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A NOVEL APPROACH TO FILTERS IN BL-ALGEBRAS THROUGH BIPOLAR FUZZY SET THEORY

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ABSTRACT. This paper introduces a novel approach to the study of filters in BL-algebras by leveraging the principles of bipolar fuzzy set theory and investigating some of their properties. Filters are essential in the structural analysis of BL-algebras, affecting their properties and applications across various fields. Moreover, Bipolar-valued fuzzy filters generated by a fuzzy set are discussed. By integrating bipolar fuzzy set concepts, we provide a new framework that enhances the representation of uncertainty and vagueness inherent in filter operations. We explore the foundational aspects of bipolar fuzzy sets and demonstrate their applicability in defining and characterizing filters within BL-algebras. Our findings highlight the potential of bipolar fuzzy set theory to enrich the understanding of filters in BL-algebras.

1. INTRODUCTION

BL-algebras come from the field of Mathematical Logic; they were created by Hajek [6] to study "Basic Logic" (or BL for short) and he widely studied many types of filters. This logic is based on continuous triangular norms (t-norms), which are important in fuzzy set theory. These algebras are similar to Lindenbaum algebras used in classical logic. The main example of a BL-algebra is the interval [0,1], which has a special structure based on a continuous t-norm. One of the pivotal concepts being examined is the classification and construction of various types of filters in BL-algebras. Filter theory facilitates the understanding of substructures within algebras, and the integration of bipolar fuzzy logic presents an exciting methodology to analyze these filters. Liu and Li [15] significantly contributed to the field by discussing fuzzy filters of BL-algebras. Moreover, their exploration of fuzzy Boolean and positive implicative filters further highlights the importance of fuzzy frameworks in this context [16]. The concept of BL-algebras has also been examined in local BL-algebras [25], in the Boolean deductive systems of BL-algebras [26] and in (Fuzzy) filters of Sheffer stroke BL-algebras [22].

Fuzzy sets, which were introduced by Zadeh [27], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. After the introduction

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of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields. These are widely scattered over many disciplines such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. The foundation for this work is built upon a variety of previous studies in the realm of fuzzy structures. For instance, the exploration of doubt N-ideals in BCK-algebras by Al-masarwah et al. [3] provides essential insights into the interplay between algebraic structures and fuzzy logic. Moreover, the work of Al-roqi et al. [4] on normal unisoft filters in R0-algebras establishes a critical framework for understanding filter properties in broader algebraic contexts. The contributions of Ejegwa and Otuwe [5] on fuzzy subgroups further enrich our understanding of fuzzy structures in algebra. In this paper, we also build upon Muhiuddin and Shum's identification of new types of fuzzy subalgebras [19] and their exploration of fuzzy ideals [20]. Furthermore, prior studies, including those by Muhiuddin and Al-roqi [21], along with the analysis of fuzzy set operations in [23, 24], highlight the diverse applicability of fuzzy concepts across different algebraic systems.

As a generalization of traditional fuzzy sets, Zhang [28] first introduced the bipolar fuzzy sets concept. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [13, 14]). Bipolar fuzzy set theory allows for a more unique representation of complexity compared to traditional fuzzy set theory. Recent studies have expanded on this notion, including the work of Al-Kadi and Muhiuddin on bipolar fuzzy BCI-implicative ideals [1] and Al-Masarwah et al. on the properties of bipolar fuzzy H-ideals [2]. The research on hemirings [7], regular ordered semigroups [8], $(\in, \in \vee q)$-Bipolar fuzzy BCK/BCI-algebras [9], Bipolar fuzzy hyper BCK-ideals in hyper BCK-algebras [10], Bipolar fuzzy UP-algebras [11], Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras [12], Bipolar-valued fuzzy soft hyper BCK algebras [17], and bipolar fuzzy KU-subalgebras [18] indicates the depth of connectivity between bipolar fuzzy logic and classical algebraic concepts, pushing the boundaries of our understanding of these systems.

This paper presents a novel approach to studying filters in BL-algebras through bipolar fuzzy set theory. Filters play a crucial role in the structural analysis of BL-algebras, influencing their properties and diverse applications. We specifically discuss bipolar-valued fuzzy filters generated by fuzzy sets, introducing a framework that improves the representation of uncertainty and vagueness in filter operations. Additionally, we examine the foundational aspects of bipolar fuzzy sets and illustrate their relevance in characterizing filters in BL-algebras.

2. PRELIMINARIES

2.1. **Basic results on BL-algebras.**

Definition 2.1. [6] A *BL-algebra* is a structure $(\Omega, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants 0, 1 such that

(i) $(\Omega, \wedge, \vee, \odot, 0, 1)$ is a bounded lattice,

(ii) $(\Omega, \odot, 1)$ is an abelian monoid, i.e., $\odot$ is commutative, associative and $\vartheta \odot 1 = 1 \odot \vartheta = \vartheta$, for all $\vartheta \in \Omega$

(iii) the following conditions hold for all $\vartheta, \kappa, \hbar \in \Omega$:

(B1) $\vartheta \odot \kappa \leq \hbar$ if and only if $\vartheta \leq \kappa \to \hbar$ (residuation),
(B2) $\vartheta \wedge \kappa = \vartheta \odot (\vartheta \to \kappa)$ (divisibility),
(B3) $(\vartheta \to \kappa) \vee (\kappa \to \vartheta) = 1$ (prelinearity).

Proposition 2.1. [6] *Let Ω be a BL-algebra. The following properties hold for all $\vartheta, \kappa, \hbar \in \Omega$:*

(a1) $\vartheta \leq \kappa$ *if and only if* $\vartheta \to \kappa = 1$,
(a2) $\vartheta \to (\kappa \to \hbar) = (\vartheta \odot \kappa) \to \hbar$,
(a3) $\vartheta \odot \kappa \leq \vartheta \wedge \kappa$,
(a4) $(\vartheta \to \kappa) \odot (\kappa \to \hbar) \leq \vartheta \to \hbar$,
(a5) $\vartheta \vee \kappa = ((\vartheta \to \kappa) \to \kappa) \wedge ((\kappa \to \vartheta) \to \vartheta)$,
(a6) $\vartheta \to (\kappa \to \hbar) = \kappa \to (\vartheta \to \hbar)$,
(a7) $\vartheta \to \kappa \leq (\hbar \to \vartheta) \to (\hbar \to \kappa)$.

Definition 2.2. [6] A nonempty subset $\widetilde{\mathcal{F}}$ of Ω is called a *filter* of Ω if it satisfies

(i) $\vartheta, \kappa \in \widetilde{\mathcal{F}}, \Longrightarrow \vartheta \odot \kappa \in \widetilde{\mathcal{F}}$
(ii) $\vartheta \in \widetilde{\mathcal{F}}$ and $\vartheta \leq \kappa \in \widetilde{\mathcal{F}} \Longrightarrow \kappa \in \widetilde{\mathcal{F}}$.

Definition 2.3. [16] A nonempty subset $\widetilde{\mathcal{F}}$ of Ω is called a *deductive system* of Ω if it satisfies

(i) $1 \in \widetilde{\mathcal{F}}$,
(ii) $\vartheta \in \widetilde{\mathcal{F}}$ and $\vartheta \to \kappa \in \widetilde{\mathcal{F}} \Longrightarrow \kappa \in \widetilde{\mathcal{F}}$.

Proposition 2.2. [26] *A non-empty subset $\widetilde{\mathcal{F}}$ of BL-algebra is a deductive system if and only if $\widetilde{\mathcal{F}}$ is a filter.*

Definition 2.4. [16] Let μ be a fuzzy set in Ω. Then μ is called a *fuzzy filter* of Ω if the following assertions are valid for all $\vartheta, \kappa \in \Omega$:

(i) $\mu(1) \geq \mu(\vartheta)$,
(ii) $\mu(\kappa) \geq \min\{\mu(\vartheta), \mu(\vartheta \to \kappa)\}$.

2.2. **Basic results on bipolar valued fuzzy sets.** In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets.

Let X be the universe of discourse. A *bipolar-valued fuzzy set* $\widetilde{\mathcal{U}}$ in X is an object having the form

$$\widetilde{\mathcal{U}} = \{(\vartheta, \widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_n}(\vartheta) \mid \vartheta \in X\}$$

where $\widetilde{\mathcal{U}_p} : X \to [0,1]$ and $\widetilde{\mathcal{U}_n} : X \to [-1,0]$ are mappings. The positive membership degree $\widetilde{\mathcal{U}_p}(\vartheta)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$, and the negative membership degree $\widetilde{\mathcal{U}_n}(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$. If $\widetilde{\mathcal{U}_p}(x) \neq 0$ and $\widetilde{\mathcal{U}_n}(x) = 0$, it is the situation that ϑ is regarded as having only positive satisfaction for $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$. If $\widetilde{\mathcal{U}_p}(\vartheta) = 0$ and $\widetilde{\mathcal{U}_n}(\vartheta) \neq 0$, it is the situation that ϑ does not satisfy the property of $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ but somewhat satisfies the counter-property of $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$. It is possible for an element ϑ to be $\widetilde{\mathcal{U}_p}(\vartheta) \neq 0$ and $\widetilde{\mathcal{U}_n}(\vartheta) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [14]). For the sake of simplicity, we shall use the symbol $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ for the bipolar-valued fuzzy set $\widetilde{\mathcal{U}} = \{(\vartheta, \widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_n}(\vartheta) \mid \vartheta \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets. For two bipolar fuzzy sets $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ and $\widetilde{\mathcal{W}} = (\Omega; \widetilde{\mathcal{W}_p}, \widetilde{\mathcal{W}_n})$ in X, we define the union of $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ and $\widetilde{\mathcal{W}} = (\Omega; \widetilde{\mathcal{W}_p}, \widetilde{\mathcal{W}_n})$ as follows:

$$\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}} = (X; \mu^P_{\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}}}, \mu^N_{\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}}}) \tag{2.1}$$

where

$$\mu^P_{\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}}} : X \to [0,1],\ \vartheta \mapsto \mu^P_{\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}}}(\vartheta) = \max\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{W}_p}(\vartheta)\},$$
$$\mu^N_{\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}}} : X \to [-1,0],\ \vartheta \mapsto \mu^N_{\widetilde{\mathcal{U}} \cup \widetilde{\mathcal{W}}}(\vartheta) = \min\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{W}_n}(\vartheta)\}.$$

3. Bipolar-valued fuzzy filters

In the following discussion, we will use Ω to represent a BL-algebra unless indicated otherwise.

Definition 3.1. Let $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ be a bipolar-valued fuzzy set in Ω. Then $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is called a *bipolar-valued fuzzy filter* of Ω if the following assertions are valid for all $\vartheta, \kappa \in \Omega$:

$$\widetilde{\mathcal{U}_p}(1) \geq \widetilde{\mathcal{U}_p}(\vartheta)\ \&\ \widetilde{\mathcal{U}_n}(1) \leq \widetilde{\mathcal{U}_n}(\vartheta), \tag{3.1}$$

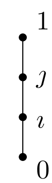
and

$$\begin{aligned} &\widetilde{\mathcal{U}_p}(\kappa) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\vartheta \to \kappa)\}, \\ &\widetilde{\mathcal{U}_n}(\kappa) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\vartheta \to \kappa)\}. \end{aligned} \tag{3.2}$$

Example 3.2. Let $\Omega = \{0, \imath, \jmath, 1\}$ be a set with the following Cayley tables and Hasse diagram:

$\odot$	0	$\imath$	$\jmath$	1
0	0	0	0	0
$\imath$	0	0	$\imath$	$\imath$
$\jmath$	0	$\imath$	$\jmath$	$\jmath$
1	0	$\imath$	$\jmath$	1

$\to$	0	$\imath$	$\jmath$	1
0	1	1	1	1
$\imath$	$\imath$	1	1	1
$\jmath$	0	$\imath$	1	1
1	0	$\imath$	$\jmath$	1



For all $\vartheta, \kappa \in \Omega$, we define $\vartheta \wedge \kappa = \min\{\vartheta, \kappa\}$ and $\vartheta \vee \kappa = \max\{\vartheta, \kappa\}$. Then $(\Omega, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra (see [25]). Define a bipolar-valued fuzzy set $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ in Ω by

$$\widetilde{\mathcal{U}_p}(1) = 0.8, \widetilde{\mathcal{U}_p}(0) = \widetilde{\mathcal{U}_p}(\imath) = 0.3, \widetilde{\mathcal{U}_p}(\jmath) = 0.6,$$
$$\widetilde{\mathcal{U}_n}(1) = -0.3, \widetilde{\mathcal{U}_n}(0) = \widetilde{\mathcal{U}_n}(\imath) = -0.9, \widetilde{\mathcal{U}_n}(\jmath) = -0.5.$$

It is easily verified that $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω.

Theorem 3.1. *A bipolar-valued fuzzy set $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω if and only if it satisfies the following assertions:*

$$\begin{aligned} &\widetilde{\mathcal{U}_p}(\hbar) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\kappa)\}, \\ &\widetilde{\mathcal{U}_n}(\hbar) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\kappa)\} \end{aligned} \tag{3.3}$$

for all $\vartheta, \kappa, \hbar \in \Omega$ with $\vartheta \to (\kappa \to \hbar) = 1$.

Proof. Suppose that $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω. Let $\vartheta, \kappa, \hbar \in \Omega$ be such that $x \to (\kappa \to \hbar) = 1$. Then

$$\begin{aligned} \widetilde{\mathcal{U}_p}(\hbar) &\geq \min\{\widetilde{\mathcal{U}_p}(\kappa), \widetilde{\mathcal{U}_p}(\kappa \to \hbar)\} \\ &\geq \min\{\widetilde{\mathcal{U}_p}(\kappa), \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\vartheta \to (\kappa \to \hbar))\}\} \\ &= \min\{\widetilde{\mathcal{U}_p}(\kappa), \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(1)\}\} \\ &= \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\kappa)\} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \widetilde{\mathcal{U}_n}(\hbar) &\leq \max\{\widetilde{\mathcal{U}_n}(\kappa), \widetilde{\mathcal{U}_n}(\kappa \to \hbar)\} \\ &\geq \max\{\widetilde{\mathcal{U}_n}(\kappa), \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\vartheta \to (\kappa \to \hbar))\}\} \\ &= \max\{\widetilde{\mathcal{U}_n}(\kappa), \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(1)\}\} \\ &= \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\kappa)\}. \end{aligned} \tag{3.5}$$

Conversely, assume that (3.3) is valid. Since $\vartheta \to (\vartheta \to 1) = 1$ for all $\vartheta \in \Omega$, we have

$$\begin{aligned} &\widetilde{\mathcal{U}_p}(1) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\vartheta)\} = \widetilde{\mathcal{U}_p}(\vartheta), \\ &\widetilde{\mathcal{U}_n}(1) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\vartheta)\} = \widetilde{\mathcal{U}_n}(\vartheta) \end{aligned} \tag{3.6}$$

for all $\vartheta \in \Omega$. Note that $(\vartheta \to \kappa) \to (\vartheta \to \kappa) = 1$ for all $\vartheta, \kappa \in \Omega$, Using (3.3) implies that

$$\begin{aligned} &\widetilde{\mathcal{U}_p}(\kappa) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta \to \kappa), \widetilde{\mathcal{U}_p}(\vartheta)\}, \\ &\widetilde{\mathcal{U}_n}(\kappa) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta \to \kappa), \widetilde{\mathcal{U}_n}(\vartheta)\} \end{aligned} \tag{3.7}$$

for all $\vartheta, \kappa \in \Omega$. Hence $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω. □

Combining (a2) and Theorem 3.1, we have the following corollary.

Corollary 3.2. *A bipolar-valued fuzzy set $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω if and only if it satisfies the following assertions:*

$$\begin{aligned} &\widetilde{\mathcal{U}_p}(\hbar) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\kappa)\}, \\ &\widetilde{\mathcal{U}_n}(\hbar) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\kappa)\} \end{aligned}$$

for all $\vartheta, \kappa, \hbar \in \Omega$ with $\vartheta \odot \kappa \leq \hbar$.

Theorem 3.3. *Let $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ be a bipolar-valued fuzzy set in Ω. Then $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω if and only if it satisfies the following assertions:*

(i) $(\forall \vartheta, \kappa \in \Omega)\ (\vartheta \leq \kappa \Rightarrow \widetilde{\mathcal{U}_p}(\vartheta) \leq \widetilde{\mathcal{U}_p}(\kappa)\ \&\ \widetilde{\mathcal{U}_n}(\vartheta) \geq \widetilde{\mathcal{U}_n}(\kappa))$,

(ii) $(\forall \vartheta, \kappa \in \Omega)\ (\widetilde{\mathcal{U}_p}(\vartheta \odot \kappa) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\kappa)\})$,

(iii) $(\forall \vartheta, \kappa \in \Omega)\ (\widetilde{\mathcal{U}_n}(\vartheta \odot \kappa) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\kappa)\})$.

Proof. Let $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ be a bipolar-valued fuzzy filter of Ω and let $\vartheta, \kappa \in \Omega$. If $\vartheta \leq \kappa$, then $\vartheta \odot \vartheta \leq \vartheta \leq \kappa$. It follows from the Corollary 3.2 that

$$\begin{aligned} &\widetilde{\mathcal{U}_p}(\kappa) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\vartheta)\} = \widetilde{\mathcal{U}_p}(\vartheta), \\ &\widetilde{\mathcal{U}_n}(\kappa) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\vartheta)\} = \widetilde{\mathcal{U}_n}(\vartheta). \end{aligned}$$

Since $\vartheta \odot \kappa \leq \vartheta \odot \kappa$, we have

$$\widetilde{\mathcal{U}_p}(\vartheta \odot \kappa) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\kappa)\},$$
$$\widetilde{\mathcal{U}_n}(\vartheta \odot \kappa) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\kappa)\}$$

by Corollary 3.2. Conversely, let $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ be a bipolar-valued fuzzy set in Ω which satisfies (i), (ii) and (iii). Let $\vartheta, \kappa, \hbar \in \Omega$ be such that $\vartheta \odot \kappa \leq \hbar$. Then

$$\widetilde{\mathcal{U}_p}(\hbar) \geq \widetilde{\mathcal{U}_p}(\vartheta \odot \kappa) \geq \min\{\widetilde{\mathcal{U}_p}(\vartheta), \widetilde{\mathcal{U}_p}(\kappa)\},$$
$$\widetilde{\mathcal{U}_n}(\hbar) \leq \widetilde{\mathcal{U}_n}(\vartheta \odot \kappa) \leq \max\{\widetilde{\mathcal{U}_n}(\vartheta), \widetilde{\mathcal{U}_n}(\kappa)\}.$$

Using Corollary 3.2, we know that $\widetilde{\mathcal{U}} = (\Omega; \widetilde{\mathcal{U}_p}, \widetilde{\mathcal{U}_n})$ is a bipolar-valued fuzzy filter of Ω. This completes the proof. □

4. CONCLUSIONS

In conclusion, this paper presents a significant advancement in the exploration of filters within BL-algebras through the innovative application of bipolar fuzzy set theory. By examining the properties of filters and introducing bipolar-valued fuzzy filters generated by fuzzy sets, we have established a new analytical framework that effectively addresses the complexities of uncertainty and vagueness associated with filter operations. The integration of bipolar fuzzy concepts not only enhances the theoretical understanding of filters but also opens new avenues for their application across various domains. Our research underscores the importance of leveraging advanced mathematical tools to deepen the insights into the structural characteristics of BL-algebras, ultimately contributing to the broader field of fuzzy logic and its practical implementations. The findings encourage further investigation into the interplay between fuzzy set theory and algebraic structures, paving the way for future studies that may yield additional theoretical and practical advancements.

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REFERENCES

[1] D. Al-Kadi and G. Muhiuddin, Bipolar fuzzy BCI-implicative ideals of BCI-algebras, Annals of Communications in Mathematics, Vol. 3, No.1 (2020), 88–96.
[2] A. Al-Masarwah and A.G. Ahmad, On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras, Eur. J. Pure Appl. Math. **11** (2018), 652-670.
[3] Anas Al-masarwah, Abd Ghafur Ahmad and G. Muhiuddin, Doubt N-ideals theory in BCK-algebras based on N-structures, Annals of Communications in Mathematics, Vol. 3, No.1 (2020), 54–62.
[4] A.M. Al-roqi, G. Muhiuddin, and S. Aldhafeeri, Normal Unisoft Filters in R_0-algebras. *Cogent Math.* **2017**, *4*, 1383006.
[5] P. A. Ejegwa and J. A. Otuwe, Frattini fuzzy subgroups of fuzzy groups Annals of Communication in Mathematics, Vol. 2 (1) (2019), 24-31
[6] P. Hájek, *Metamathematics of Fuzzy Logic,* Kluwer Academic Publishers, Dordrecht, 1998.
[7] K. Hayat, T. Mahmood and B.Y. Cao, On bipolar anti fuzzy H-ideals in hemirings, Fuzzy Inf. Eng. **9** (2017), 1–19.
[8] M. Ibrara, A. Khana and B. Davvazb, Characterizations of regular ordered semigroups in terms of (α, β)-bipolar fuzzy generalized bi-ideals, J. Intell. Fuzzy Syst. **33** (2017), 365-376.
[9] C. Jana, M. Pal and A. B. Saeid, $(\in, \in \vee q)$-Bipolar fuzzy BCK/BCI-algebras, Missouri J. Math. Sci. **29** (2017), 1–22.
[10] Y.B. Jun, M.S. Kang, H.S. Kim, Bipolar fuzzy hyper BCK-ideals in hyper BCK-algebras, Iran. J. Fuzzy Systems **8**(2), (2011), 105–120.

[11] K. Kawila, C. Udomsetchai, A. Iampan, Bipolar fuzzy UP-algebras, Math. Comput. Appl., **23**(4)(2018), 69.
[12] K. J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc. 32 (2009), 361–373.
[13] K. M. Lee, Bipolar-valued fuzzy sets and their operations, Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000), 307–312.
[14] K. M. Lee, Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, J. Fuzzy Logic Intelligent Systems 14 (2004), 125–129.
[15] L. Liu and K. Li, *Fuzzy filters of BL-algebras,* Inform. Sci. **173** (2005), 141–154.
[16] L. Liu and K. Li, *Fuzzy Boolean and positive implicative filters of BL-algebras,* Fuzzy Sets and Systems **152** (2005), 333–348.
[17] G. Muhiuddin, Habib Harizavi and Y. B. Jun, Bipolar-valued fuzzy soft hyper BCK ideals in hyper BCK algebras, Discrete Mathematics Algorithms and Applications, Vol. 12, No. 02, 2050018 (2020).
[18] G. Muhiuddin, Bipolar fuzzy KU-subalgebras/ideals of KU-algebras, Annals of Fuzzy Mathematics and Informatics, Volume 8, No. 3, (September 2014), pp. 409-418.
[19] G. Muhiuddin and K. P. Shum, New types of (α, β) - fuzzy subalgebras of BCK/BCI-algebras, International Journal of Mathematics and Computer Science, Vol. 14, No. 2, (2019) 449–464.
[20] G. Muhiuddin and Abdullah M. Al-roqi, Classifications of (alpha, beta)-fuzzy ideals in BCK/BCI–algebras, J. Math. Anal., Volume 7, Issue 6 (2016) 75–82.
[21] G. Muhiuddin and A. M. Al-roqi, Subalgebras of BCK/BCI-algebras based on (α, β)-type fuzzy sets, J. Comput. Anal. Appl. 18(6) (2015) 1057–1064.
[22] T. Oner, T. Katican, A. Borumand Saeid, (2023). (Fuzzy) filters of sheffer stroke BL-algebras. Kragujevac Journal of Mathematics, 47(1), 39–55.
[23] T. Senapati and K. P. Shum, Cubic subalgebras of BCH-algebras, Annals of Communications in Mathematics, Vol. 1, No.1 (2018), 65-73.
[24] S. Thongarsa, P. Burandate and A. Iampan, Some operations of fuzzy sets in UP-algebras with respect to a triangular norm, Annals of Communication in Mathematics, Vol. 2 (1) (2019), 1-10.
[25] E. Turunen and S. Sessa, *Local BL-algebras,* Mult-Valued Logic **6** (2001), 229–249.
[26] E. Turunen, *Boolean deductive systems of BL-algebras*, Arch Math. Logic, 40 (2001), 467-473.
[27] L. A. Zadeh, Fuzzy sets, Inf. Control 8 (1965), 338-353.
[28] W. R. Zhang, Bipolar fuzzy sets, Proceedings of Fuzzy-IEEE (1998), 835-840.

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