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MATERIALS WITH A DESIRED REFRACTION COEFFICIENT

ALEXANDER G. RAMM

ABSTRACT. Producing materials with a desired refraction coefficient is of great theoretical and practical interest. There was no general method for creating such materials, except the method, developed by the author. It was not even known that such a method do exist. The theoretical basis of this method is the asymptotic solution of the many-body wave scattering problem for many small bodies with prescribed boundary impedances. Multiple scattering is essential in our theory. The small bodies are embedded in a bounded region D, filled with a material with a known refraction coefficient $n_0(x)$. Our basic physical assumption is $a \ll d \ll \lambda$, where a is the characteristic size of the small particle, d is the minimal distance between neighboring particles, and λ is the wave length in D. The asymptotic of the solution to the above many-body scattering problem is derived for $a \to 0$.

1. INTRODUCTION

Let D be a bounded domain in $\mathbb{R}^3$ filled with a material with a known refraction coefficient $n_0(x)$. Let us embed into D many small particles D_m of a characteristic size a with boundary impedances ζ_m let d be the minimal distance between neighboring particles, and λ be the wavelength in D. We assume that

$$a \ll d \ll \lambda. \tag{1.1}$$

We assume that the boundary impedance of a small body D_m is given by the equation:

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \quad h(x) \in C(D), \tag{1.2}$$

where x_m is a point inside D_m and $\kappa \in [0, 1)$ is a constant that can be chosen by a researcher. Since D_m is small, its position can be characterized by a point x_m.

Let us formulate a recipe for producing a material with a desired refraction coefficient $n(x)$. The refraction coefficient $n(x)$ is defined by the wave equation

$$\Delta u + k^2 n^2(x) u = 0, \tag{1.3}$$

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where $u(x)$ is the wave field, $k > 0$ is the wave number. For simplicity, we assume in this paper that u is a scalar function. We also assume that D is filled in with a material whose refraction coefficient $n_0(x)$ is known.

Problem: *We want to produce in D the material with a desired refraction coefficient $n(x)$.*

Let us formulate a **recipe** for solving this problem.

Step 1. *Calculate*

$$p(x) = k^2[n_0^2(x) - n^2(x)]. \tag{1.4}$$

This step is trivial.

Step 2. *Given $p(x)$, find functions $h(x)$, Im$h(x) \leq 0$, and $N(x) \geq 0$ from the equation:*

$$4\pi h(x)N(x) = p(x), \qquad N(x) \geq 0, \quad Im\, p(x) \leq 0. \tag{1.5}$$

This step is also trivial. It has many solutions. For example, one can fix $N(x) > 0$ and define $h(x)$ by the formula:

$$h(x) = \frac{p(x)}{4\pi N(x)}, \tag{1.6}$$

provided that $\operatorname{Im} p(x) \leq 0$.

Step 3. *Distribute $N = O(\frac{1}{a^{2-\kappa}})$ small particles D_m with boundary impedances $\zeta_m = \frac{h(x_m)}{a^\kappa}$ in the domain D according to the law:*

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \quad a \to 0, \tag{1.7}$$

where Δ is any open subset of D, $\mathcal{N}(\Delta)$ is the number of small bodies in the subset Δ, $N(x)$ is the function from Step 2, the boundary impedance of the body D_m is chosen by formula (1.2)*, the function $h(x)$ in this formula is defined in Step 2, and x_m is an arbitrary fixed point inside D_m.*

Our main result is the following theorem.

Theorem 1. *The refraction coefficient of the material, obtained in the domain D after Step 3, tends to the desired refraction coefficient $n(x)$ as $a \to 0$.*

A proof of this result is not short. It is presented in the monographs [1],[2], [3], in the review paper [5], and in the author's papers cited in these references. Many other problems, based on similar ideas and methods, are presented in the above monographs: scattering of electromagnetic waves by many small bodies, scattering of heat waves by many small bodies, scattering of quantum-mechanical waves by many potentials with small supports, some inverse scattering problems, and other results.

Remark 1. *One may use the spherical particles $M(x_m, a)$ centered at the points x_m and of radius a for the creating of the materials with a desired refraction coefficient.*

Remark 2. *The total volume V_a of the embedded particles tends to zero when $a \to 0$.*

A proof is easy:

$$V_a = \frac{4}{3}\pi a^3 \times O(\frac{1}{a^{2-\kappa}}) = O(a^{1+\kappa}) \to 0, \quad a \to 0.$$

2. Additional considerations

It is known how to embed many small particles D_m at the prescribed points x_m. The size of these particles can be as small as 20 nanometers. One of the known methods is stereolitography.

The author is not familiar with the method of producing small particles with a prescribed boundary impedance. To use my recipe for creating materials with a desired refraction coefficient practically, it is necessary to develop a method for creating such small particles. Let us give some arguments showing that such particles can be prepared.

The first argument goes as follows. The wave scattering problem for one small body with a prescribed boundary impedance is:

$$\Delta u + k^2 n_0^2(x) u = 0, \tag{2.1}$$

$$u_\nu = \zeta_m u \quad on \quad S_m, \tag{2.2}$$

$$u = u_0 + v, \tag{2.3}$$

$$v_{|x|} - ikv = o(\frac{1}{|x|}), \qquad |x| \to \infty. \tag{2.4}$$

Problem (2.1)–(2.4) has a solution and this solution is unique, see [4], pp. 30–50. Therefore, the small body with the prescribed boundary impedance should exist. The condition $\mathrm{Im}\, n(x) \le 0$ is used for the proof of the uniqueness of the solution to the scattering problem (2.1)–(2.4).

The second argument for the existence of small bodies with a prescribed boundary impedance goes as follows.

Problem (2.1)–(2.4) with $\zeta = 0$ does exist. The same is true for $\zeta = \infty$. The small particles with any intermediate value of ζ should also exist.

3. Materials with a desired radiation pattern

Suppose that

$$\Delta u + k^2 n^2(x) u := \Delta u + k^2 u - q(x) u = 0, \tag{3.1}$$

where

$$q(x) = k^2[1 - n^2(x)]. \tag{3.2}$$

Recall that $n(x) = 1$ out of the bounded domain D.

The solution u to the scattering problem (3.1) exists and is unique, subject to the condition

$$u = u_0 + v, \tag{3.3}$$

where v satisfies the radiation condition and u_0 is the incident plane wave.

We assume in this section that $k > 0$ and $\alpha \in S^2$, the direction of the incident wave, are fixed. We denote the scattering amplitude $A(\beta)$,

$$A(\beta) := A_q(\beta) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-ik\beta\cdot y} q(y)u(y)dy, \tag{3.4}$$

where $u(y) := u(y, \alpha, k)$, and the dependence on α and k is omitted since α and k are fixed.

Choose an arbitrary $f(\beta) \in L^2(S^2)$, where S^2 is the unit sphere in $\mathbb{R}^3$, and an arbitrary small fixed number $\epsilon > 0$, and state the following new inverse problem:

Inverse problem. *Given $f(\beta)$ and ϵ, find $q \in L^2(D)$ such that*

$$\|f(\beta) - A_q(\beta)\|_{L^2(S^2)} < \epsilon. \tag{3.5}$$

It was not known if this problem has a solution. This problem was studied and solved in [3]. We formulate the result and refer the reader to [3] for a detailed proof.

Theorem 2. *For any $f(\beta) \in L^2(S^2)$ and an arbitrary small number $\epsilon > 0$, there exists a $q \in L^2(D)$ such that inequality* (3.5) *holds.*

Remark 3. There are infinitely many potentials satisfying (3.5). Indeed, the scattering amplitude depends continuously on the potential in the following sense:

$$\|A_{q_1} - A_{q_2}\|_{L^2(S^2)} \leq c\|q_1 - q_2\|_{L^2(D)}, \tag{3.6}$$

where $c > 0$ is a constant depending only on the bound for the norms of the potentials and on D. Therefore, small changes of the potential in $L^2(D)$ norm lead to small changes in the scattering amplitude in $L^2(S^2)$ norm in the sense (3.6). Thus, if inequality (3.5) holds for some $q \in L^(D)$, it will hold for any potential sufficiently close to q in $L^2(D)$ norm.

Remark 4. Theorem 2 can be of practical interest. For example, let $f(\beta) = 1$ in a narrow cone and $f(\beta) = 0$ outside this cone. Then, the body D with such a radiation pattern will have practical interest. The wave, scattered by this body, will be scattered mostly in the above cone. The scattered wave can be directed not as usual to the back of the body and to the front of the body, but mostly to the above cone.

4. CONCLUSION

A recipe is given for creating materials with a desired refraction coefficient by embedding many small particles with prescribed boundary impedances into a given material. The refraction coefficient can be so chosen that the resulting material will have a desired radiation pattern for a fixed wave number and a fixed direction of the incident plane wave. Materials with a prescribed radiation pattern can be created. For future developments, it is desirable to do many experiments based on the author's theory. One can change the given refraction coefficient $n_0(x)$ in a desired direction. Theoretically, the major advance is the author's (asymptotical as $a \to 0$) solution to the many-body scattering problem under the assumption $a \ll d \ll \lambda$.

5. DISCLOSURE STATEMENT.

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DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY
MANHATTAN, KS 66506, USA.
ORCID: 0000-0002-5160-4248
Email address: `ramm@ksu.edu`