ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 8, Number 1 (2025), 137-149
DOI: 10.62072/acm.2025.080111
ISSN: 2582-0818
https://www.technoskypub.com/journal/acm/

SOME PROPERTIES OF FUNDAMENTAL FORMULATION OF TRICOMPLEX POLYNOMIALS

MD. NASIRUZZAMAN* AND M. MURSALEEN

ABSTRACT. In this paper, we introduce the algebra of tricomplex numbers as in idempotent forms and tricomplex polynomials as a generalization of the field of bicomplex numbers. We describe how to define elementary functions in such an algebra, polynomials, Taylor series for tricomplex holomorphic functions, algebra of eigenvalues corresponding to an eigenvector on tricomplex space, and using a specific result, we define tricomplex polynomial, which is a better generalization of bicomplex polynomial.

1. INTRODUCTION

Complex numbers can be generalized to higher dimensions in a variety of ways. The most famous extension is provided by Hamilton's quaternions [6], which are mostly used to depict rotations in three dimensions. Quaternions, on the other hand, do not multiply commutatively. At the close of the 19th century, Corrado Segre [5] discovered another expansion by describing specific multidimensional algebras. These days, this kind of number is referred to as a multicomplex number. Both Price [1] and Fleury [8] conducted in-depth research on them. Segre [7] invented bicomplex numbers, which are a generalization of complex numbers to four real dimensions, similar to quaternions. These two number systems are different because (i) bicomplex numbers are commutative, whereas quaternions are not, and (ii) quaternions form a division algebra, whereas bicomplex numbers do not. It has been demonstrated that the bicomplex numbers system is more appealing than the quaternions for these reasons. When we define multicomplex numbers as the special higher dimensional counterparts of bicomplex numbers and tricomplex numbers, certain characteristics of bicomplex numbers are maintained. An overview of the structure of the multicomplex space $\mathbb{C}^k$ is given at the beginning of this work [1].
Crucially, we define a number of idempotent elements that will be essential for all future developments, including the zeros of characteristic polynomials on tricomplex space, the

2020 *Mathematics Subject Classification.* 16D10, 30G35, 46C05, 46C50.
Key words and phrases. Bicomplex numbers; Multicomplex number system; Tricomplex number system; Bicomplex analysis; Tricomplex Polynomial; Multicomplex Taylor series.
Received: February 22, 2025. Accepted: March 17, 2025. Published: March 31, 2025.

*Corresponding author.

characteristic root of tricomplex matrices, the convergent of a tricomplex sequence, tricomplex polynomials, tricomplex derivatives, and Taylor series representation. After that, we may demonstrate some practical characteristics of functions on $\mathbb{C}^3$. In this paper we introduce elementary functions, such as polynomials, exponentials, trigonometric functions, Taylor representation for holomorphic function,in this algebra, as well as their inverses (something that, incidentally, is not possible in the case of quaternions). We will show how these elementary functions enjoy properties that are very similar to those enjoyed by their complex counterparts. To generalize, the observation consists in looking at maps $f = (f_1, f_2)$ in a open set $U \subset \mathbb{C}^3 \to \mathbb{C}^3$ and to ask that each component f_1, f_2 be holomorphic in z_1 and in z_2 without assuming any additional relationship between them. Though both generalizations are important, and give rise to large and interesting theories. We believe that there is another, even more appropriate generalization, which so far has not received enough attention (cf.[2],[3],[4]). To this purpose, we introduce to tricomplex Cauchy-Riemann system and to apply it to pairs of holomorphic functions (f_1, f_2) in a open set $U \subset \mathbb{C}^3 \to \mathbb{C}^3$, so that the pair (f_1, f_2) can be interpreted as a map of $\mathbb{C}^3$ to itself. It is then natural to ask whether it makes any sense to consider pairs (f_1, f_2) for which the following system is satisfied:

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}$$
$$\frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}$$

The bicomplex polynomial was discussed by M.E. Luna-Elizarrara's, M. Shapiro [2], and the eigenvalues for bicomplex matrices was discussed in [9]. We generalized it one step far in tricomplex space $\mathbb{C}^3$ The algebra which one obtains is the tricomplex algebra. In this paper we show how to introduce the elementary functions, such as polynomials, characteristic polynomial functions, zeros of characteristic polynomial, on tricomplex space $\mathbb{C}^3$. We will show how these elementary functions enjoy properties that are very similar to those enjoyed by their complex counterparts. If $\mathcal{A} := \{(a_{lj}) \in \mathbb{C}^3_{m\times n} = \mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2\}$ and $\mathcal{A}u = \lambda u$ which is equivalent to

$$\begin{cases} \mathcal{A}_1 u_1 = \lambda_1 u_1, \\ \mathcal{A}_2 u_2 = \lambda_2 u_2. \end{cases}$$

Then λ is eigenvalue of the tricomplex matrix A corresponding to eigenvector u where $\lambda := \lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2 \in \mathbb{C}^3$ and $u = u_1\mathcal{I}_1 + u_2\mathcal{I}_2$. To generalize the above observation consists in looking at

$$\mathcal{A} := (a_{lj}) \in \mathbb{C}^3_{m\times n} = \mathcal{B}_{i_3}\mathcal{I}_1 + \mathcal{C}_{i_3}\mathcal{I}_2 := \mathcal{B}_{i_2}\mathcal{I}_1 + \mathcal{C}_{i_2}\mathcal{I}_2$$

where $\mathcal{B}_{i_3}, \mathcal{C}_{i_3} \in \mathbb{C}^2_{m\times n}$ and $\mathcal{B}_{i_2}, \mathcal{C}_{i_2} \in \mathbb{C}^2_{m\times n}$.

Without assuming any additional relationship between them, both generalizations are important, and give rise to large and interesting theories, we believe that there is another even more appropriate generalization, which so far has not received enough attention. For more details and recent published articles we prefer to see, for example, families of bicomplex holomorphic functions [10], bicomplex and hyperbolic numbers [11], bicomplex beta operators [12] and multicomplex numbers and their properties [13].

2. Preliminaries

Definition 2.1 (Bicomplex Numbers([1, 9])). The set of the bicomplex numbers is defined as

$$\mathbb{BC} := \{z_1 + z_2 i_2 \mid z_1, z_2 \in \mathbb{C}^1(i_1)\} \tag{2.1}$$

where i_1, i_2 are the imaginary units and governed by the rules

$$i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1 = j \tag{2.2}$$

and so,

$$j^2 = 1, i_1 j = j i_1 = -i_2, i_2 j = j i_2 = -i_1. \tag{2.3}$$

Note that

$$\mathbb{C}^1(i_k) := \{x + y i_k \mid i_k^2 = -1 \text{ and } x, y \in \mathbb{R} \text{ for } k = 1, 2\} \tag{2.4}$$

where $\mathbb{C}^1$ is the set of all complex numbers with the imaginary units i_k for $k = 1, 2$. Thus the bicomplex numbers are complex numbers with complex coefficients, which explain the name of bicomplex.

With the addition and the multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{BC}$ makes up a commutative ring (in fact they are the particular case of the so called multicomplex numbers).

Clearly the bicomplex numbers

$$\mathbb{BC} \cong \mathrm{Cl}_{\mathbb{C}}(1, 0) \cong \mathrm{Cl}_{\mathbb{C}}(0, 1) \tag{2.5}$$

are unique among the complex Clifford algebras in that they are commutative but not division algebras. It is also convenient to write the set of bicomplex numbers as

$$\mathbb{BC} := \{x_0 + x_1 i_1 + x_2 i_2 + x_3 i_1 i_2 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}. \tag{2.6}$$

We know the complex conjugation plays an important role for both algebraic and geometric properties of $\mathbb{C}^1$. So for bicomplex numbers there are three possibilities of conjugations. Let $z \in \mathbb{BC}$ and $z_1, z_2 \in \mathbb{C}^1(i_1)$, such that $z := z_1 + z_2 i_2$, then we define the three conjugation as:

$$z^{\dagger_1} = (z_1 + z_2 i_2)^{\dagger_1} = \overline{z}_1 + \overline{z}_2 i_2 \tag{2.7}$$

$$z^{\dagger_2} = (z_1 + z_2 i_2)^{\dagger_2} = z_1 - z_2 i_2 \tag{2.8}$$

$$z^{\dagger_3} = (z_1 + z_2 i_2)^{\dagger_3} = \overline{z}_1 - \overline{z}_2 i_2. \tag{2.9}$$

All the three kinds of conjugations have some of the standard properties of conjugations, such as

$$(z_1 + z_2)^{\dagger_k} = z_1^{\dagger_k} + z_2^{\dagger_k} \tag{2.10}$$

$$(z_1^{\dagger_k})^{\dagger_k} = z_1 \tag{2.11}$$

$$(z_1 . z_2)^{\dagger_k} = z_1^{\dagger_k} . z_2^{\dagger_k}. \tag{2.12}$$

We know that the product of a standard complex number with its conjugate gives the square of the Euclidean metric in $\mathbb{R}^2$. Thus the analogs of this, for bicomplex numbers, are the following. Let $z_1, z_2 \in \mathbb{C}^1(i_1)$ and $z := z_1 + z_2 i_2 \in \mathbb{BC}$, then we have:

$$\mid z \mid_{i_1}^2 = z.z^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}^1(i_1) \tag{2.13}$$

$$\mid z \mid_{i_2}^2 = z.z^{\dagger_1} = (\mid z_1 \mid^2 - \mid z_2 \mid^2) + 2Re(z_1 \overline{z}_2) i_2 \in \mathbb{C}^1(i_2) \tag{2.14}$$

$$\mid z \mid_{j}^2 = z.z^{\dagger_3} = (\mid z_1 \mid^2 + \mid z_2 \mid^2) - 2Im(z_1 \overline{z}_2) j \in \mathbb{D}. \tag{2.15}$$

Where $\mathbb{D}$ is the subalgebra of hyperbolic numbers, and is defined as

$$\mathbb{D} := \{x + yj \mid j^2 = 1, x, y \in \mathbb{R}, \} \cong \mathrm{Cl}_{\mathbb{R}}(0, 1). \tag{2.16}$$

Note that for $z_1, z_2 \in \mathbb{C}^1(i_1)\; and\; z := z_1 + z_2 i_2 \in \mathbb{BC}$, we can define the usual (Euclidean in $\mathbb{R}^4$) norm of z as $\mid z \mid = \sqrt{\mid z_1 \mid^2 + \mid z_2 \mid^2} = \sqrt{Re(\mid z \mid_j^2)}$. It is easy to verifying that $z.\frac{z^{\dagger_2}}{|z|_{i_1}^2} = 1$. Hence the inverse of z is given by

$$z^{-1} = \frac{z^{\dagger_2}}{\mid z \mid_{i_1}^2}. \tag{2.17}$$

Idempotent basis:

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers e_1 and e_2 defined as $e_1 = \frac{1+i_1 i_2}{2}$, $e_2 = \frac{1-i_1 i_2}{2}$ In fact e_1 and e_2 are hyperbolic numbers ($i_1 i_2 = i_2 i_1 = j$). They make up the so called idempotent basis of the bicomplex numbers, and one easily can check that

$$e_1^2 = e_1, e_2^2 = e_2, e_1 + e_2 = 1, e_1.e_2 = 0, e_k^{\dagger_3} = e_k \text{ (for } k = 1, 2). \tag{2.18}$$

Thus any bicomplex number can be written as

$$z = z_1 + z_2 i_2 = \alpha_1 e_1 + \alpha_2 e_2, \text{where } \alpha_1 = z_1 - z_2 i_1, \alpha_2 = z_1 + z_2 i_1. \tag{2.19}$$

Definition 2.2 (Multicomplex Numbers([1, 9])). We must firstly define the multiicomplex space in which we have to work, that will do so inductively. For the base case $k = 0$, we define $\mathbb{C}^0 := \mathbb{R}$, that is the set of all real numbers with additions, multiplication and norm being defined as usual. The case for $k = 1$ is also familiar to $\mathbb{C}^1$, which is simply the standard complex plane with arithmetic and norm usually defined. The case of $k = 2$ and $k = 3$ are familiar with $\mathbb{C}^2$ and $\mathbb{C}^3$ are the simply bicomplex plane and tricomplex plane. So we define

$$\mathbb{C}^k := \{z_1 + z_2 i_k \mid z_1, z_2 \in \mathbb{C}^{k-1}, k > 1, i_k^2 = -1\; and\; i_m i_n = i_n i_m \text{ for } m \neq n\}. \tag{2.20}$$

The arithmetic is defined in usual way and if z_1, z_2, z_3 and $z_4 \in \mathbb{C}^{k-1}$ and w_1, w_2 and $w_3 \in \mathbb{C}^k$, then

$$(z_1 + z_2 i_k) + (z_3 + z_4 i_k) = (z_1 + z_3) + (z_2 + z_4) i_k \tag{2.21}$$

$$(z_1 + z_2 i_k)(z_3 + z_4 i_k) = (z_1 z_3 - z_2 z_4) + (z_1 z_4 + z_2 z_3) i_k \tag{2.22}$$

$$w_1(w_2 + w_3) = w_1 w_2 + w_1 w_3. \tag{2.23}$$

With this definition it is simple to show that for all natural numbers k,$\mathbb{C}^k$ is a commutative ring with unity. Further, assuming have defined the norm
$\| \,.\, \|_{k-1}$: $\mathbb{C}^{k-1} \to \mathbb{R}_{\geq 0}$, we define the norm $\| \,.\, \|_k$: $\mathbb{C}^k \to \mathbb{R}_{\geq 0}$ by

$$\| z_1 + z_2 i_k \|_k^2 = \| z_1 \|_{k-1}^2 + \| z_2 \|_{k-1}^2, \tag{2.24}$$

with this definition of the norm, the space $\mathbb{C}^k$ becomes a modified Banach algebra.
Other useful representations of the multicomplex numbers can be found by repetitively applying to the multicomplex coefficients of lower dimension, that is decomposing z_1 and z_2 into lower dimension repetitively. We obtain

$$\mathbb{C}^k := \{z_{11} + z_{12} i_{k-1} + z_{21} i_k + z_{22} i_k i_{k-1} \mid z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{C}^{k-2}\}. \tag{2.25}$$

For all $x_0, \cdots, x_k, \cdots, x_{1\cdots k} \in \mathbb{R}$,

$$\mathbb{C}^k := \{x_0 + x_1 i_1 + \cdots + x_k i_k + x_{12} i_1 i_2 + \cdots + x_{k-1k} i_{k-1} i_k + \cdots + x_{1\cdots k} i_1 \cdots i_k\}. \tag{2.26}$$

It is clear that we can represent each element of $\mathbb{C}^k$ with $\binom{k}{0} + \binom{k}{1} + \cdots + \binom{k}{k}$ {where $\binom{k}{r} = \frac{k!}{r!(k-r)!}$}, coefficients in $\mathbb{R}$. One coefficients x_0 for the real part k, and coefficients $x_1, \cdots, x_k$ for the pure imaginary directions and additional coefficients corresponding to 'cross coupled'imaginary directions. We note that the cross directions do not exit in $\mathbb{R}$ or $\mathbb{C}$, but appear only in $\mathbb{C}^k$ for $k \geq 2$.

The multicomplex space for $k \geq 2$ has many idempotents elements, that is elements I with the property that $I^2 = I$

$$I_1 = \frac{1 + i_k i_{k-1}}{2} \text{ and } I_2 = \frac{1 - i_k i_{k-1}}{2} \tag{2.27}$$

$$I_1^2 = (\frac{1 + i_k i_{k-1}}{2})^2 = \frac{1 + i_k i_{k-1}}{2} = I_1 \tag{2.28}$$

$$I_2^2 = (\frac{1 - i_k i_{k-1}}{2})^2 = \frac{1 - i_k i_{k-1}}{2} = I_2 \tag{2.29}$$

$$I_1 + I_2 = (\frac{1 + i_k i_{k-1}}{2}) + (\frac{1 - i_k i_{k-1}}{2}) = 1 \tag{2.30}$$

$$I_1 I_2 = (\frac{1 + i_k i_{k-1}}{2})(\frac{1 - i_k i_{k-1}}{2}) = 0. \tag{2.31}$$

Definition 2.3 (Tricomplex Numbers ([1, 9])**.** The set of the tricomplex numbers is defined as

$$\mathbb{TC} := \{z_1 + z_2 i_3 \mid z_1, z_2 \in \mathbb{C}^2(i_2)\} \tag{2.32}$$

where i_2, i_2 are the imaginary units and governed by the rules. For all $m = 1, 2, 3, 4$ and $l = 1, 2, 3,$

$$i_m^2 = -1, i_1 i_2 = i_2 i_1 = j_1,\ i_2 i_3 = i_3 i_2 = j_2,\ i_1 i_3 = i_3 i_1 = j_3,\ j_l^2 = 1, \tag{2.33}$$

and so, With this definition it is simple to show that $\mathbb{C}^3$ is a commutative ring with unity. Further, assuming have defined the norm
$\| \, . \, \|_2$: $\mathbb{C}^2 \to \mathbb{R}_{\geq 0}$, we define the norm $\| \, . \, \|_k$: $\mathbb{C}^3 \to \mathbb{R}_{\geq 0}$ by

$$\| z_1 + z_2 i_3 \|_3^2 = \| z_1 \|_2^2 + \| z_2 \|_2^2 \tag{2.34}$$

with this definition of the norm, the space $\mathbb{C}^3$ becomes a modified Banach algebra.
Other useful representations of the tricomplex numbers can be found by repetitively applying to the tricomplex coefficients of lower dimension, that is decomposing z_1 and z_2 into lower dimension repetitively. We obtain

$$\mathbb{C}^3 := \{x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3 + x_4 i_4 + x_5 j_1 + x_6 j_2 + x_7 j_3 \mid x_0, \cdots, x_3, \cdots, x_7 \in \mathbb{R}\}. \tag{2.35}$$

In the tricomplex space the idempotents representation as follows,

$$\mathcal{I}_1 = \frac{1 + i_3 i_2}{2} \text{ and } \mathcal{I}_2 = \frac{1 - i_3 i_2}{2} \tag{2.36}$$

$$\mathcal{I}_1^2 = (\frac{1 + i_3 i_2}{2})^2 = \frac{1 + i_3 i_2}{2} = \mathcal{I}_1 \tag{2.37}$$

$$\mathcal{I}_2^2 = (\frac{1 - i_3 i_2}{2})^2 = \frac{1 - i_3 i_2}{2} = \mathcal{I}_2 \tag{2.38}$$

$$\mathcal{I}_1 + \mathcal{I}_2 = (\frac{1 + i_3 i_2}{2}) + (\frac{1 - i_3 i_2}{2}) = 1 \tag{2.39}$$

$$\mathcal{I}_1 \mathcal{I}_2 = (\frac{1 + i_3 i_2}{2})(\frac{1 - i_3 i_2}{2}) = 0. \tag{2.40}$$

Theorem 2.1. *([3]) Let X_1 and X_2 be open sets in $\mathbb{C}^1(i_1)\}$. If $f_{e_1} : X_1 \to \mathbb{C}^1(i_1)$ and $f_{e_2} : X_2 \to \mathbb{C}^1(i_1)$ be holomorphic functions of $\mathbb{C}^1(i_1)$ on X_1 and X_2 respectively. Then the function $f : X_1 \times_e X_2 \to \mathbb{C}^2$ is defined as*

$$f(z_1 + z_2 i_2) = f_{e_1}(z_1 - z_2 i_1) e_1 + f_{e_2}(z_1 + z_2 i_1) e_2, \forall z_1 + z_2 i_2 \in f : X_1 \times_e X_2 \tag{2.41}$$

which is $\mathbb{C}^2$ holomorphic on the open set $X_1 \times_e X_2$, and

$$f'(z_1+z_2i_2) = f'_{e_1}(z_1-z_2i_1)e_1 + f'_{e_2}(z_1+z_2i_1)e_2, \forall z_1+z_2i_2 \in f : X_1 \times_e X_2. \quad (2.42)$$

Definition 2.4. ([2]) Let $z_n = \alpha_n e_1 + \beta_n e_2$ for $n \geq 1$. The sequence $\{z_n\}_{n\geq 1}$ is said to be convergent component-wise if the sequence of the complex numbers $\{\alpha_n\}$ and $\{\beta_n\}$ are convergent in the complex plane to complex numbers α_0 and β_0, respectively. In this case, we write $z_n \to z_0 = \alpha_0 e_1 + \beta_0 e_2$, and we say that z_n has limit z_0.

Theorem 2.2. *([3]) Let $\{z_n = z_{1,n} + z_{2,n}i_2\}_{n=1}^{\infty} \subseteq \mathbb{C}^2$ be invertible for each positive integer n. Then $\prod_{n=1}^{\infty} z_n$ converges if and only if both $(\prod_{n=1}^{\infty} z_{1,n} - z_{2,n}i_1)$ and $(\prod_{n=1}^{\infty} z_{1,n} + z_{2,n}i_1)$ converge. Further, when convergent, we obtain the identity.*

$$\prod_{n=1}^{\infty} z_n = (\prod_{n=1}^{\infty} z_{1,n} - z_{2,n}i_1)e_1 + (\prod_{n=1}^{\infty} z_{1,n} + z_{2,n}i_1)e_2.$$

Definition 2.5 (Bicomplex holomorphic functions ([4])). Let $\mathcal{U}$ be an open set of $\mathbb{BC}$ and $z_0 \in \mathcal{U}$. Then, $f : \mathcal{U} \subset \mathbb{BC} \to \mathbb{BC}$ is said to be $\mathbb{BC}$- differentiable at z_0 with derivative equal to $f'(z_0) \in \mathbb{BC}$ if

$$\lim_{\substack{z\to z_0 \\ z-z_0\ inv.}} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0). \quad (2.43)$$

We also say that the function f is bicomplex holomorphic ($\mathbb{BC}$- holomorphic) on an open set $\mathcal{U}$ if and only if f is $\mathbb{BC}$- differentiable at each point of $\mathcal{U}$. Using $z = z_1 + z_2i_2$, a bicomplex number z can be seen as an element (z_1, z_2) of $\mathbb{C}^2$, so a function $f(z_1+z_2i_2) = f_1(z_1,z_2) + f_2(z_1,z_2)i_2$ of $\mathbb{BC}$ can be seen as a mapping $f(z_1,z_2) = (f_1(z_1,z_2), f_2(z_1,z_2))$ of $\mathbb{C}^2$.

Theorem 2.3. *([4]) Let $f : \mathcal{U} \subset \mathbb{C}^2 \to \mathbb{C}^2$ be a function, and $f(z_1+z_2i_2) = f_1(z_1,z_2) + f_2(z_1,z_2)i_2$, where $z_1, z_2 \in \mathbb{C}^1$. Then the following are equivalent*

- (*i*) *f is holomorphic in $\mathcal{U}$.*
- (*ii*) *f_1 and f_2 are holomorphic in z_1 and z_2 and satisfying the bicomplex Cauchy-Riemann equations:*

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \text{ and } \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2} \quad (2.44)$$

- (*iii*) *f can be represented, near every point $z_0 \in \mathcal{U}$, by a Taylor series.*

3. Main Results

We define the following theorem and corollaries given below in tricomplex space

Definition 3.1 (Tricomplex Polynomial). Let $z = z_1 + z_2i_3 = \alpha_1\mathcal{I}_1 + \alpha_2\mathcal{I}_2$ be a tricomplex number, where $\alpha_1 = (z_1 - z_2i_2), \alpha_2 = (z_1 + z_2i_2)$ and $\mathcal{I}_1, \mathcal{I}_2$ are idempotent basis and let $\mathcal{A}_p := \delta_p\mathcal{I}_1 + \gamma_p\mathcal{I}_2$ be tricomplex coefficients for $p = 0, \cdots, n$. Then $f(z) := \sum_{p=0}^{n} \mathcal{A}_p z^p$ is called the tricomplex polynomial and written as

$$f(z) := \sum_{p=0}^{n}(\delta_p\alpha_1^p)\mathcal{I}_1 + \sum_{p=0}^{n}(\gamma_p\alpha_2^p)\mathcal{I}_2 = f_1(\alpha_1)\mathcal{I}_1 + f_2(\alpha_2)\mathcal{I}_2.$$

If we denote the set of all r_1 and r_2 distinct roots of $f_1(\alpha_1)$ and $f_2(\alpha_2)$ by ξ_1 and ξ_2, and if we denote by ξ the set of all distinct roots of polynomial $f(z)$, then $f(z)$ has $r_1.r_2$ distinct roots and it is easy to see that $\xi := \xi_1\mathcal{I}_1 + \xi_2\mathcal{I}_2$ and so the structure of the zero set of a tricomplex polynomial $f(z)$ of degree n is fully described by the following corollary.

Corollary 3.1. *If both the polynomials $f_1(\alpha_1)$ and $f_2(\alpha_2)$ are of degree at least one, and if $\xi_1 = \{\mu_1, \cdots, \mu_\sigma\}$ has r_1 distinct roots and $\xi_2 = \{\nu_1, \cdots, \nu_\tau\}$ has r_2 distinct roots, then the set of the distinct roots of f is given by*

$$\xi := w_{s,t} = \mu_s \mathcal{I}_1 + \nu_t \mathcal{I}_2 \mid s = 1, \cdots, \sigma, \text{ and } t = 1, \cdots, \tau.$$

Example 3.2. Let $f(z) := (\frac{1+i_3 i_2}{2})z^5 + \{(-1-4i_2)+(4-2i_2)i_3\}z^4 + \{(-11+6i_2)-(12+11i_2)i_3\}z^3 + \{(\frac{29+26i_2}{2}) + (\frac{-26+47i_2}{2})i_3\}z^2 + \{(\frac{13-34i_2}{2}) + (\frac{34+13i_2}{2})i_3\}z + (\frac{-11-2i_2}{2}) + (\frac{2-11i_2}{2})$. Then
we have

$$f(z) := f_1(\alpha_1)\mathcal{I}_1 + f_2(\alpha_2)\mathcal{I}_2$$

where

$$\mathcal{I}_1 := (\frac{1+i_3 i_2}{2}),\ \mathcal{I}_2 := (\frac{1-i_3 i_2}{2})$$

$$f_1(\alpha_1) := \alpha_1^5 + (-3-8i_2)\alpha_1^4 + (-22+18i_2)\alpha_1^3 + (38+26i_2)\alpha_1^2 + (13-34i_2)\alpha_1 + (-11-2i_2)$$

$$f_2(\alpha_2) := \alpha_2^4 - 6i_2\alpha_2^3 - 9\alpha_2^2$$

$$\xi_1 := \{\mu_1 = i_2, \mu_2 = 1 + 2i_2\}$$

$$\xi_2 := \{\nu_1 = 0, \nu_2 = 3i_2\}$$

$$\xi := \{W_{s,t} = \mu_s I_1 + \nu_t I_2 \mid s, t = 1, 2\}$$

has 4 distinct roots.

$$\xi := \{\frac{i_2 - i_3}{2}, 2i_2 + i_3, (\frac{1+2i_2}{2}) + (\frac{-2+i_2}{2})i_3, (\frac{1+5i_2}{2}) + (\frac{1+i_2}{2})i_3\}.$$

Corollary 3.2. *If $f_1(\alpha_1) = 0$, then $\xi_1 = \mathbb{C}^2$ and $\xi_2 = \{\nu_1, \cdots, \nu_\tau\}$, where $\tau \le n$; and $\xi := z_t = \omega \mathcal{I}_1 + \nu_t \mathcal{I}_2 \mid \omega \in \mathbb{C}^2, t = 1, \cdots, \tau$. If $f_2(\alpha_2) = 0$, then $\xi_2 = \mathbb{C}^2$ and $\xi_1 = \{\mu_1, \cdots, \mu_\sigma\}$, where $\sigma \le n$; and*

$$\xi := z_s = \mu_s \mathcal{I}_1 + \omega \mathcal{I}_2 \mid \omega \in \mathbb{C}^2, s = 1, \cdots, \sigma.$$

Example 3.3. Let $f(z) := (1 - i_3 i_2)z^2 + i_3 - i_2$. Then
we have

$$f(z) := f_1(\alpha_1)\mathcal{I}_1 + f_2(\alpha_2)\mathcal{I}_2$$

where

$$\mathcal{I}_1 := (\frac{1+i_3 i_2}{2}),\ \mathcal{I}_2 := (\frac{1-i_3 i_2}{2})$$

$$f_1(\alpha_1) := 2(\alpha_1^2 - i_2)\mathcal{I}_1$$

$$f_1(\alpha_1) := 0$$

$$\xi := z_s = \mu_s \mathcal{I}_1 + \omega \mathcal{I}_2 = \{\pm(\frac{1+i_2}{\sqrt{2}})\mathcal{I}_1 + \omega \mathcal{I}_2 \mid \omega \in \mathbb{C}^2(\omega = \sqrt{i_2})\}.$$

Corollary 3.3. *If all the coefficients $\mathcal{A}_p$ with the exception of $\mathcal{A}_0 = \delta_0 \mathcal{I}_1 + \gamma_0 \mathcal{I}_2$ are not tricomplex multiples of $\mathcal{I}_1$ (respectively $\mathcal{I}_2$), but $\mathcal{A}_0$ has $\gamma_0 \neq 0$ (respectively $\delta_0 \neq 0$), then polynomial f has no root.*

Example 3.4. Let $f(z) := (1 - i_3 i_2)z^2 + 1 + i_3 - i_2 - i_3 i_2$. Then
we have

$$f(z) := f_1(\alpha_1)\mathcal{I}_1 + f_2(\alpha_2)\mathcal{I}_2$$

where

$$f_1(\alpha_1) = 2(\alpha_1^2 - i_2) \text{ and } f_2(\alpha_2) = 2.$$

Clearly polynomial has no root.

Corollary 3.4. *(Analogue of Fundamental Theorem of Algebra for tricomplex Polynomials)*
Let $f(z) := \sum_{p=0}^{n} \mathcal{A}_p z^p$ be tricomplex Polynomial, where $\mathcal{A}_p := \delta_p \mathcal{I}_1 + \gamma_p \mathcal{I}_2$, and $z^p = \alpha_1^p \mathcal{I}_1 + \alpha_2^p \mathcal{I}_2$, with $\alpha_1 = (z_1 - z_2 i_2), \alpha_2 = (z_1 + z_2 i_2)$. If all the coefficients $\mathcal{A}_p$ with the exception of $\mathcal{A}_0 = \delta_0 \mathcal{I}_1 + \gamma_0 \mathcal{I}_2$ are not tricomplex multiples of $\mathcal{I}_1$ (respectively $\mathcal{I}_2$), but $\mathcal{A}_0$ has $\gamma_0 \neq 0$ (respectively $\delta_0 \neq 0$), then polynomial f has no root. In all other cases f has at least one root.

Remark. *A tricomplex polynomial may not have a unique factorization into linear polynomials.*

Example 3.5. Let $f(z) := z^3 + 1$. Then
we have

$$f_1(\alpha_1) = \alpha_1^3 + 1, f_2(\alpha_2) = \alpha_2^3 + 1$$
$$\xi_1 = \{\mu_1 = -1, \mu_2 = \frac{1+\sqrt{3}i_3}{2}, \mu_3 = \frac{1-\sqrt{3}i_3}{2}\}$$
$$\xi_2 = \{\nu_1 = -1, \nu_2 = \frac{1+\sqrt{3}i_3}{2}, \nu_3 = \frac{1-\sqrt{3}i_3}{2}\}$$
$$\xi := z_{s,t} = \mu_s \mathcal{I}_1 + \nu_t \mathcal{I}_2 \mid s,t = 1,2,3$$
$$z^3 + 1 := (z+1)(z - \frac{1}{2} - \frac{\sqrt{3}}{2} i_3)(z - \frac{1}{2} + \frac{\sqrt{3}}{2} i_3)$$
$$z^3 + 1 := (z+1)(z - \frac{1}{2} - \frac{\sqrt{3}}{2} i_2)(z - \frac{1}{2} + \frac{\sqrt{3}}{2} i_2).$$

Note: It is clear from what we have indicated that the tricomplex polynomials also do not satisfy the Fundamental theorem of algebra in its original form.

Theorem 3.5. *Let $\mathbb{C}^3 := z = \{z_1 + z_2 i_3 \mid z_1, z_2 \in \mathbb{C}^2\}$ be a tricomplex number, and let $f : \mathcal{U} \subset \mathbb{C}^3 \to \mathbb{C}^3$ be a tricomplex holomorphic function in $\mathcal{U}$. Then f can be expanded in a Taylor series about a real point a as follows:*

$$f(a+h(i_1+i_2+i_3)) := f(a)+h(i_1+i_2+i_3)f'(a)+\cdots+h^n(i_1+i_2+i_3)^n \frac{f^n(a)}{(n)!}+O(h^{(n+1)}) \tag{3.1}$$

where f^n denotes the n^{th} order derivative, and

$$(i_1 + i_2 + i_3)^n := \sum_{\substack{x_1, x_2, x_3 \\ x_1 + x_2 + x_3 = n}} \frac{n!}{x_1! x_2! x_3!} i_1^{x_1} i_2^{x_2} i_3^{x_3} \tag{3.2}$$

Proof. Easy to proof. □

Theorem 3.6. *Let $f, g : \mathcal{U} \subset \mathbb{C}^3 \to \mathbb{C}^3$ be tricomplex holomorphic functions in $\mathcal{U}$ and if $f(a) = 0$ and $g(a) = 0$, but $g'(a) \neq 0$. Then*

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)} \tag{3.3}$$

and hence, in general, if $f^n(a) = 0 = g^n(a)$, but $g^{(n+1)}(a) \neq 0$. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{n+1}(a)}{g^{n+1}(a)}. \tag{3.4}$$

Proof. From Taylor series we have,

$$f(a+h(i_1+i_2+i_3)) := \sum_{r=0}^{n+1} h^r (i_1+i_2+i_3)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)}) \tag{3.5}$$

$$g(a+h(i_1+i_2+i_3)) := \sum_{r=0}^{n+1} h^r (i_1+i_2+i_3)^r \frac{g^{(r)}(a)}{r!} + O(h^{(n+2)})$$

Put $a+h(i_1+i_2+i_3) = z$.
Then $h(i_1+i_2+i_3) = z-a$.

$$f(z) := \sum_{r=0}^{n+1} (z-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})$$

$$g(z) := \sum_{r=0}^{n+1} (z-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})$$

$$\frac{f(z)}{g(z)} := \frac{\sum_{r=0}^{n+1} (w-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})}{\sum_{r=0}^{n+1} (z-a)^r \frac{f^{(r)}(a)}{r!} + O(h^{(n+2)})}.$$

If $f(a) = 0 = g(a)$, but $g'(a) \neq 0$. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f'(a)}{g'(a)}.$$

If $f'(a) = 0 = g'(a)$, but $g''(a) \neq 0$, then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f''(a)}{g''(a)}.$$

Hence in general, if $f^n(a) = 0 = g^n(a)$, but $g^{(n+1)}(a) \neq 0$. Then

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \frac{f^{n+1}(a)}{g^{n+1}(a)}.$$

□

Definition 3.6 (Tricomplex Matrices)**.** The set of $m \times n$ matrices $\mathbb{C}^3_{m\times n}$ with tricomplex entries, is denoted as $\mathcal{A} := \{(a_{lj}) \in \mathbb{C}^3_{m\times n}, 1 \leq l \leq m, 1 \leq j \leq n\} = \mathcal{B}_{i_3}\mathcal{I}_1 + \mathcal{C}_{i_3}\mathcal{I}_2 := \mathcal{B}_{i_2}\mathcal{I}_1 + \mathcal{C}_{i_2}\mathcal{I}_2$, where $\mathcal{B}_{i_3}, \mathcal{C}_{i_3} \in \mathbb{C}^2_{m\times n}$ and $\mathcal{B}_{i_2}, \mathcal{C}_{i_2} \in \mathbb{C}^2_{m\times n}$ and $\mathcal{I}_1 = \frac{1+i_3 i_2}{2}, \mathcal{I}_2 = \frac{1-i_3 i_2}{2}$.

Definition 3.7 (Eigenvalues for Tricomplex Matrices)**.** let $\mathcal{A} := \{(a_{lj}) \in \mathbb{C}^3_{m\times n} = \mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2\}$ and $\mathcal{A}u = \lambda u$ which is equivalent to

$$\begin{cases} \mathcal{A}_1 u_1 = \lambda_1 u_1, \\ \mathcal{A}_2 u_2 = \lambda_2 u_2. \end{cases}$$

Then λ is called the eigenvalue of the tricomplex matrix $\mathcal{A}$ corresponding to eigenvector u where $\lambda := \lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2 \in \mathbb{C}^3$ and $u = u_1\mathcal{I}_1 + u_2\mathcal{I}_2$. If λ is not a zero divisor and $u_1 \neq 0, u_2 \neq 0$ then λ is an eigenvalue of $\mathcal{A}$ if and only if λ_1 and λ_2 be an eigenvalue of $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponding to eigenvector of u_1 and u_2.

We define and prove the following theorem given bellow.

Theorem 3.7. *let* $\mathcal{A} := \{(a_{lj}) \in \mathbb{C}^3_{m\times n} = \mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2\}$ *and* $\mathcal{A}u = \lambda u$ *which is equivalent to*

$$\begin{cases} \mathcal{A}_1 u_1 = \lambda_1 u_1, \\ \mathcal{A}_2 u_2 = \lambda_2 u_2. \end{cases}$$

Where $\lambda = \lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2 \in \mathbb{C}^3$ *and* $u = u_1\mathcal{I}_1 + u_2\mathcal{I}_2$*. Then tricomplex matrix* $\mathcal{A}$ *has* $\{\lambda = p_1.q_1\}\}$ *distinct eigenvalues if and only if* $\mathcal{A}_1$ *has* $\{\lambda_1 = p_1\}$ *distinct eigenvalues and* $\mathcal{A}_2$ *has* $\{\lambda_2 = q_1\}$ *distinct eigenvalues.*

Proof. We have $\lambda = \{\alpha_s\mathcal{I}_1 + \beta_t\mathcal{I}_2 \mid 1 \le s \le p_1, 1 \le t \le q_1\}$

$$= \{\alpha_1, \alpha_2, \cdots, \alpha_{p_1}\}\mathcal{I}_1 + \{\beta_1, \beta_2, \cdots, \beta_{q_1}\}\mathcal{I}_2 = \lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2$$

$$\mathcal{A}u = \lambda u \Rightarrow (\mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2)u = (\lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2)u$$

$$\begin{cases} \mathcal{A}_1 u_1 = \lambda_1 u_1, \\ \mathcal{A}_2 u_2 = \lambda_2 u_2. \end{cases}$$

Conversely: If $\lambda_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_{p_1}\}, \lambda_2 = \{\beta_1, \beta_2, \cdots, \beta_{q_1}\}$

$$\mathcal{A}u = (\mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2)u = (\lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2)u$$

$$\mathcal{A}u = \lambda u.$$

Implies that

$$\lambda = \{\lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2 = \alpha_s\mathcal{I}_1 + \beta_t\mathcal{I}_2 \mid 1 \le s \le p_1, 1 \le t \le q_1\}$$

□

Example 3.8.

$$\mathcal{A}_{2,2} = \begin{pmatrix} 1 - i_2 + i_3 + i_2 i_3 & 1 + i_2 + i_3 - i_2 i_3 \\ 1 - i_2 - i_3 + i_2 i_3 & -1 + i_2 + i_3 - i_2 i_3 \end{pmatrix}$$

$$\mathcal{A} = \mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2$$

$$\mathcal{A} = \begin{pmatrix} 2(1 - i_2) & 0 \\ 2 & -2 \end{pmatrix}\mathcal{I}_1 + \begin{pmatrix} 0 & 2(1 + i_2) \\ -2i_2 & 2i_2 \end{pmatrix}\mathcal{I}_2$$

$$det(\mathcal{A} - \lambda I) := \lambda^2 - 2i_3\lambda + 4(i_2 - 1)$$

$$det(\mathcal{A}_1 - \lambda_1 I) := \lambda_1^2 + 2i_2\lambda_1 + 4(i_2 - 1)$$

$$det(\mathcal{A}_2 - \lambda_2 I) := \lambda_2^2 - 2i_3\lambda_2 + 4(i_2 - 1)$$

$\lambda = \{(-i_2+\sqrt{3 - 4i_2})\mathcal{I}_1+(i_2+\sqrt{3 - 4i_2})\mathcal{I}_2, (-i_2+\sqrt{3 - 4i_2})\mathcal{I}_1+(i_2-\sqrt{3 - 4i_2})\mathcal{I}_2, (-i_2-\sqrt{3 - 4i_2})\mathcal{I}_1 + (i_2 + \sqrt{3 - 4i_2})\mathcal{I}_2, (-i_2 - \sqrt{3 - 4i_2})\mathcal{I}_1 + (i_2 - \sqrt{3 - 4i_2})\mathcal{I}_2\}$

Theorem 3.8. *let* $\mathcal{A} := \{(a_{lj}) \in \mathbb{C}^3_{n\times n} = (\alpha_{lj})\mathcal{I}_1 + (\beta_{lj})\mathcal{I}_2, 1 \le l, j \le n\}$ *be any tricomplex matrix and* $det(\lambda I_n - \mathcal{A})$ *be the characteristic polynomial, then the matrix* $\mathcal{A}$ *is zero of* $det(\lambda I_n - \mathcal{A})$*.*

Proof. We have

$$det(\lambda I_n - \mathcal{A}) := det(\lambda_1 I_n - \alpha_{lj})\mathcal{I}_1 + det(\lambda_2 I_n - \beta_{lj})\mathcal{I}_2,$$

where

$$det(\lambda I_n - \mathcal{A}) = \sum_{p=0}^{n} a_p\lambda^p = (\sum_{p=0}^{n} \delta_p\lambda_1^p)\mathcal{I}_1 + (\sum_{p=0}^{n} \gamma_p\lambda_2^p)\mathcal{I}_2$$

$$det(\lambda I_n - \mathcal{A}) = (\lambda I_n - \mathcal{A}).Adj(\lambda I_n - \mathcal{A}) = Adj(\lambda I_n - \mathcal{A}).(\lambda I_n - \mathcal{A}).$$

And

$$Adj(\lambda I_n - \mathcal{A}) = \sum_{p=0}^{n-1} \omega_p \lambda^p = (\sum_{p=0}^{n-1} \phi_p \lambda_1^p)\mathcal{I}_1 + (\sum_{p=0}^{n-1} \psi_p \lambda_2^p)\mathcal{I}_2.$$

Take

$$\mathcal{A} = \mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2, \lambda = \lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2.$$

Then we have

$$\begin{gathered}\phi_{n-1} = \delta_n I\\ \phi_{n-2} - \mathcal{A}_1\phi_{n-1} = \delta_{n-1} I\\ \phi_{n-3} - \mathcal{A}_1\phi_{n-2} = \delta_{n-2} I\\ \vdots \ddots \vdots\\ \phi_0 - \mathcal{A}_1\phi_1 = \delta_1 I\\ -\mathcal{A}_1\phi_0 = \delta_0 I\end{gathered}$$

And

$$\begin{gathered}\psi_{n-1} = \gamma_n I\\ \psi_{n-2} - \mathcal{A}_1\psi_{n-1} = \gamma_{n-1} I\\ \psi_{n-3} - \mathcal{A}_1\psi_{n-2} = \gamma_{n-2} I\\ \vdots \ddots \vdots\\ \psi_0 - \mathcal{A}_1\psi_1 = \gamma_1 I\\ -\mathcal{A}_1\psi_0 = \gamma_0 I\end{gathered}$$

Multiplying by $\mathcal{A}_1^n, \mathcal{A}_1^{n-1}, \cdots, \mathcal{A}_1, I$

$$\begin{gathered}\mathcal{A}_1^n\phi_{n-1} = \mathcal{A}_1^n\delta_n I\\ \mathcal{A}_1^{n-1}\phi_{n-2} - \mathcal{A}_1^n\phi_{n-1} = \mathcal{A}_1^{n-1}\delta_{n-1} I\\ \mathcal{A}_1^{n-2}\phi_{n-3} - \mathcal{A}_1^{n-1}\phi_{n-2} = \mathcal{A}_1^{n-2}\delta_{n-2} I\\ \vdots \ddots \vdots\\ \mathcal{A}_1\phi_0 - \mathcal{A}_1^2\phi_1 = \mathcal{A}_1\delta_1 I\\ -\mathcal{A}_1\phi_0 = \delta_0 I\end{gathered}$$

$$\delta_n\mathcal{A}_1^n + \delta_{n-1}\mathcal{A}_1^{n-1} + \cdots + \delta_1\mathcal{A}_1 + \delta_0 I = 0 \tag{3.6}$$

Similarly multiplying by $\mathcal{A}_2^n, \mathcal{A}_2^{n-1}, \cdots, \mathcal{A}_2, I$
We have

$$\gamma_n\mathcal{A}_2^n + \gamma_{n-1}\mathcal{A}_2^{n-1} + \cdots + \gamma_1\mathcal{A}_2 + \psi_0 I = 0 \tag{3.7}$$

From above equation we have

$$a_n\mathcal{A}^n + a_{n-1}\mathcal{A}^{n-1} + \cdots + a_1\mathcal{A} + a_0 I = 0 \tag{3.8}$$

□

Theorem 3.9. *let $\mathcal{A} := \{(a_{lj}) \in \mathbb{C}^3_{n\times n} = \mathcal{A}_1\mathcal{I}_1 + \mathcal{A}_2\mathcal{I}_2\}$ be any tricomplex matrix, then $\mathcal{A}$ is zero of $det(\lambda I_n - \mathcal{A})$ if and only if $\mathcal{A}_1$ is zero of $det(\lambda_1 I_n - \mathcal{A}_1)$ and $\mathcal{A}_2$ is zero of $det(\lambda_2 I_n - \mathcal{A}_2)$ where $\lambda = \lambda_1\mathcal{I}_1 + \lambda_2\mathcal{I}_2$.*

Proof. Very simple, can be easily □

Example 3.9. From above example clearly

$$f(\mathcal{A}) := \mathcal{A}^2 - 2i_3\mathcal{A} + 4(i_2 - 1) = 0$$

$$f_1(\mathcal{A}_1) := \mathcal{A}_1^2 + 2i_2\mathcal{A}_1 + 4(i_2 - 1) = 0$$
$$f_2(\mathcal{A}_2) := \mathcal{A}_2^2 - 2i_3\mathcal{A}_2 + 4(i_2 - 1) = 0.$$

4. Conclusion & Observation

We conclude that in our work, we introduce the algebraic form of tricomplex numbers, which is a generalization of the field of bicomplex numbers. We describe how to define some basic formulas in the form of algebra polynomials, Taylor series for tricomplex holomorphic functions, algebra of eigenvalues and eigenvectors. We define a tricomplex polynomial as a better generalization of a bicomplex polynomial. Moreover, we can apply the properties of Multicomplex polynomials to generalize tricomplex and bicomplex polynomials.

5. Acknowledgment

The authors of the manuscript gratefully thank reviewers for their positive feedback and relevant constructive comments.

Availability of data and materials: Not applicable.
Competing interests: The author declares no competing interests.
Funding: Not applicable.

References

[1] G.B. Price, An Introduction to Multicomplex Spaces and Functions, Marcel Dekker, 1991.
[2] M.E Luna-Elizarrara's, M.Shapiro, Bicomplex numbers and their elementary functions CUBO A Mathematical Journal Vol.14, No 02, (61-80). June 2012.
[3] F. L. Reid and A. Robert and V, Gorder, A Multicomplex Riemann Zeta Function, Adv. Appl. Cliff ord Algebras 23 (2013), 237-251
[4] G. Lantoine, R. P. Russell and T. Dargent, Using Multicomplex Variables for Automatic Computation of High-order Derivatives, ACM Transactions on Mathematical Software, Vol. 38, No. 3, Article 16, April 2012, 21 pages.
[5] C. Segre, The real representation of complex elements and hyperalgebraic entities (Italian). Mathematische Annalen 40 (1892), 413-467.
[6] W. R. Hamilton, On Quaternions; or on a new System of Imaginaries in Algebra, The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Vol. 15, 1844, pp. 489-495.
[7] C. Segre, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, Mathematische Annalen, Vol. 40, No. 3, 1892, pp. 413-467.
[8] N. Fleury, M. R. Detraubenberg and R. M. Yamaleev, Commutative Extended Complex Numbers and Connected Trigonometry, Journal of Mathematical Analysis and Applications, Vol. 180, Dec. 1993, pp. 431-457.
[9] Daniel Alpay, Maria Elena Luna-Elizarraras, Michael Shapiro and Daniele C. Struppa Basics of Functional Analysis with Bicomplex Scalars, and Bicomplex Schur Analysis.
[10] K.S. Charak, D. Rochon and N. Sharma, Normal families of bicomplex holomorphic functions, ArXiv:0806.4403v1 (2008).
[11] D. Rochon and M. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, Analele Universitatii Oradea, Fasc. Matematica 11 (2004), 71-110.
[12] M. Mursaleen, M. Nasiruzzaman and H.M. Srivastava, Approximation by bicomplex beta operators in compact BC-disks, Math. Meth. Appl. Sci., 2016, 39 2916–2929.
[13] M. Nasiruzzaman and M. Mursaleen, The multicomplex numbers and their properties on some elementary functions Thai Journal of Mathematics Volume 21 Number 1 (2023) Pages 77–100.

Md. Nasiruzzaman
Department of Mathematics, Faculty of Science, University of Tabuk, PO Box-4279, Tabuk-71491, Saudi Arabia
ORCID: 0000-0003-4823-0588
Email address: nasir3489@gmail.com

M. Mursaleen[1,2]

[1] Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamil Nadu, India
ORCID: 0000-0003-4128-0427

[2] Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan
Email address: mursaleenm@gmail.com