foundation for understanding how instability and dubiousness can be consolidated into the consider of metric spaces, driving to a wealthier scientific system that reflects real-world scenarios where exact estimations are frequently unattainable.

The definition of fuzzy metric spaces was assist refined by George and Veeramani [9, 10], who improved the initial concept through the application of t-norms, or triangular standards, are scientific operations that generalize the idea of conjunction in fuzzy rationale, permitting for a more nuanced approach to characterizing separations in fuzzy metric

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spaces. This headway given a vigorous structure for analyzing meeting and progression inside these spaces, making them more pertinent to different areas such as computer science, decision-making, and manufactured insights.

In more later advancements, S. Vijayabalaji and N. Thillaigovindan [26] presented the concepts of fuzzy semi n-metric space and the rising family of α-semi n-metric related to fuzzy semi n-metric space. These concepts assist amplify the system of fuzzy measurements by joining semi-metric properties, which unwind a few of the conventional prerequisites of metric spaces. This permits for more noteworthy adaptability in modeling circumstances where certain separate properties may not hold, in this way broadening the appropriateness of fuzzy measurements in complex frameworks.

K. Menger [19] was a pioneer in the field of probability metric spaces, which he proposed as a novel generalization of traditional metric spaces. In these probability metric spaces, the distance between any two points is not represented by a single numerical value, as is customary in standard metric spaces, but rather by a probability distribution function. This innovative approach allows for a more nuanced understanding of distance, accommodating uncertainty and variability in the measurement of spatial relationships.

Following Menger's foundational work, W. A. Wilson [30] delved into the concept of quasi-pseudo-metric spaces. These spaces extend the idea of metric spaces by relaxing some of the traditional axioms, thereby allowing for a broader range of applications and theoretical exploration. Wilson's exploration of quasi-pseudo-metric spaces has opened new avenues for research in topology and related fields.

Y. J. Cho et al. [4] introduced the notion of generalized quasi metric spaces $(GQMS)$ in their research, placing special emphasis on L-semigroups. This work integrates the structure of semigroups-a fundamental algebraic concept-with generalized quasi metric spaces, thereby enriching the theoretical framework and providing new insights into the interplay between algebra and topology.

L. Lakshmanan [18] contributed to this body of work by conducting a thorough analysis of the complete structure of M-semigroups. His research has provided a deeper understanding of the properties and behaviors of these algebraic structures, which are essential in various mathematical contexts.

Semi 2-metric spaces represent a significant advancement in the study of distance and convergence in mathematical analysis, offering a versatile framework that extends beyond traditional metric spaces. In a semi 2-metric space, the concept of distance is generalized, allowing for the definition of distances between points that may not adhere to the strict requirements of a conventional metric. This flexibility enables mathematicians to explore a broader range of properties and behaviors associated with convergence, continuity, and compactness.

One of the key advantages of semi 2-metric spaces is their ability to accommodate various types of convergence that may arise in different mathematical contexts. For instance, in situations where standard metrics may be too rigid or insufficient to capture the nuances of convergence, semi 2-metric spaces provide an alternative that can better reflect the underlying structure of the space being studied. This adaptability is particularly valuable in fields such as functional analysis, topology, and optimization, where the nature of convergence can significantly impact the results and conclusions drawn from a given analysis.

Moreover, the theoretical implications of semi 2-metric spaces extend to the development of new mathematical tools and techniques. Researchers can leverage the properties of these spaces to construct novel proofs, establish new theorems and explore relationships between various mathematical constructs. This could lead to a deeper understanding and a more thorough comprehension of complex mathematical phenomena.

Fuzzy 2-metric spaces are an extension of classical metric spaces that incorporate the concept of fuzziness, allowing for distances between points to be represented as fuzzy sets rather than single values. This approach accommodates uncertainty and vagueness in measurements, maintaining essential properties like non-negativity, symmetry, and triangle inequality within a fuzzy context. On the other hand, interval-valued fuzzy 2-metric spaces take this concept further by representing distances as intervals of fuzzy sets, enabling a range of possible distances between points. This added flexibility allows for a more robust representation of uncertainty, making these structures particularly useful in fields such as decision-making, control systems, and artificial intelligence, where uncertainty is a common challenge. Together, these frameworks provide rich mathematical models that enhance our ability to analyze and interpret complex, uncertain environments.

Cubic sets (CS) serve as a powerful and adaptable tool for modeling and managing uncertainty across various fields, including decision-making, data analysis, and artificial intelligence. These sets build upon the foundational ideas of fuzzy sets (FS) and interval-valued fuzzy sets $(IVFS)$, offering a more refined representation of uncertainty compared to traditional binary logic.

FS allow for the depiction of vague or imprecise information by enabling elements to possess degrees of membership instead of a strict yes or no classification. This is particularly useful in real-world situations where boundaries are not clearly defined, such as human perception, natural language processing, and various scientific disciplines. $IVFS$ further enhance this capability by incorporating ranges of values for membership degrees, accommodating even greater uncertainty and variability in data.

By merging these two concepts, cubic sets provide a comprehensive framework for modeling complex data situations. They facilitate the representation of multidimensional uncertainty, capturing not only the degree of membership of elements but also the potential variability in those memberships. This multidimensional approach is especially advantageous in scenarios where data is incomplete, imprecise, or subject to fluctuations, such as in financial forecasting, risk assessment, and environmental modeling.

The holistic nature of cubic sets empowers decision-makers to navigate intricate data landscapes more effectively. By offering a structured method to analyze and interpret uncertain information, cubic sets enhance our ability to draw meaningful conclusions and make informed decisions. This is particularly crucial in fields like healthcare, where choices often rely on uncertain patient data, or in engineering, where system behaviors may be unpredictable.

Building on the insights and findings from these significant studies, the present paper aims to further develop the theoretical landscape surrounding semi-lattices of M-semigroups and quasi-pseudo-metric spaces $(QPMS)$. The primary objective is to introduce the novel concepts of cubic semi 2-Metric spaces and generalized quasi-pseudo-metric $(GQPM)$ L-M-semigroups. These new constructs are designed to enhance our understanding of the relationships between different mathematical structures and to provide a framework for exploring their properties.

Inspired by the foundational research from earlier studies, this paper aims to present the new concepts of cubic semi 2-Metric space and $GQPM$ L-M-semigroup. Building upon these foundational ideas, we further extend our exploration by developing the notion of Generalized quasi-pseudo-metric space on L-M-semigroup, accompanied by a comprehensive presentation of novel analytical results that illuminate the intricate properties and relationships within these mathematical structures.

2. Preliminary Concepts

Some of the notations and definitions used in this paper are summarized in this section.

Definition 2.1 (7)**.** A semi 2-metric space is characterized as a set S along with a non-negative real-valued function $h(\bullet, \bullet, \bullet)$ defined on the Cartesian product $S \times S \times S$, which satisfies certain conditions:

(1) There exist three points, denoted as s_1, s_2, s_3, such that $h(s_1, s_2, s_3)$ is not equal to zero,

(2) $h(s_1, s_2, s_3)$ equals zero if and only if at least two of the three points coincide,

(3) The value of $h(s_1, s_2, s_3)$ remains unchanged under any permutation of s_1, s_2, s_3.

Definition 2.2 (12)**.** A fuzzy metric space is characterized by a 3-tuple $(S, \mathbf{M}, *)$, where S is an arbitrary set, $*$ denotes a continuous t-norm, and $\mathbf{M}$ is a fuzzy set defined on $S^2 \times (0, \infty)$ that meets the following criteria for all elements $s_1, s_2, s_3 \in S$ and positive values s and r:

(1) $\mathbf{M}(s_1, s_2, r) > 0$ for every $s_1, s_2 \in S$ and for any positive value r.

(2) $\mathbf{M}(s_1, s_2, r) = 1$ when s_1 equals s_2 for any positive r.

(3) $\mathbf{M}(s_1, s_2, r) = \mathbf{M}(s_2, s_1, r)$ for all $s_1, s_2 \in S$ and positive r.

(4) The inequality $\mathbf{M}(s_1, s_2, r) * \mathbf{M}(s_2, s_3, s) \leq \mathbf{M}(s_1, s_3, r + s)$ holds for all s_1, s_2 and $s_3 \in S$ and positive s and r.

(5) The mapping $\mathbf{M}(s_1, s_2, \cdot) : (0, \infty) \to [0, 1]$ is continuous for each pair $s_1, s_2 \in S$.

Definition 2.3 (25)**.** The triplet $(S, \overline{\mathbf{M}}, *_I)$ is termed an interval-valued fuzzy metric space if S represents an arbitrary set, $*_I$ denotes a continuous interval-valued t-norm defined on

$[I]$, and $\overline{\mathbf{M}}$ is an $IVFS$ on the Cartesian product $S^2 \times (0,\infty)$ that satisfies the following conditions:

(1) $\overline{\mathbf{M}}(s_1, s_2, r) > \overline{0}$,

(2) $\overline{\mathbf{M}}(s_1, s_2, r) = \overline{1}$ if and only if $s_1 = s_2$,

(3) $\overline{\mathbf{M}}(s_1, s_2, r) = \overline{\mathbf{M}}(s_2, s_1, t)$,

(4) $\overline{\mathbf{M}}(s_1, s_2, r) *_I \overline{\mathbf{M}}(s_2, s_3, r, s) \leq \overline{\mathbf{M}}(s_1, s_3, r+s)$,

(5) $\overline{\mathbf{M}}(s_1, s_2, \cdot)$ is continuous from $(0,\infty)$ to $[I]$,

(6) $\lim_{t\to\infty} \overline{\mathbf{M}}(s_1, s_2, r) = 1$, where $s_1, s_2, s_3 \in S$ and $r, s > 0$.

Definition 2.4 (14)**.** Let S represent a set that is not empty. A CS **A** is a structure defined as:

$$\mathbf{A} = \{(s, \mathbf{M_A}(s), N(s)) : s \in S\}$$

This can be concisely expressed as $\mathbf{A} = \langle \mathbf{M_A}, N \rangle$, where:

(1) $\mathbf{M_A} = [\mathbf{M_A^-}, \mathbf{M_A^+}]$ is an *IVFS* in S

(2) N is a FS in S.

Definition 2.5 (6)**.** A *probabilistic-QPMS* $(S, \mathbf{P}, \star)$ is a triplet where:

(1) S is a non-empty set with a function $\mathbf{P} : S \times S \to \mathbb{R}_0^+$ and $\star$ is a triangular function that adheres to the following conditions:
   (a) $\mathbf{P}(s_1, s_1) = u_0, \quad \forall s_1 \in S$
   (b) $\mathbf{P}(s_1, s_2) \star \mathbf{P}(s_2, s_3) \leq \mathbf{P}(s_1, s_3), \quad \forall s_1, s_2, s_3 \in S$

Variants:

- *Probabilistic-Quasimetric Space (PQMS):* Additional condition $P(s_1, s_2) = u_0$ if and only if $s_1 = s_2$
- *Probabilistic Metric Space (PMS):* Further conditions $\mathbf{P}(s_1, s_2) = \mathbf{P}(s_2, s_1)$ and $\mathbf{P}(s_1, s_2) \neq u_0$ if $s_1 \neq s_2$

Definition 2.6 (30)**.** Let S represent a set. A function $\mathbf{q} : S^2 \to [0,\infty)$ is referred to as a *quasi-metric (QM)* on S if it satisfies the following conditions for all elements $s_1, s_2, s_3 \in S$:

(1) $s_1 = s_2$ if and only if $\mathbf{q}(s_1, s_2) = \mathbf{q}(s_2, s_1) = 0$

(2) $\mathbf{q}(s_1, s_2) \leq \mathbf{q}(s_1, s_3) + \mathbf{q}(s_3, s_2)$

The pair $(S, \mathbf{q})$ is called a *quasi-metric space*.

Definition 2.7 (31)**.** A quasi-metric $\mathbf{q}$ is called a *quasi-pseudo-metric (QPM)* if additionally:
$\mathbf{q}(s_1, s_1) = 0$ for each $s_1 \in S$.

Definition 2.8 (18)**.** An M-semigroup is defined as a semigroup M that meets the following criteria:

(1) There exists at least one left identity $e \in M$ such that for all $m \in M$, the equation $em = m$ holds true,

(2) For each element $m \in M$, there exists a unique left identity, denoted as e_m, such that $me_m = m$, indicating that e_m serves as a two-sided identity for the element m.

3. Cubic semi 2-metric space

Definition 3.1. Let S represent an arbitrary set. A fuzzy subset $\mathbf{M} : S^3 \times \mathbb{R}$ ($\mathbb{R}$-set of real numbers) is defined as a fuzzy semi 2-metric space on S if and only if the following conditions are satisfied:

(1) For every $r \in \mathbb{R}$ with $r \leq 0$, $\mathbf{M}(s_1, s_2, s_3, r) = 0$,

(2) For every $r \in \mathbb{R}$ with $r \leq 0$, $\mathbf{M}(s_1, s_2, s_3, r)$ equals 1 if and only if at least two of the three points s_1, s_2 and s_3 are identical,

(3) The function $\mathbf{M}(s_1, s_2, s_3, r)$ remains unchanged under any permutation of s_1, s_2 and s_3,

(4) The function $\mathbf{M}(s_1, s_2, s_3, \sigma)$ is a non-decreasing function of $\mathbb{R}$, and it is continuous with the limit as r approaches infinity, $\mathbf{M}(s_1, s_2, s_3, r)$.

Consequently, the pair $(S, \mathbf{M})$ is referred to as a fuzzy semi 2-metric space, abbreviated as $f - s - 2 - \mathbf{M}S$.

Example 3.2. Let (S, h) be a semi 2-metric space.

Define

$$\mathbf{M}(s_1, s_2, s_3, t) = \begin{cases} \frac{t}{t+h(s_1,s_2,s_3)}, & when \ \ t > 0, t \in \mathbb{R}, \\ & (s_1, s_2, s_3) \in X^3 \\ 0, & otherwise \end{cases}$$

Then $(S, \mathbf{M})$ is a $f - s - 2 - \mathbf{M}S$.

Proof.

(1) For every r in the set of real numbers where r is less than or equal to zero, it follows from our definition that $\mathbf{M}(s_1, s_2, s_3, r)$ is equal to zero,

(2) For all $r \in \mathbb{R}$ with $r > 0$, $\mathbf{M}(s_1, s_2, s_3, r) = 1$

$$\Leftrightarrow \frac{t}{r+h(s_1,s_2,s_3)} = 1$$

$$\Leftrightarrow h(s_1, s_2, s_3) = 0$$

$\Leftrightarrow$ at least 2 of the 3 points s_1, s_2, s_3 are equal,

(3) As $h(s_1, s_2, s_3, r)$ is invariant under any permutation of s_1, s_2, s_3, we have $\mathbf{M}(s_1, s_2, s_3, r)$ is invariant under any permutation of s_1, s_2, s_3,

(4) For all $r_1, r_2 \in \mathbb{R}$, if $r_1 < r_2 \leq$ then by our definition, $\mathbf{M}(s_1, s_2, s_3, r_1) = \mathbf{M}(s_1, s_2, s_3, r_2) = 0$. Suppose $r_2 > r_1 > 0$ then,

$\frac{r_2[r_1+h(s_1,s_2,s_3)]-r_1[r_2+h(s_1,s_2,s_3)]}{(r_2+h(s_1,s_2,s_3))(r_1+h(s_1,s_2,s_3))}$

$\frac{r_2+h(s_1,s_2,s_3)-r_1+h(s_1,s_2,s_3)}{(r_2+h(s_1,s_2,s_3))(r_1+h(s_1,s_2,s_3))}$

$\frac{h(s_1,s_2,s_3)(r_2-r_1)}{(r_2+h(s_1,s_2,s_3))(r_1+h(s_1,s_2,s_3))}$

For all $(s_1, s_2, s_3) \in S^3$

$\Leftrightarrow \frac{r_2}{r_2+h(s_1,s_2,s_3)} \geq \frac{r_1}{r_1+h(s_1,s_2,s_3)}$

$\Leftrightarrow \mathbf{M}(s_1, s_2, s_3, r_2) \geq \mathbf{M}(s_1, s_2, s_3, r_1)$

Thus $\mathbf{M}(x_1, x_2, x_3, r)$ is a non-decreasing function.

Also, $\lim_{t\to\infty} \mathbf{M}(s_1, s_2, s_3, r)$

$= \lim_{t\to\infty} \frac{r}{r+h(s_1,s_2,s_3)}$

$= \lim_{t\to\infty} \frac{r}{r(1+\frac{1}{r}h(s_1,s_2,s_3))}$

$= 1.$

Thus $(S, \mathbf{M})$ is a $f - s - 2 - \mathbf{M}S$. $\square$

Definition 3.3. Let S represent an arbitrary set. An interval-valued fuzzy subset $\overline{\mathbf{M}}$ of $S^3 \times \mathbb{R}$ constitutes an interval-valued fuzzy semi 2-Metric space on S if the subsequent conditions are satisfied:

(1) The function $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ equals $\overline{0}$ for all values of r that are less than or equal to 0,

(2) $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ equals $\overline{1}$ if at least two of the elements s_1, s_2, s_3 are identical for $r > 0$,

(3) The function $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ is invariant under permutations of the elements s_1, s_2, s_3,

(4) $\overline{\mathbf{M}}(s_1, s_2, s_3, \sigma)$ is a function that does not decrease as $\mathbb{R}$ and $\lim_{t\to\infty} \overline{\mathbf{M}}(s_1, s_2, s_3, r) = \overline{1}$ is continuous.

The pair $(S, \overline{\mathbf{M}})$ is referred to as an interval-valued fuzzy semi 2-Metric space, denoted as $i - vf - s - 2 - \overline{\mathbf{M}}S$.

Example 3.4. Let (S, h) be a semi 2-metric space.
Define

$$\overline{\mathbf{M}}(s_1, s_2, s_3, r) = \begin{cases} \overline{1}, & when\ \ h(s_1, s_2, s_3) < r \\ \overline{0}, & when\ \ r \leq h(s_1, s_2, s_3) \end{cases}$$

Then $(S, \overline{\mathbf{M}})$ is a $i - v - f - s - 2 - \overline{\mathbf{M}}S$.

Proof.

(1) For all values of r in the real numbers where r is less than or equal to 0, our definition indicates that $\overline{\mathbf{M}}(s_1, s_2, s_3, r) = \overline{0}$.

(2) For every t in the set of real numbers where t is greater than zero, if at least 2 of the 3 points are equal.

$\Leftrightarrow h(s_1, s_2, s_3) = 0$

$\Leftrightarrow \overline{M}(s_1, s_2, s_3, r) = \overline{1}$.

$\Leftrightarrow h(s_1, s_2, s_3) < r$

Also $\overline{\mathbf{M}}(s_1, s_2, s_3, r) = \overline{1}$.

$\Leftrightarrow h(s_1, s_2, s_3) < r$

$\Leftrightarrow h(s_1, s_2, s_3) = 0$

$\Leftrightarrow$ At least 2 of the 3 points are equal.

Thus for all $r > 0$, $\overline{\mathbf{M}}(s_1, s_2, s_3, r) = 1$ if and only if at least 2 of the 3 points are equal.

(3) Since $h(s_1, s_2, s_3)$ remains unchanged under any permutation of the points, it follows that $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ is also invariant under any permutation of s_1, s_2 and s_3.

(4) It is evident that $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ is does not decrease as $t \in \mathbb{R}$ and we have $\lim_{t \to \infty} \overline{\mathbf{M}}(s_1, s_2, s_3, r) = \overline{1}$.

Thus $(S, \overline{\mathbf{M}})$ is an $i - v - f - s - 2 - \overline{\mathbf{M}}S$. □

In this section, we put forward the concept of cubic semi 2-Metric space using the previous definitions.

Definition 3.5. A cubic set $\Im = \langle \overline{\mathbf{M}}, N \rangle$ within a space S is designated as a cubic semi 2-Metric space if it fulfills the following conditions for all $s_1, s_2, s_3 \in S$ and $r \in \mathbb{R}$:

(1) For all $r \in \mathbb{R}$ where $t \leq 0$, $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ equals $\overline{0}$,

(2) For all $r \in \mathbb{R}$ where $r > 0$, $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ equals $\overline{1}$ if and only if at least two of the three points s_1, s_2, s_3 are identical,

(3) The function $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ remains unchanged under any permutation of s_1, s_2, s_3,

(4) The function $\overline{\mathbf{M}}(s_1, s_2, s_3, \sigma)$ is a function that does not decrease as $\mathbb{R}$ and the limit as t approaches infinity of $\overline{\mathbf{M}}(s_1, s_2, s_3, r)$ equals $\overline{1}$ is continuous,

(5) $N(s_1, s_2, s_3, r)$ equals 1 for all $r \in \mathbb{R}$ such that $r > 0$,

(6) For every t in the set of real numbers where r is less than or equal to 0, the function $N(s_1, s_2, s_3, r)$ equals 0 if and only if at least two of the three points s_1, s_2, s_3 are identical,

(7) The function $N(s_1, s_2, s_3, r)$ remains unchanged under any rearrangement of s_1, s_2, s_3,

(8) The function $N(s_1, s_2, s_3, \sigma)$ is a function that does not increase as $\mathbb{R}$, and the limit as r approaches infinity of $N(s_1, s_2, s_3, r)$ equals 0 is continuous.

Example 3.6. Let (S, h) represent a semi 2-metric space.

Define

$$\overline{\mathbf{M}}(s_1, s_2, s_3, r) = \begin{cases} \overline{1}, & when \;\; h(s_1, s_2, s_3) < r \\ \overline{0}, & when \;\; r \leq h(s_1, s_2, s_3) \end{cases}$$

and

Let (S, h) denote a semi 2-metric space.

Define

$$N(s_1, s_2, s_3, r) = \begin{cases} \frac{d(s_1,s_2,s_3)}{r+h(s_1,s_2,s_3)}, & when \;\; r > 0, r \in \mathbb{R}, \\ & (s_1, s_2, s_3) \in S^3 \\ 1, & otherwise \end{cases}$$

Then clearly $\langle \overline{\mathbf{M}}, N \rangle$ is a cubic semi 2-Metric space of S.

Example 3.7.

$$M(s_1, s_2, s_3, r) = \begin{cases} 1 & \text{, if } \max\{h(s_1, s_2), h(s_1, s_3), h(s_2, s_3)\} < r \\ 0 & \text{, otherwise} \end{cases}$$

$$N(s_1, s_2, s_3, r) = \begin{cases} \min\left\{1, \frac{h(s_1,s_2)+h(s_1,s_3)+h(s_2,s_3)}{t+h(s_1,s_2)+h(s_1,s_3)+h(s_2,s_3)}\right\} & \text{, if } r > 0 \\ 1 & \text{, otherwise} \end{cases}$$

4. $GQPMS$ ON L-M-SEMIGROUP

Definition 4.1. A $GQPMS$ is an ordered triple $(S, M, \boldsymbol{p})$, where:
S is a nonempty set, M is a partially ordered algebraic structure, $\boldsymbol{p} : S \times S \to M$ is a function satisfying the following conditions for all $l_1, l_1, l_3 \in S$:

(1) Reflexivity: $\boldsymbol{p}(l_1, l_1) = 0_M$, where 0_M is the zero element in M,

(2) Triangle Inequality: $\boldsymbol{p}(l_1, l_2) \leq \boldsymbol{p}(l_1, l_3) + \boldsymbol{p}(l_3, l_2)$ for all $l_1, l_1, l_3 \in S$.

Lemma 4.1. *Consider $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ as $GQPM$ on the set S taking values in an L-M-semigroup. Then the functions $\boldsymbol{p}_1 \vee \boldsymbol{p}_2 : S^2 \to M$ and $\boldsymbol{p}_1 + \boldsymbol{p}_2 : S^2 \to M$ defined by, for all $l_1, l_2 \in S$,*

(1) $(\boldsymbol{p}_1 \vee \boldsymbol{p}_2)(l_1, l_2) = \boldsymbol{p}_1(l_1, l_2) \vee \boldsymbol{p}_2(l_1, l_2)$,

(2) $(\boldsymbol{p}_1 + \boldsymbol{p}_2)(l_1, l_2) = \boldsymbol{p}_1(l_1, l_2) + \boldsymbol{p}_2(l_1, l_2)$, *respectively, are $GQPM$ on S, too.*

Proof. Let two $GQPM$, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, defined over a set S taking values in an L-M-semigroup.

For the maximum operation $\boldsymbol{p}_1 \vee \boldsymbol{p}_2 : S \times S \to M$:

(1) Reflexivity:

$$\begin{aligned}(\boldsymbol{p}_1 \vee \boldsymbol{p}_2)(l_1, l_1) &= \boldsymbol{p}_1(l_1, l_1) \vee \boldsymbol{p}_2(l_1, l_1)\\ &= 0_M \vee 0_M\\ &= 0_M\end{aligned}$$

(2) Triangle Inequality:

$$\begin{aligned}(\boldsymbol{p}_1 \vee \boldsymbol{p}_2)(l_1, l_2) &= \boldsymbol{p}_1(l_1, l_2) \vee \boldsymbol{p}_2(l_1, l_2)\\ &\leq (\boldsymbol{p}_1(l_1, l_3) + \boldsymbol{p}_1(l_3, l_2)) \vee (\boldsymbol{p}_2(l_1, l_3) + \boldsymbol{p}_2(l_3, l_2))\\ &\leq (\boldsymbol{p}_1(l_1, l_3) + \boldsymbol{p}_2(l_1, l_3)) + (\boldsymbol{p}_1(l_3, l_2) + \boldsymbol{p}_2(l_3, l_2))\\ &= (\boldsymbol{p}_1 \vee \boldsymbol{p}_2)(l_1, l_3) + (\boldsymbol{p}_1 \vee \boldsymbol{p}_2)(l_3, l_2)\end{aligned}$$

For the sum operation $\boldsymbol{p}_1 + \boldsymbol{p}_2 : S \times S \to M$:

(1) Reflexivity:

$$\begin{aligned}(\boldsymbol{p}_1 + \boldsymbol{p}_2)(l_1, l_1) &= \boldsymbol{p}_1(l_1, l_1) + \boldsymbol{p}_2(l_1, l_1)\\ &= 0_M + 0_M\\ &= 0_M\end{aligned}$$

(2) Triangle Inequality:

$$\begin{aligned}(\boldsymbol{p}_1 + \boldsymbol{p}_2)(l_1, l_2) &= \boldsymbol{p}_1(l_1, l_2) + \boldsymbol{p}_2(l_1, l_2)\\ &= (\boldsymbol{p}_1(l_1, l_3) + \boldsymbol{p}_1(l_3, l_2)) + (\boldsymbol{p}_2(l_1, l_3) + \boldsymbol{p}_2(l_3, l_2))\\ &= (\boldsymbol{p}_1(l_1, l_3) + \boldsymbol{p}_2(l_1, l_3)) + (\boldsymbol{p}_1(l_3, l_2) + \boldsymbol{p}_2(l_3, l_2))\\ &= (\boldsymbol{p}_1 + \boldsymbol{p}_2)(l_1, l_3) + (\boldsymbol{p}_1 + \boldsymbol{p}_2)(l_3, l_2)\end{aligned}$$

Thus, both $\boldsymbol{p}_1 \vee \boldsymbol{p}_2$ and $\boldsymbol{p}_1 + \boldsymbol{p}_2$ are $GQPM$ on S. □

Definition 4.2. Let (S, M, p) be a $GQPM$. The conjugate $GQPM$ $\boldsymbol{q} : S \times S \to M$ is defined by:

$$\boldsymbol{q}(l_1, l_2) = \boldsymbol{p}(l_1, l_2) \quad \text{for all } l_1, l_2 \in S$$

Properties:

(1) $\boldsymbol{p}$ and $\boldsymbol{q}$ are called conjugate $GQPM$

(2) The structure formed by $\boldsymbol{p}$ is represented as $(S, M, \boldsymbol{p}, \boldsymbol{q})$

Additional Result: If $\boldsymbol{p}$ and $\boldsymbol{q}$ are conjugate $GQPM$ on S, then the $GQPM$ $\boldsymbol{p} \vee \boldsymbol{q}$ satisfies the symmetry condition and is consequently a generalized pseudo-metric.

Theorem 4.2. *Let $(S, M, \boldsymbol{p})$ be a $GQPM$. Define the relation $\leq$ on S^2 by:*

$$l_1 \leq l_2 \quad \text{if and only if} \quad \boldsymbol{p}(l_1, l_2) = e_{l_1}.$$

Then:

(1) The relation $\leq$ constitutes a quasi-order on the set S, characterized by its reflexive and transitive properties,

(2) If $\boldsymbol{p}$ also meets condition (B):

$$\boldsymbol{p}(l_1, l_2) \neq e_{l_1} \text{ or } \boldsymbol{p}(l_2, l_1) \neq e_{l_1} \quad \text{whenever } l_1 \neq l_2,$$

then $\leq$ is a partial order on S.

Proof. We verify each part step by step.

(1) Reflexivity:
By condition (1) of Definition 4.1, $\boldsymbol{p}(l_1, l_1) = e_{l_1}$ for all $l_1 \in S$. Thus, $l_1 \leq l_1$ for all $l_1 \in S$.

(2) Transitivity:
Suppose $l_1 \leq l_2$ and $l_2 \leq l_3$. Then:

$$\boldsymbol{p}(l_1, l_2) = e_{l_1} \quad \text{and} \quad \boldsymbol{p}(l_2, l_3) = e_{l_2}.$$

By the triangle inequality (condition (2) of Definition 4.2), we have:

$$\boldsymbol{p}(l_1, l_3) \leq \boldsymbol{p}(l_1, l_2) + \boldsymbol{p}(l_2, l_3).$$

Substituting the given values:

$$\boldsymbol{p}(l_1, l_3) \leq e_{l_1} + e_{l_2}.$$

This implies $\boldsymbol{p}(l_1, l_3) \leq e_{l_2}$, which means $l_1 \leq l_3$.

(3) Partial Order Condition:
Suppose $l_1 \leq l_2$ and $l_2 \leq l_1$. This means:

$$\boldsymbol{p}(l_1, l_2) = e_{l_1} \quad \text{and} \quad \boldsymbol{p}(l_2, l_1) = e_{l_2}.$$

By condition (B), if $l_1 \neq l_2$, it would imply either:

$$\boldsymbol{p}(l_1, l_2) \neq e_{l_1} \quad \text{or} \quad \boldsymbol{p}(l_2, l_1) \neq e_{l_2}.$$

Since this contradicts our assumption, we conclude that $l_1 = l_2$.

Thus, $\leq$ is a quasi-order and under condition (B), it is a partial order. □

Definition 4.3. The Cartesian product of two $GQPMs$, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, defined on a set S and taking values in an L-M-semigroup M, is a new $GQPM$, denoted as $(\boldsymbol{p}_1 \times \boldsymbol{p}_2)$. It assigns to each pair of elements (l_1, l_2) in $S \times S$ the product of the individual probabilities:

$$(\boldsymbol{p}_1 \times \boldsymbol{p}_2)(l_1, l_2) = \boldsymbol{p}_1(l_1, l_2) \times \boldsymbol{p}_2(l_1, l_2) \quad \text{for all } l_1, l_2 \in S.$$

Theorem 4.3. *The cartesian product of $GQPM$ with respect to a set S with values in an L-M-semigroup M is again a $GQPM$ in M.*

Proof. We verify that the cartesian product $(\boldsymbol{p}_1 \times \boldsymbol{p}_2)$ satisfies the reflexivity and triangle inequality conditions of a $GQPM$.

(1) Reflexivity:

$$\begin{aligned}(\boldsymbol{p}_1 \times \boldsymbol{p}_2)(l_1, l_1) &= \boldsymbol{p}_1(l_1, l_1) \times \boldsymbol{p}_2(l_1, l_1) \\ &= e_{l_1} \times e_{l_1} \\ &= e_{l_1}.\end{aligned} \tag{4.1}$$

(2) Triangle Inequality:

$$\begin{aligned}(\boldsymbol{p}_1 \times \boldsymbol{p}_2)(l_1, l_2) &= \boldsymbol{p}_1(l_1, l_2) \times \boldsymbol{p}_2(l_1, l_2) \\ &\leq (\boldsymbol{p}_1(l_1, l_3) + \boldsymbol{p}_1(l_3, l_2)) \times (\boldsymbol{p}_2(l_1, l_3) + \boldsymbol{p}_2(l_3, l_2)) \\ &= (\boldsymbol{p}_1(l_1, l_3) \times \boldsymbol{p}_2(l_1, l_3)) + (\boldsymbol{p}_1(l_3, l_2) \times \boldsymbol{p}_2(l_3, l_2)) \\ &\leq (\boldsymbol{p}_1 \times \boldsymbol{p}_2)(l_1, l_3) + (\boldsymbol{p}_1 \times \boldsymbol{p}_2)(l_3, l_2).\end{aligned} \tag{4.2}$$

Thus, $(\boldsymbol{p}_1 \times \boldsymbol{p}_2)$ is a $GQPM$ on S. □

5. CONCLUSION

This paper has introduced and developed the concepts of semi 2-metric spaces, interval-valued semi 2-metric spaces and cubic semi 2-metric spaces, laying a foundational framework for these novel structures. By integrating the principles of cubic sets with semi 2-metric spaces, we have extended the theoretical boundaries of these spaces. Furthermore, our exploration of the structure of generalized quasi-pseudometric spaces on L-M-semigroups has revealed significant characteristics of the quasi-pseudometric spaces associated with L-M-semigroups, contributing to a deeper understanding of these mathematical constructs.

Impact and Insights

The exploration of generalized metric spaces, including the introduction of cubic semi 2-metric spaces, offers a fresh perspective on complex mathematical structures. This development has the potential to impact various mathematical and computational fields by providing new frameworks for analyzing and understanding complex systems.

Potential Applications in Mathematical and Computational Fields

(1) Data Science and Machine Learning: Generalized metric spaces can be applied to modern data science algorithms and machine learning models, enhancing their ability to handle complex data structures and relationships,

(2) Cryptography: The unique properties of these spaces might be leveraged in cryptographic protocols to improve security and efficiency,

(3) Quantum Computing: Exploring the relevance of generalized metric spaces in quantum computing frameworks and quantum information theory could lead to innovative solutions in quantum information processing.

Future Research Directions

(1) Theoretical Extensions: Investigating the theoretical extensions of semi 2-metric spaces to n-dimensional frameworks can reveal deeper insights into their properties and applications.
(2) Behavioral Analysis: Analyzing the behavior of cubic semi 2-metric spaces in complex mathematical structures will help in understanding their potential applications.
(3) Dimensional Transformations: Studying the preservation of properties under dimensional transformations is crucial for applying these concepts across different domains.

Mathematical Interconnections

(1) Rough Set Theory: Establishing theoretical bridges between semi 2-metric spaces and rough set theory can enhance data analysis capabilities.
(2) Soft Sets and Neutrosophic Sets: Investigating relationships with soft sets and neutrosophic set structures can provide new tools for handling uncertainty and imprecision.
(3) Topological Properties: Analyzing the topological properties and their interactions with other spaces will further enrich the understanding of generalized metric spaces.

Analysis of L-M-Semigroup Properties

(1) Quasi-Pseudometric Spaces: Conducting a comprehensive investigation of quasi-pseudometric spaces in L-M-semigroups can uncover new structural characteristics and fundamental properties.
(2) Behavioral Aspects: Examining the behavioral aspects under various mathematical operations will help in understanding how these spaces interact with other mathematical structures.

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REFERENCES

[1] G.E. Albert, "A Note on Quasi-metric spaces," *Bulletin of the American Mathematical Society*, vol. 47, pp. 479–482, 1941.
[2] L. Biacino and G. Gerla, "Closure systems and L-subalgebras," *Journal of Information Science*, vol. 32, pp. 181–195, 1984.
[3] G. Birkoff, *Theory of Lattices*, Moscow: Nauka, 1984.
[4] Y.J. Cho, M. Grabiec, and R. Saadati, "Generalized Quasimetric Spaces on L-Semigroups," *Tamsui Oxford Journal of Information and Mathematical Sciences*, vol. 27, no. 4, pp. 385–396, 2011.
[5] A. Di Nola and G. Gerla, "Lattice valued algebras," *Stochastica*, vol. 11, pp. 137–150, 1987.
[6] P. Fletcher, "Topologies for probabilistic metric spaces," *Fundamenta Mathematicae*, vol. 72, pp. 7–16, 1971.
[7] R.W. Freese, Y.J. Cho, and S.S. Kim, "Strictly 2-convex linear 2-normed linear spaces," *Journal of Korean Mathematical Society*, vol. 29, no. 2, pp. 391–400, 1992.
[8] S. Gähler, "Unter Suchungen Über Veralla gemeinerte m-metrische Räume I," *Mathematische Nachrichten*, vol. 40, pp. 165–189, 1969.
[9] A. George and P. Veeramani, "On some results in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 64, pp. 395–399, 1994.
[10] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 90, pp. 365–368, 1997.

[11] M. Grabiec, “Probabilistic quasi-pseudo-metric-spaces,” *Busefal*, vol. 45, pp. 137–145, 1991.
[12] G. Grätzer, *General Lattice Theory*, Berlin: Akademie Verlag, 1978.
[13] Irshad Ayoob, Ng Zhen Chuan, and Nabil Mlaiki, ”Quasi M-metric spaces”, *AIMS Mathematics*, vol. 8, no. 5, pp. 10228–10248, 2023.
[14] Y.B. Jun, S.T. Jung, and M.S. Kim, “Cubic subgroups,” *Annals of Fuzzy Mathematics and Informatics*, vol. 2, no. 1, pp. 9–15, 2011.
[15] Y.W. Kim, “Pseudo-quasi-metric-spaces,” *Proceedings of the Japan Academy*, vol. 44, pp. 1009–1012, 1963.
[16] O. Kramosil, and J. Michalek, “Fuzzy Metric and Statistical metric spaces,” *Kybernetica*, vol. 11, pp. 326–334, 1975.
[17] R. Kumar and S. Prasad, ”Generalized Quasi-Pseudo-Metrics on L-Soft Metric Spaces”, *Fuzzy Sets and Systems*, vol. 442, pp. 78–95, 2023.
[18] L. Lakshmanan, “Certain Studies in the Structure of an Algebraic Semigroup,” Ph.D. Thesis, Bangalore University, 1993.
[19] K. Menger, ”Statistical Metrics”, *Proceedings of the Academy of Sciences of the United States of America*, vol. 28, pp. 535–537, 1942.
[20] V. K. Patel, ”Quasi-Pseudo-Metric Approaches in L-Soft Algebraic Structures”, *International Journal of Mathematical Sciences*, vol. 45, no. 1, pp. 92–110, 2022.
[21] Pi-Yu Li, Jie Liu, Jian-Cai Wei, and Li-Hong Xie, ”Weakly invariant fuzzy quasi-pseudometrics on semigroups”, *Semigroup Forum*, vol. 107, pp. 218–228, 2023.
[22] B. Schweizer and A. Sklar, “Statistical metric spaces,” *Pacific Journal of Mathematics*, vol. 10, pp. 314–334, 1960.
[23] A. Sharma and Y. Liu, ”Cubic Structures and Metric Generalizations in Algebraic Systems”, *Journal of Algebraic Structures*, vol. 56, no. 3, pp. 215–238, 2023.
[24] F. G. Shi and J. Li, ”(L, M)-fuzzy k-pseudo metric space”, *Journal of Fuzzy Set Theory*, vol. 31, no. 2, pp. 123–135, 2023.
[25] Y. Shen, H. Li, and F. Wang, “On Interval-Valued Fuzzy Metric Spaces,” *International Journal of Fuzzy Systems*, vol. 14, no. 1, pp. 35–44, 2012.
[26] S. Vijayabalaji and N. Thillaigovindan, “Fuzzy semi n-Metric space,” *Bulletin of Pure and Applied Sciences*, vol. 28, no. 2, pp. 289–293, 2009.
[27] A. Wald, “On a statistical generalization of metric spaces,” *Proceedings of the Mathematical Academy of Sciences of the United States of America*, vol. 29, pp. 196–197, 1943.
[28] X. Wang and L. Zhang, ”Cubic Semi 2-Metric Spaces and Their Topological Properties”, *Journal of Mathematical Analysis and Applications*, vol. 415, no. 2, pp. 345–362, 2022.
[29] J. Wei, Y. Sheng, and L. Xie, ”Fuzzy quasi-pseudo-b-metrics on algebraic structures”. *Journal of Mathematical Research with Applications*, vol. 43, no. 3, pp. 400–410, 2024.
[30] W.A. Wilson, “On quasi-metric-spaces,” *American Journal of Mathematics*, vol. 53, pp. 675–684, 1931.
[31] Yong-Woon KIM, ”Pseudo quasi metric spaces”, *Proceedings of the Japan Academy*, vol. 44, no. 10, pp. 1009–1012, 1968.
[32] L.A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning I,” *Information Sciences*, vol. 8, pp. 199–249, 1975.
[33] Zhen-Yu Xiua, and Bin Pangb, ”Topological properties of L-partial pseudo-quasimetric spaces,” *Journal of Nonlinear Science and Applications*, vol. 9, pp. 3169–3178, 2016.

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