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A NEW APPROACH ON PMS-ALGEBRAS

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ABSTRACT. In this article, we expose some new results about PMS-algebras. Thus, by connecting the concepts of subalgebras and ideals as well as ideals and right congruences in this class of logical algebras, we expand the paradigm related to PMS-algebras.

1. INTRODUCTION

Logical algebras, as particularly important phenomena both in logic and algebra as well as in theoretical computer sciences, are the subject of study by many researchers in the last more than half a century. Thus Y. Imai and K. Iséki 1966 in [6] introduced BCK/BCI-algebras. In 1983, Hu and Li introduced ([5]) the notion of a BCH-algebra, which is a generalization of the notions of BCK- and BCI-algebras. Then many classes of logical algebras began to appear, such as, for example, BH-algebras (1998, [8]), BF-algebras (2007, [17]), BRK-algebras (2012, [10]), QI-algebras (2017, [2]), BA-algebras (2019, [12]), JU-algebras (2009, [11] and 2024, [13]) and Bd-algebras (2022, [3]). In the spirit of the previous one, in 2016 the concept of PMS-algebra was introduced in [14] by P. M. Sithar Selvam and K. T. Nagalakshmi. Although this class of logical algebras has not attracted much interest from the academic community, it seems that there is a justification for understanding both its internal architecture and the properties of its substructures due to the existence of a certain similarity with the properties of GK-algebra ([4]). In this sense, the texts [1, 15, 16] deal with the application of fuzzy techniques on the substructures of this class of logical algebras, while the text [9] is focused on looking at PMS-algebra within a soft environment.

In this paper, we look at both the internal architecture of PMS-algebras and the properties of sub-algebras, ideals and congruences in them. In the first subsection of the main part of this paper, we establish both the existence of a natural congruence on this class of logical algebras and its basic properties. Then we prove that the direct product of any family of

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PMS-algebras is again a PMS-algebra. Sub-algebras and ideals in PMS-algebras as well as their mutual relations are discussed in the next subsection. Thus, it is shown that every ideal is a sub-algebra and vice versa. In the last subsection we look at the connections between ideals and (right) congruences in this class of logical algebras. The mentioned connection is in the following sense: While every ideal in a PMS-algebra generates a congruence in such an algebra, to create an ideal it is sufficient for the corresponding equivalence to be a right congruence.

2. Preliminaries

Definition 2.1. ([14]) A PMS-algebra is an algebra $\mathfrak{A} =: (A, \cdot, 0)$ of type $(2, 0)$ satisfying the following axioms:

(M) $(\forall x \in A)(0 \cdot x = x)$.

(PMS) $(\forall x, y, z \in A)((x \cdot z) \cdot (y \cdot z) = y \cdot x)$.

We denote this axiomatic system by **PMS** and the corresponding algebraic system generated by them as PMS-algebra.

Notions and notations used in this text that are not defined in it are taken from the literature, mostly from [7].

3. The main results

3.1. **A different view of the properties of PMS-algebras.** The next statement follows directly from (M):

Proposition 3.1. *Let $\mathfrak{A}$ be a PMS-algebra. Then*

(1) $0 \cdot 0 = 0$.

Proof. If we put $x = 0$ in (M), we get (1). □

Although the content of the following proposition has already been presented to the public, for example, in [15], page 154, and in [9], Proposition 2.2, we demonstrate a proof of its claims since the original source ([14]) is not available. Now we have:

Proposition 3.2. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a PMS-algebra. Then*

(Re) $(\forall x \in A)(x \cdot x = 0)$.

(2) $(\forall x, y \in A)((x \cdot y) \cdot y = x)$.

(3) $(\forall x, y \in A)(x \cdot (y \cdot x) = y \cdot 0)$

(4) $(\forall x, y \in A)((x \cdot y) \cdot 0 = y \cdot x)$.

(5) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (z \cdot y) \cdot x)$.

Proof. (Re): If we put $x = 0$, $y = 0$ and $z = x$ in (PMS), we get $(0 \cdot x) \cdot (0 \cdot x) = 0 \cdot 0$. from which it follows that $x \cdot x = 0$ with respect to (M) and (1).

(2): If we put $y = 0$ and $z = y$ in (PMS), we get $(x \cdot y) \cdot (0 \cdot x) = 0 \cdot x$. Then $(x \cdot y) \cdot y = x$ according to (M).

(3): If we put $x = 0$ and $z = x$ in (PMS), we get $(0 \cdot x) \cdot (y \cdot x) = y \cdot 0$. Thus, we have $x \cdot (y \cdot x) = y \cdot 0$ according to (M).

(4): If we put $z = y$ in (PMS), we get $(x \cdot y) \cdot (y \cdot y) = y \cdot x$, ie, we get $(x \cdot y) \cdot 0 = y \cdot x$ with respect to (Re).

(5): For arbitrary $x, y, z, w \in A$, we have $(x \cdot y) \cdot z = (z \cdot w) \cdot ((x \cdot y) \cdot w)$ by (PMS). In particular, for $w = y$, from here we get $(x \cdot y) \cdot z = (z \cdot y) \cdot ((x \cdot y) \cdot y) = (z \cdot y) \cdot x$ with application (2). □

Corollary 3.3. *A non-trivial PMS-algebra cannot be commutative.*

Proof. Let the PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ have at least two distinct elements, say $x, y \in A$ and $x \neq y$. From here we get $(x \cdot y) \cdot y \neq (y \cdot x) \cdot x$ according to (2), which means that $\mathfrak{A}$ is not a commutative algebra. □

If we define ([15], Definition 2.2) the relation $\preccurlyeq$ on a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ with

$$(\forall x, y \in A)(x \preccurlyeq y \iff x \cdot y = 0),$$

then we have that holds:

Proposition 3.4. *This induced relation in any PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ satisfies the following conditions:*

(6) $(\forall x \in A)(x \preccurlyeq x)$.

(7) $(\forall x, y, z \in A)((x \preccurlyeq z \wedge y \preccurlyeq z) \implies (y \preccurlyeq x \wedge x \preccurlyeq y))$.

Proof. Presence of (Re) ensures validity of (6).

Let $x, y, z \in A$ be such that $x \preccurlyeq z$ and $y \preccurlyeq z$. This means $x \cdot z = 0$ and $y \cdot z = 0$. Then $y \cdot x = (x \cdot z) \cdot (y \cdot z) = 0 \cdot 0 = 0$ and $x \cdot y = (y \cdot z) \cdot (x \cdot z) = 0 \cdot 0 = 0$ with respect (Re) and (PMS). So, $x \preccurlyeq y$ and $y \preccurlyeq x$. □

Corollary 3.5. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a PMS-algebra. Then*

(8) $(\forall x, y \in A)(x \preccurlyeq y \implies y \preccurlyeq x)$.

Proof. (8) immediately follows from (7) if we take $z = y$. □

Remark. *Formula (8) can be written in the following form*

(9) $(\forall x, y \in A)(x \cdot y = 0 \iff y \cdot x = 0)$.

The previous thinking, assuming that the reader understands what is meant by the term congruence, is the basis for the next theorem:

Theorem 3.6. *The relation $\equiv_{\preccurlyeq}$ on a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$, defined by*

$$(\forall x, y \in A)(x \equiv_{\preccurlyeq} y \iff (x \preccurlyeq y \wedge y \preccurlyeq x)),$$

is a congruence on $\mathfrak{A}$.

Proof. It is immediately clear that

$$x \equiv_{\preccurlyeq} y \iff x \cdot y = 0$$

by (9). Since the reflexivity and symmetry of this relation are obvious, let us first prove the transitivity of the relation $\preccurlyeq$. Let $x, y, z \in A$ be such that $x \preccurlyeq y$ and $y \preccurlyeq z$. This means $x \cdot y = 0$ and $y \cdot z = 0$. Then $z \cdot y = 0$ by (9). Thus $x \cdot z = (z \cdot y) \cdot (x \cdot y) = 0 \cdot 0 = 0$ with respect (PMS) and (Re). So, $x \preccurlyeq z$.

The right compatibility of this relation with the operation in $\mathfrak{A}$ can be proved as follows: If $x, y, z \in A$ are such that $x \cdot y = 0$, then we have $0 = x \cdot y = (y \cdot z) \cdot (x \cdot z)$ according to (PMS). Thus $x \cdot z \preccurlyeq y \cdot z$ according (9). On the other hand, we have

$$(z \cdot x) \cdot (z \cdot y) = ((z \cdot y) \cdot x) \cdot z = ((x \cdot y) \cdot z) \cdot z = x \cdot y = 0$$

applying (5) twice and taking into account (2). So, we have $z \cdot x \preccurlyeq z \cdot y$. □

Corollary 3.7. *Let* $\mathfrak{A} =: (A, \cdot, 0)$ *be a PMS-algebra. Then*

$$(\forall x, y \in A)((x \cdot z = y \cdot z \lor z \cdot x = z \cdot y) \implies x \equiv_{\preccurlyeq} y).$$

Proof. Let $x, y, z \in A$ be such that $x \cdot z = y \cdot z$. Then $(x \cdot z) \cdot (y \cdot z) = 0$ by (Re). Thus $y \cdot x = 0$ by (PMS). Hence $x \equiv_{\preccurlyeq} y$ according to Theorem 3.6.

Let $x, y, z \in A$ be such that $z \cdot x = z \cdot y$. Then $(z \cdot x) \cdot (z \cdot y) = 0$ by (Re). From here we get $((z \cdot y) \cdot x) \cdot z = 0$ with respect to (5). Applying (5) again, we get $((x \cdot y) \cdot z) \cdot z = 0$. We got $x \cdot y = 0$ taking into account (2). Hence, $x \equiv_{\preccurlyeq} y$ according to Theorem 3.6. □

The previous result can be seen as right and left cancellability in PMS-algebras.

In what follows, we deal with the creation of the direct product PMS-algebras. Let $\mathfrak{X} =: \{(A_i, \cdot_i, 0_i) : i \in I\}$ be a family of PMS-algebras. If on the set

$$\prod_{i \in I} A_i = \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $*$ as follows

$$(\forall f, g \in \prod_{i \in I} A_i)(\forall \in I)((f * g)(i) =: f(i) \cdot_i g(i)),$$

we created the structure $(\prod_{i \in I} A_i, *, f_0)$, where f_0 was chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i).$$

Before we start working with direct products of PMS-algebras, we say that the operation determined in this way is well-defined. If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

Theorem 3.8. *The direct product of any family of PMS-algebras, determined as above, is a PMS-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of PMS-algebra:

Let $f, g, h \in \prod_{i \in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(M) $(f_0 * f)(i) = f_0(i) \cdot_i f(i) = 0_i \cdot_i f(i) = f(i)$.

(PMS) Considering that

$$\begin{aligned}((f * h) * (g * h))(i) &= (f(i) \cdot_i h(i)) \cdot_i (g(i) \cdot_i h(i)) \\ &= g(i) \cdot_i f(i) = (g \odot f)(i),\end{aligned}$$

we have that (PMS) is a valid formula for the observed structure.

Therefore, the structure $(\prod_{i \in I} A_i, *, f_0)$ is a PMS-algebra. □

3.2. **Sub-algebras and ideals.**

Definition 3.1. A non-empty subset S of a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is a sub-algebra in $\mathfrak{A}$ if holds

(S1) $(\forall x, y \in A)((x \in S \land y \in S) \implies x \cdot y \in S)$.

We denote the family of all subalgebras of the algebra $\mathfrak{A}$ by $\mathfrak{S}(A)$.

The family $\mathfrak{S}(A)$ is not empty because $\{0\} \in \mathfrak{S}(A)$ and $A \in \mathfrak{S}(A)$.

It is easy to conclude that every sub-algebra in the PMS-algebra $\mathfrak{A}$ satisfies the condition

(S0) $0 \in S$.

Indeed, since S is a nonempty subset of A, there exists some $x \in A$ such that $x \in S$. Then $0 = x \cdot x \in S$ according to (Re) and (S1).

Proposition 3.9. *For every sub-algebra S in a PMS-algebra $\mathfrak{A}$ the following $S \cdot 0 = \{x \cdot 0 : x \in S\} \subseteq S$ holds.*

Proof. Taking into account (S0) and (S1), we conclude that $x \in S$ and $0 \in S$ implies $x \cdot 0 \in S$. □

Definition 3.2. Let $\mathfrak{A} =: (A, \cdot, 0)$ be a PMS-algebra. A subset J of A is an ideal in $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$.

(J1) $(\forall x, y, z \in A)((x \cdot y \in J \wedge x \cdot z \in J) \implies y \cdot z \in J)$.

The family of all ideals in the algebra A is denoted by $\mathfrak{J}(A)$.

It can be immediately proven that:

Theorem 3.10. *Every ideal in a PMS-algebra $\mathfrak{A}$ is a sub-algebra in $\mathfrak{A}$.*

Proof. If we put $x = 0$ in (J1), we get (S1) with respect to (M). Thus, $\mathfrak{J}(A) \subseteq \mathfrak{S}(A)$. □

In addition to the previous one, we have:

Proposition 3.11. *Let J be an ideal in a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Them:*

(J2) $(\forall x, y \in A)(x \cdot y \in J \implies y \cdot x \in J)$.

(J3) $(\forall x, y, z \in A)((x \cdot (y \cdot z) \in J \wedge x \cdot z \in J) \implies y \in J)$.

(J4) $(\forall x, y, z \in A)(((x \cdot y) \cdot z \in J \wedge x \in J) \implies y \cdot z \in J)$.

(J5) $(\forall x, y, z \in A)(((x \cdot z) \cdot y \in J \wedge y \cdot z \in J) \implies x \in J)$.

(J6) $(\forall y, z \in A)((y \cdot z \in J \wedge z \in J) \implies y \in J)$.

Proof. (J2): If we put $z = x$ in (J1), we get

$$(x \cdot y \in J \wedge x \cdot x = 0 \in J) \implies y \cdot x \in J.$$

(J3): If we put $y = y \cdot z$ in (J1), we get $(x \cdot (y \cdot z) \in J \wedge x \cdot z \in J) \implies (y \cdot z) \cdot z \in J$. From here we get $(x \cdot (y \cdot z) \in J \wedge x \cdot z \in J) \implies y \in J$ according to (2).

(J4) If we put $x = x \cdot y$ in (J1), we get $((x \cdot y) \cdot y \in J \wedge (x \cdot y) \cdot z \in J) \implies y \cdot z \in J$. From here we get $(x \in J \wedge (x \cdot y) \cdot z \in J) \implies y \cdot z \in J$ according to (2).

(J5): If we put $x = x \cdot z$ in (J1), we get $(((x \cdot z) \cdot y \in J \wedge y \cdot z \in J) \implies (x \cdot z) \cdot z \in J$. From here we get $(((x \cdot z) \cdot y \in J \wedge y \cdot z \in J) \implies x \in J$ according to (2).

(J6): If we put $x = 0$ and $y = y \cdot z$ in (J1), we get

$$(0 \cdot (y \cdot z) \in J \wedge 0 \cdot z \in J) \implies (y \cdot z) \cdot z \in J.$$

From here, taking into account (M) and (2), we get $(y \cdot z \in J \wedge z \in J) \implies y \in J$ which proves the validity of (J6). □

Corollary 3.12. *Let J be an ideal in a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then*

(J7) $(\forall y, x \in A)((y \preccurlyeq z \wedge z \in J) \implies y \in J)$.

Proof. The validity of (J7) follows directly from presence of (J6). □

Now we can prove:

Theorem 3.13. *Any sub-algebra in a PMS-algebra is an ideal in it.*

Proof. Let S be a sub-algebra in a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ and let $x, y, z \in A$ be such that $x \cdot y \in S$ and $x \cdot z \in S$. Then $(x \cdot y) \cdot (x \cdot z) \in S$ by (S1). This $((x \cdot z) \cdot y) \cdot x \in J$ in accordance (5). Hence, $((y \cdot z) \cdot x) \cdot x \in J$ by (5) again. This means that $y \cdot z \in J$ with respect to (2). So, the sub-algebra S is an ideal in $\mathfrak{A}$. Therefore, $\mathfrak{S}(A) \subseteq \mathfrak{J}(A)$. □

Example 3.3. Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows

$\cdot$	0	a	b	c
0	0	a	b	c
a	b	0	a	b
b	a	b	0	a
c	c	a	b	0

It is easy to verify that $\mathfrak{A} =: (A, \cdot, 0)$ is a PMS-algebra ([9], Example 3.2).

Subsets $S_0 = \{0\}$, $S_3 = \{0, c\}$ and $S_4 = \{0, a, b\}$ are sub-algebras in $\mathfrak{A}$. So, $\mathfrak{S}(A) = \{S_0, S_3, S_4, A\}$. Also, $\mathfrak{J}(A) = \{S_0, S_3, S_4, A\}$. Other subsets of the set A are not sub-algebras in $\mathfrak{A}$. For illustration, the subset $S_1 = \{0, a\}$ is not a sub-algebra in $\mathfrak{A}$ because, for example, we have $a \cdot 0 = b \notin S_1$. Also, the subset $S_6 = \{0, b, c\}$ is not a sub-algebra in $\mathfrak{A}$ because, for example, we have $b \cdot c = a \notin S_6$. □

Also, we have:

Proposition 3.14. *Let $\{(A_i, \cdot_i, 1_i) : i \in I\}$ be a family of PMS-algebras, K be a subset of I and let J_i be an ideal in $(A_i, \cdot_i, 1_i)$ for each $i \in K$. Then $\prod_{i \in I} T_i$, where $T_i = J_i$ for $i \in K$ and $T_j = A_j$ for $j \neq i \in K$, is an ideal in the PMS-algebra $\prod_{i \in I} A_i$.*

Proof. If $K = \emptyset$, then $\prod_{i \in I} T_i = \prod_{i \in I} A_i$, so $\prod_{i \in I} T_i$ is certainly an ideal in $\prod_{i \in I} A_i$. Assume, therefore, that $K \neq \emptyset$. It is sufficient, according to Theorem 3.13, to prove that $\prod_{i \in I} T_i$ satisfies the conditions (S1).

Let $x, y \in \prod_{i \in I} A_i$ be such that $x \in \prod_{i \in I} T_i$ and $y \in \prod_{i \in I} T_i$. This means $x(i) \in J_i$ and $y(i) \in J_i$ for each $i \in K$. Then $(x * y)(i) = x(i) \cdot_i y_i \in J_i$ since J_i is a sub-algebra in $(A_i, \cdot_i, 1_i)$ for each $i \in K$. Hence $x * y \in \prod_{i \in I} T_i$.

As shown, $\prod_{i \in I} T_i$ is an ideal in $\prod_{i \in I} A_i$. □

Example 3.4. Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

It is easy to verify that $\mathfrak{A} =: (A, \cdot, 0)$ is a PMS-algebra ([9], Example 2.5). Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$ and $J_3 = \{0, c\}$ are ideals in $\mathfrak{A}$.

Structure $\mathfrak{A} \times \mathfrak{A} = (A \times A, *, (0, 0))$ is a PMS-algebra in accordance with Proposition 3.14 where $(x, y) * (u, v) = (x \cdot u, y \cdot v)$ for arbitrary $x, y, u, v \in A$. Subsets $J_i \times A$, $A \times J_i$ and $J_i \times J_i$ are ideals in $\mathfrak{A} \times \mathfrak{A}$ for arbitrary $i \in \{0, 1, 2, 3\}$. □

At the end of this subsection, let us to prove:

Theorem 3.15. *The family $\mathfrak{J}(A)$ of all ideals in a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is a complete lattice.*

Proof. Let $\{J_i : i \in I\}$ be a family of ideals in a PMS-algebra $\mathfrak{A}$. Since it is obvious that $0 \in \cap_{i \in I} J_i$ holds, it remains to prove the validity of (S1) for the set $\cap_{i \in I} J_i$. Let $x, y \in A$

be such that $x \in \cap_{i\in I} J_i$ and $y \in \cap_{i\in I} J_i$. Then $x \in J_i$ and $y \in J_i$ for each $i \in I$. Thus $x \cdot y \in J_i$ for each $i \in I$ by (S1). Hence, $x \cdot y \in \cap_{i\in I} J_i$.

If we denote by $\mathcal{Z}$ the family of all ideals of the algebra $\mathfrak{A}$ that contain the set $\cup_{i\in I} S_i$, then $\cap\mathcal{Z}$ is an ideal in $\mathfrak{A}$ according to the first part of this proof.

If we put $\sqcap_{i\in I} J_i = \cap_{i\in I} J_i$ and $\sqcup_{i\in I} J_i = \cap\mathcal{Z}$, then $(\mathfrak{J}(A), \sqcap, \sqcup)$ is a complete lattice. □

3.3. **Ideals and congruences.** For an equivalence ρ on the support A of the PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ we say that it is a right congruence on $\mathfrak{A}$ if the following holds

$$(\forall x, y, z \in A)((x, y) \in \rho \implies (x \cdot z, y \cdot z) \in \rho).$$

The concept of left congruence is defined analogously. For an equivalence on the PMS-algebra $\mathfrak{A}$, we say that it is a congruence on $\mathfrak{A}$ if it is a left and right congruence on $\mathfrak{A}$. We denote the family of all (right) congruences on a PMS-algebra $\mathfrak{A}$ by $(\mathfrak{Q}_r(A))$ $\mathfrak{Q}(A)$. It can be shown, by analogy with Theorem 3.16, that the families $\mathfrak{Q}_r(A)$ and $\mathfrak{Q}(A)$ are complete lattices.

The following theorem relates any ideal in the PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$ to a congruence on $\mathfrak{A}$.

Theorem 3.16. *For a given ideal J of the PMS-algebra $\mathfrak{A} =: (a, \cdot, 0)$, the relation ρ_J, defined as follows*

$$(\forall x, y \in A)((x, y) \in \rho_J \iff x \cdot y \in J),$$

is a congruence on $\mathfrak{A}$ such that for the kernel $[0]_{\rho_J} =: \{x \in A : (x, 0) \in \rho_J\}$ of the congruence ρ_J the following $[0]_{\rho_J} = J$ holds.

Proof. The relation ρ_J is reflexive due to (J0) and symmetric due to (J2). It remains to prove transitivity. Let $x, y, z \in A$ be such that $(x, y) \in \rho_J$ and $(y, z) \in \rho_J$. This means $x \cdot y \in J$ and $y \cdot z \in J$. Then $z \cdot y \in J$ by (J2). Thus $xz = (y \cdot z) \cdot (x \cdot z) \in J$ by (S1) since J is a sub-algebra in A. So, $(x, z) \in \rho_J$.

For arbitrary elements $x, y, z \in A$ such that $(x, y) \in \rho_J$, i.e. such that $x \cdot y \in J$, according to (PMS), we have $(y \cdot z) \cdot (x \cdot z) = x \cdot y \in J$. So, $(x * z, y * z) \in \rho_J$. Therefore, ρ_J is a right congruence on $\mathfrak{A}$. On the other hand, since for arbitrary $x, y, z \in A$ such that $(x, y) \in \rho$, i.e. such that $x \cdot y \in J$, we have $(z \cdot x) \cdot (z \cdot y) = ((z \cdot y) \cdot x) \cdot z = ((x \cdot y) \cdot z) \cdot z = x \cdot y \in J$, we conclude that $(z \cdot x, z \cdot y) \in \rho$ holds. Thus, ρ is a left congruence on $\mathfrak{A}$.

For arbitrary $u \in [0]_{\rho_J}$, we have $(u, 0) \in \rho_J$ which means $u = 0 \cdot u \in J$. Thus, $[0]_{\rho_J} \subseteq J$. It is obvious that the reverse inclusion also applies. Thus, $[0]_{\rho_J} = J$. □

The previous theorem realized the correspondence $\mathfrak{J}(A) \longrightarrow \mathfrak{Q}(A)$. According to what has been proven, we have the correspondence

$$J \longmapsto \rho_J \longmapsto [0]_{\rho_J} = J$$

for an arbitrary ideal J in a PMS-algebra $\mathfrak{A}$.

Example 3.5. Let $A = \{0, a, b, c\}$ as in Example 3.4. Then $\mathfrak{A} =: (A, \cdot, 0)$ be a PMS-algebra. The ideal $J = \{0, c\}$ in $\mathfrak{A}$ generates the congruence $\rho_J = \{(0, 0), (0, c), (a, a), (b, b), (c, c), (c, 0)\}$ on $\mathfrak{A}$.

In this example, it is obvious that $\rho_J = (J \cup J) \cup \equiv_{\preccurlyeq}$ holds. It is quite justified to ask: Can every congruence ρ on a PMS-algebra $\mathfrak{A}$ be represented in this form $\rho = (J \times J) \cup \equiv_{\preccurlyeq}$ for some ideal J in $\mathfrak{A}$?

For the reversal of Theorem 3.16, we have:

Theorem 3.17. *Let ρ be a right congruence on a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then the kernel $[0]_\rho =: \{x \in A : (x, 0) \in \rho\}$ of ρ is an ideal in $\mathfrak{A}$.*

Proof. Due to the reflexivity of the relation ρ, we have $0 \in [0]_\rho$. Let $x, y \in A$ be such that $x \in [0]_\rho$ and $y \in [0]_\rho$. This means $(x, 0) \in \rho$ and $(y, 0) \in \rho$. Then $(x \cdot y, 0 \cdot y) = (x \cdot y, y) \in \rho$ since ρ is a right congruence. This $(x \cdot y, 0) \in \rho$ by transitivity of ρ. Hence $x \cdot y \in [0]_\rho$. This proves the property (S1) for the set $[0]_\rho$. Now, according to Theorem 3.13, we conclude that $[0]_\rho$ is an ideal in $\mathfrak{A}$. □

The previous theorem describes the correspondence $\mathfrak{Q}_r(A) \longrightarrow \mathfrak{J}(A)$. However, we can prove that any right congruence on PMS-algebra $\mathfrak{A}$ is also a left congruence on $\mathfrak{A}$. Indeed, let $x, y, z \in A$ be such that $(x, y) \in \rho$. Then $(x \cdot z, y \cdot z) \in \rho$ and $((x \cdot z) \cdot 0, (y \cdot z) \cdot 0) \in \rho$. From where, according to (4), we get $(z \cdot x, z \cdot y) \in \rho$. This makes it possible to prove correspondence

$$\rho \longmapsto [0]_\rho \longmapsto \rho_{[0]_\rho} = \rho$$

for an arbitrary relation ρ on the PMS-algebra $\mathfrak{A}$.

Let us point out that the ideal $\{0\}$ generates the congruence $\equiv_\preccurlyeq$ and vice versa, the kernel of this congruence is the ideal $\{0\}$.

Further on, if ρ is a congruence on a PMS-algebra $\mathfrak{A} =: (A, \cdot, 0)$, then we can create the structure $\mathfrak{A}/\rho =: (A/\rho, \odot, [0]_\rho)$ where the operation $\odot$ is defined as follows

$$(\forall x, y \in A)([x]_\rho \odot [y]_\rho =: [x \cdot y]_\rho).$$

For this purpose, it is necessary to prove that $\odot$ is a well-defined operation on $\mathfrak{A}/\rho$. Let $x, y, u, v \in A$ be arbitrary elements such that $[x]_\rho = [u]_\rho$ and $[y]_\rho = [v]_\rho$. This means $(x, u) \in \rho$ and $(y, v) \in \rho$. From here we get $(x \cdot y, u \cdot y) \in \rho$ and $(u \cdot y, u \cdot v) \in \rho$ because ρ is a congruence on $\mathfrak{A}$. Then $(x \cdot y, u \cdot v) \in \rho$. This means $[x]_\rho \odot [y]_\rho = [u]_\rho \odot [v]_\rho$ which means that the operation $\odot$ is well-defined.

Now we can prove the following theorem:

Theorem 3.18. *If ρ is a congruence on a PMS-algebra $\mathfrak{A} = (A, \cdot, 0)$, then $\mathfrak{A}/\rho$ is a PMS-algebra again.*

Proof. The proof can be demonstrated by direct verification, so we will leave it out. □

Example 3.6. (a) Let $A = \{0, a, b, c\}$ as in Example 3.3. Then $\mathfrak{A} =: (A, \cdot, 0)$ be a PMS-algebra. The subset $J = \{0, a, b\}$ is an ideal in $\mathfrak{A}$. If ρ_J is the congruence on $\mathfrak{A}$ generated by the ideal J, then $\mathfrak{A}/\rho_J = (\{[0]_{\rho_J}, [c]_{\rho_J}\}, \odot, [0]_{\rho_J})$ is the quotient PMS-algebra.

(b) Let $A = \{0, a, b, c\}$ as in Example 3.4. Then $\mathfrak{A} = (A, \cdot, 0)$ be a PMS-algebra. The subset $J = \{0, c\}$ is an ideal in $\mathfrak{A}$. If ρ_J is the congruence on $\mathfrak{A}$ generated by the ideal J (see Example 3.5), then $\mathfrak{A}/\rho_J = (\{[0]_{\rho_J}, [a]_{\rho_J}, [b]_{\rho_J}\}, \odot, [0]_{\rho_J})$ is the quotient PMS-algebra. □

The following theorem illustrates the existence of the objects and processes mentioned above. In what follows, we need the following definition:

Definition 3.7. ([9], Definition 2.5) Let $\mathfrak{A} =: (A, \cdot, 0_A)$ and $\mathfrak{B} =: (B, \star, 0_B)$ be PMS-algebras. A mapping $f : A \longrightarrow B$ is called homomorphism if the following holds:

(f1) $(\forall x, y \in A)(f(x \cdot y) = f(x) \star f(y))$.

It can immediately be concluded that it is also valid

(f0) $f(0_A) = 0_B$.

Indeed, for an arbitrary $x \in A$, we have $f(0_A) = f(x \cdot x) = f(x) \star f(x) = 0_B$ by (Re).

We denote the homomorphism between PMS-algebras $\mathfrak{A}$ and $\mathfrak{B}$ by $f : \mathfrak{A} \longrightarrow \mathfrak{B}$. The terms epimorphism and monomorphism are understood here in the usual way.

The following theorem is related to the previous one:

Theorem 3.19. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism between PMS-algebras.*

(a) *If S is a sub-algebra in $\mathfrak{A}$, then $f(S)$ is a sub-algebra in $\mathfrak{B}$.*

(b) *$f(\mathfrak{A})$ is a sub-algebra in $\mathfrak{B}$.*

(c) *If K is an ideal in $\mathfrak{B}$, then $f^{-1}(K) = \{x \in A : f(x) \in K\}$ is an ideal in $\mathfrak{A}$.*

(d) *Kernel $Ker(f) = f^{-1}(\{1_B\})$ of the homomorphism f is an ideal in $\mathfrak{A}$.*

(e) *The relation ρ_f, defined as follows*

$$(\forall x, y \in A)((x, y) \in \rho_f \iff f(x) = f(y)),$$

is a congruence on $\mathfrak{A}$.

(f) *The function $\pi : \mathfrak{A} \longrightarrow \mathfrak{A}/\rho$, defined by $\pi(x) =: [x]_\rho$ is an epimorphism.*

(g) *The function $g : \mathfrak{A}/\rho \longrightarrow f(\mathfrak{A})$, defined by $g([x]_\rho) =: f(x)$ is a monomorphism.*

(h) *The homomorphism f can be represented as a superposition $f = i \circ g \circ \pi$, where $i : f(\mathfrak{A}) \longrightarrow \mathfrak{B}$ is an inclusion.*

Proof. Since the proofs of all the above statements can be obtained by direct verification, we omit them. □

Example 3.8. Let $\mathfrak{A} =: (A, \cdot, 0)$ be a PMS-algebra. Let us put $\varphi(x) =: x \cdot 0$ for an arbitrary $x \in A$ and prove that the correspondence, defined in this way, is a homomorphism between PMS-algebras. First, it is clear that the correspondence, determined in this way, is a function from $\mathfrak{A}$ to $\mathfrak{A}$. Second, it is obvious that $\varphi(0) = 0 \cdot 0 = 0$ holds due to (1). For arbitrary $x, y \in A$, according to (PMS), we have $(x \cdot 0) \cdot (y \cdot 0) = y \cdot x$. From here, taking into account (4), we get $(x \cdot 0) \cdot (y \cdot 0) = (x \cdot y) \cdot 0$. This proves that $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ holds. As shown, φ is a homomorphism. □

4. FINAL COMMENTS

The algebraic structure known as PMS-algebra, introduced in 2018, is the focus of this paper. As is usual in the research of logical algebras, the author's attention is devoted not only to the internal architecture of this structure, but also to the properties of its sub-structures, such as sub-algebras and ideals. Roughly speaking, the author believes, with a high degree of conviction, that the entire material presented in Section 3 is novel. With deep conviction, the author believes that Theorem 3.1, Theorem 3.2, as well as pairs of interrelated theorems 3.10 and 3.13, and 3.19 and 3.18 are novelties in this domain. The author is convinced that the material on PMS-algebras, presented in this article, can be an inspiration for further and deeper research of both the sub-components and the properties of those components, as well as some extensions of this class of algebras, including pseudo and hyper extensions.

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