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INTEGRAL INEQUALITIES UNDER DIVERSE PARAMETRIC PRIMITIVE EXPONENTIAL-WEIGHTED INTEGRAL INEQUALITY ASSUMPTIONS

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ABSTRACT. In this article, we establish several integral inequalities under new parametric primitive exponential-weighted integral inequality assumptions. The generality and versatility of these assumptions allow our results to extend and unify existing frameworks in the literature. In total, eight theorems are formulated. To ensure completeness and accessibility, detailed proofs are given for all the theorems.

1. INTRODUCTION

Integral inequalities are among the fundamental tools of mathematical analysis. They play a central role in several disciplines, including differential equations, probability theory, optimization, mathematical physics, numerical analysis and functional analysis. Their main utility lies in estimating integrals, providing bounds, and exploring the properties of functions. Detailed discussions of some of the most famous integral inequalities can be found in [11, 3, 22, 2, 23, 9].

Beyond the classical results, recent research has introduced some innovations by considering original and generalized assumptions. Notable contributions in this area are detailed in [16, 14, 24, 18, 12, 19, 15, 20, 21, 13, 4, 5, 10, 1, 17, 7, 8, 6]. Among these, the use of primitive integral inequality assumptions provides a modern perspective.

An example of such progress is presented in [12, Theorem 2.4]. This result states that, for $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, a differentiable non-decreasing function $f : [a, b] \mapsto [0, +\infty)$, and an integrable function $g : [a, b] \mapsto [0, +\infty)$, if the following primitive integral inequality holds:

$$\int_u^b f(t)dt \leq \int_u^b g(t)dt, \tag{1.1}$$

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for any $u \in [a, b]$, then for any $\mu, \nu \geq 0$ such that $\mu + \nu \geq 1$, we have

$$\int_a^b f^{\mu}(t) g^{\nu}(t) dt \leq \int_a^b g^{\mu+\nu}(t) dt.$$

This result is inspired by earlier work in [16, 14, 24]. It has also served as a basis for more detailed investigations of integral inequalities under assumptions similar to those of Equation (1.1). For further developments, see [18, 15, 19, 20, 21, 8]. In particular, the study in [8] makes assumptions of the following form:

$$\int_u^b f(t) dt \leq w(u) \int_u^b g(t) dt,$$

for any $u \in [a, b]$, where $w : [a, b] \mapsto [0, +\infty)$ denotes a particular weight function. Based on this form, several integral inequalities are proved, contributing to the interest in using primitive integral inequality assumptions.

In 2025, integral inequalities continue to be an active area of research, with theoretical advances enriching the field. In this article, we support this claim by presenting some new integral inequalities characterized by different parametric primitive exponential-weighted assumptions. To give an idea, the main assumption in our first result has the following form:

$$\int_u^b e^{-\theta t} f(t) dt \leq e^{(\beta-\theta)u} \int_u^b e^{-\beta t} g(t) dt,$$

for any $u \in [a, b]$, where β and θ are two parameters. Obviously, by taking $\beta = \theta = 0$, it reduces to Equation (1.1). This demonstrates the generality of our framework, which extends existing methods and allows the derivation of more versatile results. The article goes further in this direction, with different parametric primitive exponential-weighted assumptions, leading to original integral inequalities. From another point of view, the considered assumptions define new classes of functions for which the established integral inequalities provide effective tools for deriving bounds and estimating integrals. In total, eight theorems are stated and proved in detail, each of which has some potential for application.

The structure of the article is as follows: Section 2 presents our first series of integral inequalities, under new types of parametric primitive exponential-weighted assumptions. Section 3 completes Section 2 with further integral inequalities under other types of assumptions. Finally, Section 4 concludes the article and discusses possible extensions.

2. First series of integral inequalities

2.1. **Main theorem.**

Under certain assumptions, including an original parametric primitive exponential-weighted integral inequality assumption, the theorem below establishes a simple integral inequality.

In this statement, and throughout the article, it is assumed that the integrals introduced converge, which is not guaranteed a priori, especially if $a \to -\infty$ or $b \to +\infty$.

Theorem 2.1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \mapsto [0, +\infty)$ be a differentiable non-decreasing function and $g, h : [a, b] \mapsto [0, +\infty)$ be two functions. We suppose that there exist two constants $\beta, \theta \geq 0$ with $\beta \geq \theta$ such that, for any $u \in [a, b]$, we have*

$$\int_u^b e^{-\theta t} g(t) dt \leq e^{(\beta-\theta)u} \int_u^b e^{-\beta t} h(t) dt. \tag{2.1}$$

Then we have

$$\int_a^b f(t)g(t)dt \leq \int_a^b f(t)h(t)dt.$$

If $\theta \geq \beta$ and the inequality in Equation (2.1) *is reversed, then the final inequality is also reversed.*

Proof. Since f is differentiable, taking into account the parameter θ in Equation (2.1) and the exponential function, for any $t \in [a, b]$, we have

$$e^{\theta t}f(t) = \left[e^{\theta t}f(t) - e^{\theta a}f(a)\right] + e^{\theta a}f(a) = \int_a^t [e^{\theta u}f(u)]'du + e^{\theta a}f(a)$$
$$= \int_a^t [\theta f(u) + f'(u)]e^{\theta u}du + e^{\theta a}f(a).$$

This implies that

$$f(t) = e^{-\theta t}\int_a^t [\theta f(u) + f'(u)]e^{\theta u}du + e^{\theta a}e^{-\theta t}f(a). \tag{2.2}$$

Using this special integral decomposition and changing the order of integration, which is possible thanks to the Fubini-Tonelli theorem, we get

$$\int_a^b f(t)g(t)dt = \int_a^b \left\{e^{-\theta t}\int_a^t [\theta f(u) + f'(u)]e^{\theta u}du + e^{\theta a}e^{-\theta t}f(a)\right\}g(t)dt$$
$$= \int_a^b \int_a^t e^{-\theta t}[\theta f(u) + f'(u)]e^{\theta u}g(t)dudt + e^{\theta a}f(a)\int_a^b e^{-\theta t}g(t)dt$$
$$= \int_a^b \int_u^b e^{-\theta t}[\theta f(u) + f'(u)]e^{\theta u}g(t)dtdu + e^{\theta a}f(a)\int_a^b e^{-\theta t}g(t)dt$$
$$= \int_a^b [\theta f(u) + f'(u)]e^{\theta u}\left[\int_u^b e^{-\theta t}g(t)dt\right]du + e^{\theta a}f(a)\int_a^b e^{-\theta t}g(t)dt. \tag{2.3}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $0 \leq \theta \leq \beta$, $f'(u) \geq 0$ for any $u \in [a, b]$ (since f is differentiable and non-decreasing), $g(t) \geq 0$ for any $t \in [a, b]$, and the assumption in Equation (2.1) applied twice (with $u = a$ for the second application), we obtain

$$\int_a^b [\theta f(u) + f'(u)]e^{\theta u}\left[\int_u^b e^{-\theta t}g(t)dt\right]du + e^{\theta a}f(a)\int_a^b e^{-\theta t}g(t)dt$$
$$\leq \int_a^b [\beta f(u) + f'(u)]e^{\theta u}\left[e^{(\beta-\theta)u}\int_u^b e^{-\beta t}h(t)dt\right]du$$
$$+ e^{\theta a}f(a)\left[e^{(\beta-\theta)a}\int_a^b e^{-\beta t}h(t)dt\right]$$
$$= \int_a^b [\beta f(u) + f'(u)]e^{\beta u}\int_u^b e^{-\beta t}h(t)dtdu + e^{\beta a}f(a)\int_a^b e^{-\beta t}h(t)dt$$
$$= \int_a^b \int_u^b e^{-\beta t}[\beta f(u) + f'(u)]e^{\beta u}h(t)dtdu + e^{\beta a}f(a)\int_a^b e^{-\beta t}h(t)dt. \tag{2.4}$$

Changing the order of integration once again and noticing that, with the exact arguments that those used to establish Equation (2.2) with β instead of θ,

$$f(t) = e^{-\beta t} \int_a^t [\beta f(u) + f'(u)] e^{\beta u} du + e^{\beta a} e^{-\beta t} f(a),$$

we have

$$\begin{aligned}
&\int_a^b \int_u^b e^{-\beta t} [\beta f(u) + f'(u)] e^{\beta u} h(t) dt du + e^{\beta a} f(a) \int_a^b e^{-\beta t} h(t) dt \\
&= \int_a^b \int_a^t e^{-\beta t} [\beta f(u) + f'(u)] e^{\beta u} h(t) du dt + e^{\beta a} f(a) \int_a^b e^{-\beta t} h(t) dt \\
&= \int_a^b \left[e^{-\beta t} \int_a^t [\beta f(u) + f'(u)] e^{\beta u} du + e^{\beta a} e^{-\beta t} f(a) \right] h(t) dt \\
&= \int_a^b f(t) h(t) dt. \qquad (2.5)
\end{aligned}$$

Combining Equations (2.3), (2.4) and (2.5) gives

$$\int_a^b f(t) g(t) dt \leq \int_a^b f(t) h(t) dt,$$

which is the desired inequality.

If $\theta \geq \beta$ and the inequality in Equation (2.1) is reversed, then the inequality in Equation (2.4) (second line) is reversed, implying that the final inequality is also reversed. This concludes the proof of Theorem 2.1. □

As discussed in the introductory section, this theorem generalizes some frameworks in [12, 18, 15, 19, 20, 21], thanks to the use of several intermediate functions and the parametric primitive exponential-weighted assumption in Equation (2.1).

We can take another look at this assumption by introducing the "truncated Laplace transform" as follows:

$$\mathcal{L}(k; \epsilon, u) = \int_u^b e^{-\epsilon t} k(t) dt, \qquad (2.6)$$

where $k : [a, b] \mapsto \mathbb{R}$, $\epsilon \geq 0$ and $u \in [a, b]$. The assumption then becomes

$$\mathcal{L}(g; \theta, u) \leq e^{(\beta - \theta) u} \mathcal{L}(h; \beta, u),$$

for any $u \in [a, b]$, or, equivalently,

$$e^{\theta u} \mathcal{L}(g; \theta, u) \leq e^{\beta u} \mathcal{L}(h; \beta, u),$$

for any $u \in [a, b]$, Given the extensive literature on the Laplace transform, this new formulation may inspire ideas beyond the scope of the study.

From another point of view, the assumptions considered in Theorem 2.1 define new classes of functions for which the established integral inequalities provide effective tools for evaluating integrals.

Our approach thus combines the originality of the assumptions with the generality and flexibility of the integral inequality obtained. Further results, still based on parametric primitive exponential-weighted assumptions, are presented in the remainder of this section.

2.2. **Additional results.** Under certain assumptions, including an original parametric primitive exponential-weighted integral inequality assumption involving the power of a function, the theorem below shows a simple integral inequality.

Theorem 2.2. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \mapsto [0, +\infty)$ be a differentiable non-decreasing function and $g : [a, b] \mapsto [0, +\infty)$ be a function. We suppose that there exist three constants $\theta \geq 0$, $\tau \in \mathbb{R}\backslash\{-1\}$ and $\omega \geq 0$ such that, for any $u \in [a, b]$, we have*

$$\int_u^b e^{-\theta t} g(t)dt \leq \omega e^{-\theta u} f^{\tau}(u), \tag{2.7}$$

with $f(a) \neq 0$ for $\tau < 0$.

Then we have

$$\int_a^b f(t)g(t)dt \leq \omega \left[\theta \int_a^b f^{\tau+1}(u)du + \frac{1}{\tau+1} f^{\tau+1}(b) + \frac{\tau}{\tau+1} f^{\tau+1}(a)\right].$$

Proof. First, note that the decomposition in Equation (2.3) is still valid. So we have

$$\int_a^b f(t)g(t)dt = \int_a^b [\theta f(u) + f'(u)]e^{\theta u} \left[\int_u^b e^{-\theta t} g(t)dt\right] du + e^{\theta a} f(a) \int_a^b e^{-\theta t} g(t)dt. \tag{2.8}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $\theta \geq 0$, $f'(u) \geq 0$ for any $u \in [a, b]$ (since f is differentiable and non-decreasing), $g(t) \geq 0$ for any $t \in [a, b]$, the assumption in Equation (2.7) applied twice and $\tau \in \mathbb{R}\backslash\{-1\}$, we get

$$\begin{aligned}
&\int_a^b [\theta f(u) + f'(u)]e^{\theta u} \left[\int_u^b e^{-\theta t} g(t)dt\right] du + e^{\theta a} f(a) \int_a^b e^{-\theta t} g(t)dt \\
&\leq \int_a^b [\theta f(u) + f'(u)]e^{\theta u} \left[\omega e^{-\theta u} f^{\tau}(u)\right] du + e^{\theta a} f(a) \left[\omega e^{-\theta a} f^{\tau}(a)\right] \\
&= \omega \left[\theta \int_a^b f^{\tau+1}(u)du + \int_a^b f'(u) f^{\tau}(u)du + f^{\tau+1}(a)\right] \\
&= \omega \left\{\theta \int_a^b f^{\tau+1}(u)du + \left[\frac{1}{\tau+1} f^{\tau+1}(u)\right]_{u=a}^{u=b} + f^{\tau+1}(a)\right\} \\
&= \omega \left[\theta \int_a^b f^{\tau+1}(u)du + \frac{1}{\tau+1} f^{\tau+1}(b) - \frac{1}{\tau+1} f^{\tau+1}(a) + f^{\tau+1}(a)\right] \\
&= \omega \left[\theta \int_a^b f^{\tau+1}(u)du + \frac{1}{\tau+1} f^{\tau+1}(b) + \frac{\tau}{\tau+1} f^{\tau+1}(a)\right].
\end{aligned} \tag{2.9}$$

Combining Equations (2.8) and (2.9) gives the desired upper bound. This ends the proof of Theorem 2.2. □

From this result, it is interesting to see how the complex interplay between the parameters a, b, θ, ω and τ, linked by Equation (2.7), ends up characterizing an original upper bound.

It can also be noted that, since f is non-decreasing and positive, the following assumption is not of interest: there exist three constants $\theta \geq 0$, $\tau \in \mathbb{R}$ and $\omega \geq 0$ such that, for any $u \in [a, b]$, we have

$$\int_u^b e^{-\theta t} g(t)dt \geq \omega e^{-\theta u} f^{\tau}(u).$$

Indeed, in this case, taking $u = b$, we find that

$$0 = \int_b^b e^{-\theta t} g(t)dt \geq \omega e^{-\theta b} f^{\tau}(b) \geq 0,$$

so that $f(b) = 0$ and, since f is non-decreasing, we necessarily have $f(u) = 0$ for any $u \in [a, b]$. The final inequality becomes trivial in this case.

The statement below completes Theorem 2.2 by examining the special case "$\tau = -1$".

Theorem 2.3. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \mapsto (0, +\infty)$ be a differentiable non-decreasing function and $g : [a, b] \mapsto [0, +\infty)$ be a function. We suppose that there exist two constants $\theta \geq 0$ and $\omega \geq 0$ such that, for any $u \in [a, b]$, we have*

$$\int_u^b e^{-\theta t} g(t)dt \leq \omega e^{-\theta u} \frac{1}{f(u)}. \tag{2.10}$$

Then we have

$$\int_a^b f(t)g(t)dt \leq \omega \left\{ 1 + \theta(b - a) + \log\left[\frac{f(b)}{f(a)}\right] \right\}.$$

Proof. Re-using Equation (2.3), we obtain

$$\int_a^b f(t)g(t)dt = \int_a^b [\theta f(u) + f'(u)]e^{\theta u} \left[\int_u^b e^{-\theta t} g(t)dt\right] du + e^{\theta a} f(a) \int_a^b e^{-\theta t} g(t)dt. \tag{2.11}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $\theta \geq 0$, $f'(u) \geq 0$ for any $u \in [a, b]$ (since f is differentiable and non-decreasing), $g(t) \geq 0$ for any $t \in [a, b]$, and the assumption in Equation (2.10) applied twice, we get

$$\begin{aligned}
&\int_a^b [\theta f(u) + f'(u)]e^{\theta u} \left[\int_u^b e^{-\theta t} g(t)dt\right] du + e^{\theta a} f(a) \int_a^b e^{-\theta t} g(t)dt \\
&\leq \int_a^b [\theta f(u) + f'(u)]e^{\theta u} \left[\omega e^{-\theta u} \frac{1}{f(u)}\right] du + e^{\theta a} f(a) \left[\omega e^{-\theta a} \frac{1}{f(a)}\right] \\
&= \omega \left[\theta \int_a^b du + \int_a^b f'(u) \frac{1}{f(u)} du + 1\right] \\
&= \omega \left[\theta(b - a) + \{\log[f(u)]\}_{u=a}^{u=b} + 1\right] = \omega \{\theta(b - a) + \log[f(b)] - \log[f(a)] + 1\} \\
&= \omega \left\{ 1 + \theta(b - a) + \log\left[\frac{f(b)}{f(a)}\right] \right\}.
\end{aligned} \tag{2.12}$$

Combining Equations (2.11) and (2.12) gives the desired upper bound. This ends the proof of Theorem 2.3. □

We emphasize the original form of the upper bound, where the logarithm of the ratio of the main function f taken at the extremes of $[a, b]$ plays an important role.

Under certain general assumptions, including an original parametric primitive exponential-weighted integral inequality assumption involving the derivative of an intermediate function, the theorem below presents a specific integral inequality.

Theorem 2.4. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \mapsto [0, +\infty)$ be a differentiable non-decreasing function, $g : [a, b] \mapsto [0, +\infty)$ be a function and $h : [0, +\infty) \to \mathbb{R}$ be a differentiable function. We suppose that there exists a constant $\theta \geq 0$ such that, for any*

$u \in [a, b]$, *we have*

$$\int_u^b e^{-\theta t} g(t) dt \leq h'[e^{\theta u} f(u)]. \tag{2.13}$$

Then we have

$$\int_a^b f(t)g(t)dt \leq h[e^{\theta b} f(b)] - h[e^{\theta a} f(a)] + e^{\theta a} f(a) h'[e^{\theta a} f(a)].$$

If the inequality in Equation (2.13) *is reversed, then the final inequality is also reversed.*

Proof. We can re-use Equation (2.3), which gives

$$\int_a^b f(t)g(t)dt = \int_a^b [\theta f(u) + f'(u)] e^{\theta u} \left[\int_u^b e^{-\theta t} g(t) dt \right] du + e^{\theta a} f(a) \int_a^b e^{-\theta t} g(t) dt. \tag{2.14}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $\theta \geq 0$, $f'(u) \geq 0$ for any $u \in [a, b]$ (since f is differentiable and non-decreasing), $g(t) \geq 0$ for any $t \in [a, b]$, the assumption in Equation (2.13) applied twice and a suitable primitive of $[e^{\theta u} f(u)]' h'[e^{\theta u} f(u)]$, we get

$$\begin{aligned}
&\int_a^b [\theta f(u) + f'(u)] e^{\theta u} \left[\int_u^b e^{-\theta t} g(t) dt \right] du + e^{\theta a} f(a) \int_a^b e^{-\theta t} g(t) dt \\
&\leq \int_a^b [\theta f(u) + f'(u)] e^{\theta u} h'[e^{\theta u} f(u)] du + e^{\theta a} f(a) h'[e^{\theta a} f(a)] \\
&= \int_a^b [e^{\theta u} f(u)]' h'[e^{\theta u} f(u)] du + e^{\theta a} f(a) h'[e^{\theta a} f(a)] \\
&= \left\{ h[e^{\theta u} f(u)] \right\}_{u=a}^{u=b} + e^{\theta a} f(a) h'[e^{\theta a} f(a)] \\
&= h[e^{\theta b} f(b)] - h[e^{\theta a} f(a)] + e^{\theta a} f(a) h'[e^{\theta a} f(a)].
\end{aligned} \tag{2.15}$$

Combining Equations (2.14) and (2.15) gives the desired upper bound.

If the inequality in Equation (2.13) is reversed, then the inequality in Equation (2.15) is reversed, and the final inequality is also reversed.

This ends the proof of Theorem 2.4. □

Some special cases of this theorem are highlighted below to emphasize its versatility and flexibility.

- Within the framework of Theorem 2.4, if we take $h(x) = e^x$, then Equation (2.13) becomes

  $$\int_u^b e^{-\theta t} g(t) dt \leq e^{e^{\theta u} f(u)}$$

  and the final inequality becomes

  $$\int_a^b f(t)g(t)dt \leq e^{e^{\theta b} f(b)} - e^{e^{\theta a} f(a)} + e^{\theta a} f(a) e^{e^{\theta a} f(a)}.$$

- Within the framework of Theorem 2.4, if we take $h(x) = \omega \log(x)$ with $\omega \geq 0$, then Equation (2.13) becomes

  $$\int_u^b e^{-\theta t} g(t) dt \leq \omega \frac{1}{e^{\theta u} f(u)} = \omega e^{-\theta u} \frac{1}{f(u)},$$

which corresponds to the parametric primitive exponential-weighted assumption in Equation (2.10), and the final inequality becomes

$$\int_a^b f(t)g(t)dt \leq \omega \log[e^{\theta b} f(b)] - \omega \log[e^{\theta a} f(a)] + e^{\theta a} f(a) \omega \frac{1}{e^{\theta a} f(a)}$$
$$= \omega \left\{\theta b + \log[f(b)] - \theta a - \log[f(a)] + 1\right\} = \omega \left\{1 + \theta(b-a) + \log\left[\frac{f(b)}{f(a)}\right]\right\}.$$

We recognize the result of Theorem 2.3. In this sense, Theorem 2.4 can be seen as one of its possible generalizations.

- Within the framework of Theorem 2.4, if we take $h(x) = \log(1+x)$, then Equation (2.13) becomes
$$\int_u^b e^{-\theta t} g(t)dt \leq \frac{1}{1 + e^{\theta u} f(u)}$$
and the final inequality becomes
$$\int_a^b f(t)g(t)dt \leq \log[1 + e^{\theta b} f(b)] - \log[1 + e^{\theta a} f(a)] + e^{\theta a} f(a) \frac{1}{1 + e^{\theta a} f(a)}$$
$$= \log\left[\frac{1 + e^{\theta b} f(b)}{1 + e^{\theta a} f(a)}\right] + \frac{e^{\theta a} f(a)}{1 + e^{\theta a} f(a)}.$$
- Within the framework of Theorem 2.4, if we take $h(x) = \arctan(x)$, then Equation (2.13) becomes
$$\int_u^b e^{-\theta t} g(t)dt \leq \frac{1}{1 + e^{2\theta u} f^2(u)}$$
and the final inequality becomes
$$\int_a^b f(t)g(t)dt \leq \arctan[e^{\theta b} f(b)] - \arctan[e^{\theta a} f(a)] + e^{\theta a} f(a) \frac{1}{1 + e^{2\theta a} f^2(a)}.$$

These are only a few examples. Theorem 2.4 can be applied to other situations thanks to the adaptability of the function h.

3. Other integral inequalities

In this section, we revisit the results of the above section with a slight modification of the parametric primitive exponential-weighted integral inequality assumption, and show various new integral inequalities.

3.1. **Main theorem.** Under certain assumptions, including a new parametric primitive exponential-weighted integral inequality assumption different from that in Theorem 2.1, the statement below shows a simple integral inequality.

Theorem 3.1. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \mapsto [0, +\infty)$ be a differentiable non-increasing function and $g, h : [a, b] \mapsto [0, +\infty)$ be two functions. We suppose that there exist two constants $\beta, \theta \leq 0$ with $\theta \geq \beta$ such that, for any $u \in [a, b]$, we have*
$$\int_a^u e^{-\theta t} g(t)dt \leq e^{(\beta-\theta)u} \int_a^u e^{-\beta t} h(t)dt. \tag{3.1}$$
Then we have
$$\int_a^b f(t)g(t)dt \leq \int_a^b f(t)h(t)dt.$$

If $\beta \geq \theta$ and the inequality in Equation (3.1) *is reversed, then the final inequality is also reversed.*

Proof. Since f is differentiable, taking into account the parameter θ in Equation (3.1) and the exponential function, for any $t \in [a, b]$, we have

$$e^{\theta t} f(t) = e^{\theta b} f(b) - \left[e^{\theta b} f(b) - e^{\theta t} f(t)\right] = e^{\theta b} f(b) - \int_t^b [e^{\theta u} f(u)]' du$$
$$= e^{\theta b} f(b) - \int_t^b [\theta f(u) + f'(u)] e^{\theta u} du,$$

so that

$$f(t) = e^{\theta b} e^{-\theta t} f(b) - e^{-\theta t} \int_t^b [\theta f(u) + f'(u)] e^{\theta u} du. \tag{3.2}$$

Using this special integral decomposition and changing the order of integration, which is possible thanks to the Fubini-Tonelli theorem, we get

$$\int_a^b f(t) g(t) dt = \int_a^b \left\{ e^{\theta b} e^{-\theta t} f(b) - e^{-\theta t} \int_t^b [\theta f(u) + f'(u)] e^{\theta u} du \right\} g(t) dt$$
$$= e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t) dt - \int_a^b \int_t^b e^{-\theta t} [\theta f(u) + f'(u)] e^{\theta u} g(t) du dt$$
$$= e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t) dt - \int_a^b \int_a^u e^{-\theta t} [\theta f(u) + f'(u)] e^{\theta u} g(t) dt du$$
$$= e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t) dt + \int_a^b [(-\theta) f(u) - f'(u)] e^{\theta u} \left[\int_a^u e^{-\theta t} g(t) dt\right] du. \tag{3.3}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $\beta \leq \theta \leq 0$ so that $0 \leq -\theta \leq -\beta$, $f'(u) \leq 0$ for any $u \in [a, b]$ (since f is differentiable and non-increasing), $g(t) \geq 0$ for any $t \in [a, b]$, and the assumption in Equation (3.1) applied twice, we get

$$e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t) dt + \int_a^b [(-\theta) f(u) - f'(u)] e^{\theta u} \left[\int_a^u e^{-\theta t} g(t) dt\right] du$$
$$\leq e^{\theta b} f(b) \left[e^{(\beta-\theta) b} \int_a^b e^{-\beta t} h(t) dt\right]$$
$$+ \int_a^b [(-\beta) f(u) - f'(u)] e^{\theta u} \left[e^{(\beta-\theta) u} \int_a^u e^{-\beta t} h(t) dt\right] du$$
$$= e^{\beta b} f(b) \int_a^b e^{-\beta t} h(t) dt + \int_a^b [(-\beta) f(u) - f'(u)] e^{\beta u} \int_a^u e^{-\beta t} h(t) dt du$$
$$= e^{\beta b} f(b) \int_a^b e^{-\beta t} h(t) dt + \int_a^b \int_a^u e^{-\beta t} [(-\beta) f(u) - f'(u)] e^{\beta u} h(t) dt du. \tag{3.4}$$

Changing the order of integration once again and noticing that, with the exact arguments that those used to establish Equation (3.2) with β instead of θ,

$$f(t) = e^{\beta b} e^{-\beta t} f(b) - e^{-\beta t} \int_t^b [\beta f(u) + f'(u)] e^{\beta u} du,$$

we get

$$\begin{aligned}
&e^{\beta b}f(b)\int_a^b e^{-\beta t}h(t)dt+\int_a^b\int_a^u e^{-\beta t}[(-\beta)f(u)-f'(u)]e^{\beta u}h(t)dtdu\\
&=e^{\beta b}f(b)\int_a^b e^{-\beta t}h(t)dt+\int_a^b\int_t^b e^{-\beta t}[(-\beta)f(u)-f'(u)]e^{\beta u}h(t)dudt\\
&=\int_a^b\left[e^{\beta b}e^{-\beta t}f(b)-e^{-\beta t}\int_t^b[\beta f(u)+f'(u)]e^{\beta u}du\right]h(t)dt\\
&=\int_a^b f(t)h(t)dt. \qquad (3.5)
\end{aligned}$$

Combining Equations (3.3), (3.4) and (3.5) gives

$$\int_a^b f(t)g(t)dt\le\int_a^b f(t)h(t)dt,$$

which is the desired inequality.

If $\beta\ge\theta$ and the inequality in Equation (3.1) is reversed, then the inequality in Equation (3.4) (second line) is reversed, implying that the final inequality is also reversed. This concludes the proof of Theorem 3.1. □

The difference between this theorem and Theorem 2.1 lies in the monotonicity assumptions on f and in the parametric primitive exponential-weighted assumptions considered. In short, if we compare the assumptions in Equations (2.1) and (3.1), $\int_a^u e^{-\theta t}g(t)dt$ has replaced $\int_u^b e^{-\theta t}g(t)dt$, and $\int_a^u e^{-\beta t}h(t)dt$ has replaced $\int_u^b e^{-\beta t}h(t)dt$, resulting in a common integral inequality at the end.

Again, the originality of the assumptions is combined with the generality and flexibility of the integral inequality obtained. Further results, still based on parametric primitive exponential-weighted assumptions, are presented in the remainder of this section.

3.2. **Additional results.**

Under certain assumptions, including an original parametric primitive exponential-weighted integral inequality assumption involving the power of a function different from that in Theorem 2.2, the theorem below shows a simple integral inequality.

Theorem 3.2. *Let $a,b\in\mathbb{R}\cup\{\pm\infty\}$ with $a<b$, $f:[a,b]\mapsto[0,+\infty)$ be a differentiable non-increasing function and $g:[a,b]\mapsto[0,+\infty)$ be a function. We suppose that there exist three constants $\theta\le 0$, $\tau\in\mathbb{R}\backslash\{-1\}$ and $\omega\ge 0$ such that, for any $u\in[a,b]$, we have*

$$\int_a^u e^{-\theta t}g(t)dt\le\omega e^{-\theta u}f^{\tau}(u), \qquad (3.6)$$

with $f(b)\neq 0$ for $\tau<0$.

Then we have

$$\int_a^b f(t)g(t)dt\le\omega\left[\frac{\tau}{\tau+1}f^{\tau+1}(b)+\frac{1}{\tau+1}f^{\tau+1}(a)-\theta\int_a^b f^{\tau+1}(u)du\right].$$

Proof. Re-using Equation (3.3), we have

$$\begin{aligned}
\int_a^b f(t)g(t)dt&=e^{\theta b}f(b)\int_a^b e^{-\theta t}g(t)dt\\
&+\int_a^b[(-\theta)f(u)-f'(u)]e^{\theta u}\left[\int_a^u e^{-\theta t}g(t)dt\right]du. \qquad (3.7)
\end{aligned}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a,b]$, $\theta \leq 0$, $f'(u) \leq 0$ for any $u \in [a,b]$ (since f is differentiable and non-increasing), $g(t) \geq 0$ for any $t \in [a,b]$, the assumption in Equation (3.6) applied twice and $\tau \in \mathbb{R}\backslash\{-1\}$, we get

$$\begin{aligned}
&e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t)dt + \int_a^b [(-\theta)f(u) - f'(u)]e^{\theta u} \left[\int_a^u e^{-\theta t} g(t)dt\right] du \\
&\leq e^{\theta b} f(b) \left[\omega e^{-\theta b} f^{\tau}(b)\right] + \int_a^b [(-\theta)f(u) - f'(u)]e^{\theta u} \left[\omega e^{-\theta u} f^{\tau}(u)\right] du \\
&= \omega \left[f^{\tau+1}(b) - \theta \int_a^b f^{\tau+1}(u)du - \int_a^b f'(u) f^{\tau}(u)du\right] \\
&= \omega \left\{f^{\tau+1}(b) - \theta \int_a^b f^{\tau+1}(u)du - \left[\frac{1}{\tau+1} f^{\tau+1}(u)\right]_{u=a}^{u=b}\right\} \\
&= \omega \left[f^{\tau+1}(b) - \theta \int_a^b f^{\tau+1}(u)du - \frac{1}{\tau+1} f^{\tau+1}(b) + \frac{1}{\tau+1} f^{\tau+1}(a)\right] \\
&= \omega \left[\frac{\tau}{\tau+1} f^{\tau+1}(b) + \frac{1}{\tau+1} f^{\tau+1}(a) - \theta \int_a^b f^{\tau+1}(u)du\right]. \qquad (3.8)
\end{aligned}$$

Combining Equations (3.7) and (3.8) gives the desired upper bound. This ends the proof of Theorem 3.2. □

If we compare this theorem and Theorem 2.2, the monotonicity assumption on f has changed, and $\int_a^u e^{-\theta t} g(t)dt$ has replaced $\int_u^b e^{-\theta t} g(t)dt$ in the parametric primitive exponential-weighted integral inequality assumption considered.

Note that, since f is non-increasing and positive, the following assumption is not of interest: there exist three constants $\theta \leq 0$, $\tau \in \mathbb{R}$ and $\omega \geq 0$ such that, for any $u \in [a,b]$, we have

$$\int_a^u e^{-\theta t} g(t)dt \geq \omega e^{-\theta u} f^{\tau}(u).$$

Indeed, in this case, taking $u = a$, we get

$$0 = \int_a^a e^{-\theta t} g(t)dt \geq \omega e^{-\theta a} f^{\tau}(a) \geq 0,$$

so that $f(a) = 0$ and, since f is non-increasing, we necessarily have $f(u) = 0$ for any $u \in [a,b]$, which is not an interesting case.

The statement below fills the gap in Theorem 3.2 by considering the case "$\tau = -1$".

Theorem 3.3. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a,b] \mapsto (0,+\infty)$ be a differentiable non-increasing function and $g : [a,b] \mapsto [0,+\infty)$ be a function. We suppose that there exist two constants $\theta \leq 0$ and $\omega \geq 0$ such that, for any $u \in [a,b]$, we have*

$$\int_a^u e^{-\theta t} g(t)dt \leq \omega e^{-\theta u} \frac{1}{f(u)}. \qquad (3.9)$$

Then we have

$$\int_a^b f(t)g(t)dt \leq \omega \left\{1 - \theta(b-a) + \log\left[\frac{f(a)}{f(b)}\right]\right\}.$$

Proof. Re-using Equation (3.3), we have

$$\int_a^b f(t)g(t)dt = e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t)dt + \int_a^b [(-\theta)f(u) - f'(u)]e^{\theta u} \left[\int_a^u e^{-\theta t} g(t)dt\right] du. \tag{3.10}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $\theta \leq 0$, $f'(u) \leq 0$ for any $u \in [a, b]$ (since f is differentiable and non-increasing), $g(t) \geq 0$ for any $t \in [a, b]$, and the assumption in Equation (3.9) applied twice, we get

$$\begin{aligned}
& e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t)dt + \int_a^b [(-\theta)f(u) - f'(u)]e^{\theta u} \left[\int_a^u e^{-\theta t} g(t)dt\right] du \\
& \leq e^{\theta b} f(b) \left[\omega e^{-\theta b} \frac{1}{f(b)}\right] + \int_a^b [(-\theta)f(u) - f'(u)]e^{\theta u} \left[\omega e^{-\theta u} \frac{1}{f(u)}\right] du \\
& = \omega \left[1 - \theta \int_a^b du - \int_a^b f'(u) \frac{1}{f(u)} du\right] \\
& = \omega \left[1 - \theta(b-a) - \{\log[f(u)]\}_{u=a}^{u=b}\right] = \omega \{1 - \theta(b-a) - \log[f(b)] + \log[f(a)]\} \\
& = \omega \left\{1 - \theta(b-a) + \log\left[\frac{f(a)}{f(b)}\right]\right\}. \qquad (3.11)
\end{aligned}$$

Combining Equations (3.10) and (3.11) gives the desired upper bound. This ends the proof of Theorem 3.3. □

The result is similar to that in Theorem 2.3. The only changes are the sign before θ and the main logarithmic term.

Under certain assumptions, including an original parametric primitive exponential-weighted integral inequality assumption involving the derivative of an intermediate function different from that in Theorem 2.4, the theorem below presents a specific integral inequality.

Theorem 3.4. *Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $f : [a, b] \mapsto [0, +\infty)$ be a differentiable non-increasing function, $g : [a, b] \mapsto [0, +\infty)$ be a function and $h : [0, +\infty) \to \mathbb{R}$ be a differentiable function. We suppose that there exists a constant $\theta \leq 0$ such that, for any $u \in [a, b]$, we have*

$$\int_a^u e^{-\theta t} g(t)dt \leq h'[e^{\theta u} f(u)]. \tag{3.12}$$

Then we have

$$\int_a^b f(t)g(t)dt \leq e^{\theta b} f(b) h'[e^{\theta b} f(b)] - h[e^{\theta b} f(b)] + h[e^{\theta a} f(a)].$$

If the inequality in Equation (3.12) *is reversed, then the final inequality is also reversed.*

Proof. The necessary assumptions being satisfied, we can re-use Equation (3.3). We have

$$\int_a^b f(t)g(t)dt = e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t)dt + \int_a^b [(-\theta)f(u) - f'(u)]e^{\theta u} \left[\int_a^u e^{-\theta t} g(t)dt\right] du. \tag{3.13}$$

Using $f(u) \geq 0$ and $e^{\theta u} \geq 0$ for any $u \in [a, b]$, $\theta \leq 0$, $f'(u) \leq 0$ for any $u \in [a, b]$ (since f is differentiable and non-increasing), $g(t) \geq 0$ for any $t \in [a, b]$, and the assumption in Equation (3.12) applied twice, we get

$$\begin{aligned}
& e^{\theta b} f(b) \int_a^b e^{-\theta t} g(t) dt + \int_a^b [(-\theta) f(u) - f'(u)] e^{\theta u} \left[\int_a^u e^{-\theta t} g(t) dt \right] du \\
& \leq e^{\theta b} f(b) h'[e^{\theta b} f(b)] + \int_a^b [(-\theta) f(u) - f'(u)] e^{\theta u} h'[e^{\theta u} f(u)] du \\
& = e^{\theta b} f(b) h'[e^{\theta b} f(b)] - \int_a^b [e^{\theta u} f(u)]' h'[e^{\theta u} f(u)] du \\
& = e^{\theta b} f(b) h'[e^{\theta b} f(b)] - \left\{ h[e^{\theta u} f(u)] \right\}_{u=a}^{u=b} \\
& = e^{\theta b} f(b) h'[e^{\theta b} f(b)] - h[e^{\theta b} f(b)] + h[e^{\theta a} f(a)]. \qquad (3.14)
\end{aligned}$$

Combining Equations (3.13) and (3.14) gives the desired upper bound.

If the inequality in Equation (3.12) is reversed, then the inequality in Equation (3.14) is reversed, and the final inequality is also reversed.

This ends the proof of Theorem 3.4. □

Thanks to the arbitrary choice of h, this theorem can be adapted to many situations where an integral inequality involving the product of two functions is needed. Similar to what is done after the statement of Theorem 2.4, numerous examples can be given.

4. Conclusions and ideas for future research

In this article, we have presented and proved eight theorems on new integral inequalities. They are innovative in two complementary aspects: (i) their overall forms, and (ii) the parametric primitive exponential-weighted assumptions made. These results generalize existing frameworks and provide greater flexibility in the choice of functions involved. We thus enrich the theory of integral inequalities by offering a versatile approach that can be applied to a wider range of mathematical problems. Some ideas for future research are outlined below:

- *Refinement of the assumptions:* The introduced parametric primitive exponential-weighted assumptions can be further refined to derive sharper inequalities. The investigation of optimal parameter ranges for specific applications is a natural extension.
- *Multidimensional generalizations:* Extending our results to functions of several variables could address problems in multivariable calculus.
- *Numerical applications:* Developing numerical methods based on our inequalities could improve computational approaches in optimization problems, particularly those involving integral constraints.
- *Use of the truncated Laplace transform:* Using the truncated Laplace transform, as introduced in Equation (2.6), may lead to new mathematical developments, including possible other types of integral inequalities.

In addition to these ideas, we expect that our results and technical proofs will inspire further research on integral inequalities.

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