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# SMOOTH SYMMETRIZED AND PERTURBED HYPERBOLIC TANGENT REAL AND COMPLEX, ORDINARY AND FRACTIONAL NEURAL NETWORK APPROXIMATIONS OVER INFINITE DOMAINS

GEORGE A. ANASTASSIOU

ABSTRACT. In this article we study the univariate quantitative smooth approximation, real and complex, ordinary and fractional under differentiation of functions. The approximators here are neural network operators activated by the symmetrized and perturbed hyperbolic tangent function. All domains used are of the whole real line. The neural network operators here are of quasi-interpolation type: the basic ones, the Kantorovich type ones, and of the quadrature type. We give pointwise and uniform approximations with rates. We finish with interesting illustrations.

## 1. INTRODUCTION

The author in [1] and [2], see 'Chapters 2-5' was the first to establish neural network approximation to continuous functions with rates by very specific neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by using the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" activation functions are assumed to be of compact support.

Again the author inspired by [14], continued his studies on neural network approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent types which resulted into [5] - [9], by treating both the univariate and multivariate cases. The author also treated the corresponding fractional cases [10], [11].

The author here performs symmetrized and perturbed hyperbolic tangent activated neural network approximations to differentiated functions from $\mathbb{R}$ into $\mathbb{R}$.

We present real and complex, ordinary and fractional quasi-interpolation quantitative approximations. We derive very tight Jackson type inequalities.

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Real feed-forward neural networks (FNNs) with one hidden layer, the ones we use here, are mathematically expressed by

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle \alpha_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $\alpha_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle \alpha_j \cdot x \rangle$ is the inner product of $\alpha_j$ and $x$, and $\sigma$ is the activation function of the network. For a detailed study in neural networks in general read [18], [19], [20].

## 2. Basics

Initially we follow [12], pp. 455-460.

Our perturbed hyperbolic tangent activation function here to be used is

$$g_{q,\lambda}(x) := \frac{e^{\lambda x} - q e^{-\lambda x}}{e^{\lambda x} + q e^{-\lambda x}}, \quad \lambda, q > 0, \ x \in \mathbb{R}. \tag{1}$$

Here $\lambda$ is the parameter and $q$ is the deformation coefficient.

For more details read Chapter 18 of [12]: "$q$-deformed and $\lambda$-Parametrized Hyperbolic Tangent based Banach space Valued Ordinary and Fractional Neural Network Approximation".

'The Chapters 17 and 18' of [12] motivate our current work.

The proposed "symmetrization method" aims to use half data feed to our multivariate neural networks.

We will employ the following density function

$$M_{q.\lambda}(x) := \frac{1}{4}\left(g_{q,\lambda}(x+1) - g_{q,\lambda}(x-1)\right) > 0, \tag{2}$$

$\forall x \in \mathbb{R}; q, \lambda > 0$.

We have that

$$M_{q,\lambda}(-x) = M_{\frac{1}{q},\lambda}(x), \ \forall\, x \in \mathbb{R}; \ q, \lambda > 0, \tag{3}$$

and

$$M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x), \ \forall\, x \in \mathbb{R}; \ q, \lambda > 0. \tag{4}$$

Adding (3) and (4) we obtain

$$M_{q,\lambda}(-x) + M_{\frac{1}{q},\lambda}(-x) = M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x), \tag{5}$$

a key to this work.

So that

$$\Phi(x) := \frac{M_{q,\lambda}(x) + M_{\frac{1}{q},\lambda}(x)}{2} \tag{6}$$

is an even function, symmetric with respect to the $y$-axis.

By (18.18) of [12], we have

$$\begin{aligned} &M_{q,\lambda}\left(\tfrac{\ln q}{2\lambda}\right) = \tfrac{\tanh(\lambda)}{2}, \\ &\text{and} \\ &M_{\frac{1}{q},\lambda}\left(-\tfrac{\ln q}{2\lambda}\right) = \tfrac{\tanh(\lambda)}{2}, \ \lambda > 0. \end{aligned} \tag{7}$$

sharing the same maximum at symmetric points.

By Theorem 18.1, p. 458 of [12], we have that

$$\sum_{i=-\infty}^{\infty} M_{q,\lambda}(x-i) = 1,\ \forall\, x \in \mathbb{R}, \lambda, q > 0, \quad \text{and} \quad \sum_{i=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x-i) = 1,\ \forall\, x \in \mathbb{R}, \lambda, q > 0. \tag{8}$$

Consequently, we derive that

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1,\ \forall\, x \in \mathbb{R}. \tag{9}$$

By Theorem 18.2, p. 459 of [12], we have that

$$\int_{-\infty}^{\infty} M_{q,\lambda}(x)\,dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} M_{\frac{1}{q},\lambda}(x)\,dx = 1, \tag{10}$$

so that

$$\int_{-\infty}^{\infty} \Phi(x)\,dx = 1, \tag{11}$$

therefore $\Phi$ is a density function.

By Theorem 18.3, p. 459 of [4], we have:

Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q, \lambda > 0$. Then

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} M_{q,\lambda}(nx-k) < 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}} = Te^{-2\lambda n^{(1-\alpha)}}, \tag{12}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

Similarly, we get that

$$\sum_{k=-\infty}^{\infty} M_{\frac{1}{q},\lambda}(nx-k) < Te^{-2\lambda n^{(1-\alpha)}}. \tag{13}$$

Consequently we obtain that

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} \Phi(nx-k) < Te^{-2\lambda n^{(1-\alpha)}}, \tag{14}$$

where $T := 2\max\left\{q, \frac{1}{q}\right\} e^{4\lambda}$.

An essential property follows:

**Theorem 1.** *([13]) Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then*

$$\int_{\{u \in \mathbb{R} : |nx-u| \geq n^{1-\alpha}\}} \Phi(nx-u)\,du < \frac{\left(q + \frac{1}{q}\right)}{e^{2\lambda(n^{1-\alpha}-1)}},\ \ q, \lambda > 0. \tag{15}$$

In particular, by [13], we have that

$$\Phi(x) < \left(q + \frac{1}{q}\right) \lambda e^{-2\lambda(x-1)}, \ \forall\, x \geq 1. \tag{16}$$

We need,

**Definition 2.** *In this article we study the smooth approximation properties of the following interpolation neural network operators acting on $f \in C_B(\mathbb{R})$ (continuous and bounded functions):*

*(i) the basic ones*

$$B_n(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k), \ \forall\, x \in \mathbb{R}, n \in \mathbb{N}, \tag{17}$$

*(ii) the Kantorovich type operators*

$$C_n(f,x) := \sum_{k=-\infty}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) \Phi(nx-k), \ \forall\, x \in \mathbb{R}, n \in \mathbb{N}, \tag{18}$$

*(iii) let $\theta \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{\theta} w_r = 1$, $k \in \mathbb{Z}$, and*

$$\delta_{nk}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right), \tag{19}$$

*we consider also the quadrature type operators*

$$D_n(f,x) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, \Phi(nx-k), \ \forall\, x \in \mathbb{R}, n \in \mathbb{N}. \tag{20}$$

*We will be using the first modulus of continuity:*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in \mathbb{R}: \\ |x-y| \leq \delta}} |f(x) - f(y)|, \ \delta > 0, \tag{21}$$

*where $f \in C(\mathbb{R})$ which is bounded and or uniformly continuous.*

## 3. Main Results

Here we study the smooth approximation properties of neural network operators $B_n$, $C_n$, $D_n$ under differentiation.

**Theorem 3.** *Here $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $N \in \mathbb{N}$, $f \in C^N(\mathbb{R})$, with $f^{(i)} \in C_B(\mathbb{R})$, $i = 0, 1, ..., N$; $x \in \mathbb{R}$. Then*

*(i)*

$$\left| B_n(f,x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} B_n\left((\cdot - x)^j\right)(x) \right| \leq$$

$$\frac{\omega_1\left(f^{(N)}, \frac{1}{n^{\beta}}\right)}{n^{\beta N} N!} + \frac{4\left\|f^{(N)}\right\|_{\infty}}{(n\lambda)^N} \left(q + \frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)} =: M^s, \tag{22}$$

*(ii) assume all $f^{(j)}(x) = 0$, $j = 1, ..., N$, we have that*

$$|B_n(f,x) - f(x)| \leq M^s, \tag{23}$$

*at high speed $n^{-\beta(N+1)}$,*

*(iii)*

$$\left|B_n(f,x)-f(x)\right| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\beta j}}+\frac{1}{n^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j}2e^{2\lambda}j!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]+M^s, \tag{24}$$

*and*

*(iv)*

$$\left\|B_n(f)-f\right\|_\infty \leq \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\beta j}}+\frac{1}{n^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j}2e^{2\lambda}j!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]+M^s. \tag{25}$$

*Proof.* By Taylor's theorem we have ($x \in \mathbb{R}$)

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{26}$$

It follows

$$f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^j+$$

$$\Phi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Hence

$$B_n(f,x)=\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}B_n\left((\cdot-x)^j\right)(x)+ \tag{27}$$

$$\sum_{k=-\infty}^{\infty}\Phi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Call

$$R:=\sum_{k=-\infty}^{\infty}\Phi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{28}$$

(I) Let $\left|\frac{k}{n}-x\right|<\frac{1}{n^\beta}$. Then

i) case $\frac{k}{n}\geq x$:

$$|R| \leq \sum_{\begin{cases} k=-\infty \\ :\left|\frac{k}{n}-x\right|<\frac{1}{n^\beta}\end{cases}}^{\infty}\Phi(nx-k)\int_x^{\frac{k}{n}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt \tag{29}$$

$$\leq \sum_{\begin{cases} k=-\infty \\ :\left|\frac{k}{n}-x\right|<\frac{1}{n^\beta}\end{cases}}^{\infty}\Phi(nx-k)\,\omega_1\left(f^{(N)},\frac{1}{n^\beta}\right)\frac{\left(\frac{k}{n}-x\right)^N}{N!}$$

$$\leq \omega_1\left(f^{(N)},\frac{1}{n^\beta}\right)\frac{1}{N!n^{\beta N}}.$$

We found that

$$|R|\,|_{\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}} \leq \omega_1\left(f^{(N)},\frac{1}{n^{\beta}}\right)\frac{1}{N!n^{\beta N}}. \tag{30}$$

(ii) case $\frac{k}{n} < x$: then

$$|R| \leq \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k)\left|\int_{\frac{k}{n}}^{x}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|$$

$$\leq \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k)\int_{\frac{k}{n}}^{x}\left|f^{(N)}(x)-f^{(N)}(t)\right|\frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!}dt \tag{31}$$

$$\leq \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k)\,\omega_1\left(f^{(N)},\frac{1}{n^{\beta}}\right)\frac{\left(x-\frac{k}{n}\right)^{N}}{N!}$$

$$\leq \omega_1\left(f^{(N)},\frac{1}{n^{\beta}}\right)\frac{1}{N!n^{\beta N}}.$$

Consequently, we have proved that

$$|R|\,|_{\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}} \leq \frac{\omega_1\left(f^{(N)},\frac{1}{n^{\beta}}\right)}{N!n^{\beta N}}. \tag{32}$$

Next, we see ($\frac{k}{n} \geq x$)

$$|R| \leq \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k)\left|\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|$$

$$\leq \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k)\int_{x}^{\frac{k}{n}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt$$

$$\leq 2\left\|f^{(N)}\right\|_{\infty} \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k)\frac{\left(\frac{k}{n}-x\right)^{N}}{N!} \tag{33}$$

$$= \frac{2\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!} \sum_{\left\{\begin{array}{c} k=-\infty \\ :|nx-k|\geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(nx-k)(k-nx)^{N}.$$

In case $\frac{k}{n} < x$, we get

$$|R| \leq \sum_{\left\{\begin{array}{c} k=-\infty \\ : \left|\frac{k}{n}-x\right| \geq \frac{1}{n^{\beta}} \end{array}\right.}^{\infty} \Phi(nx-k) \left| \int_{\frac{k}{n}}^{x} \left( f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(nx-k) \int_{\frac{k}{n}}^{x} \left| f^{(N)}(x) - f^{(N)}(t) \right| \frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_{\infty} \sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(nx-k) \frac{\left(x-\frac{k}{n}\right)^{N}}{N!} \tag{34}$$

$$= \frac{2\left\|f^{(N)}\right\|_{\infty}}{n^N N!} \sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(nx-k)(nx-k)^N .$$

Consequently, it holds

$$|R| \,|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^{\beta}}} \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{n^N N!} \sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(|nx-k|) |nx-k|^N . \tag{35}$$

Next, we treat

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(|nx-k|) |nx-k|^N \overset{(16)}{\leq}$$

$$\left(\mu := \left(q + \frac{1}{q}\right) \lambda\right)$$

$$\mu \sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} e^{-2\lambda(|nx-k|-1)} |nx-k|^N =$$

$$\mu e^{2\lambda} \sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} e^{-2\lambda|nx-k|} |nx-k|^N =: (*) . \tag{36}$$

Notice that (set $\overline{\mu} := 2\lambda$)

$$e^{\frac{\overline{\mu}|nx-k|}{2}} = \sum_{\lambda=0}^{\infty} \frac{\left(\frac{\overline{\mu}|nx-k|}{2}\right)^{\lambda}}{\lambda!} \geq \left(\frac{\overline{\mu}\,|nx-k|}{2}\right)^N \frac{1}{N!}. \tag{37}$$

Therefore we have

$$\left(\frac{\overline{\mu}\,|nx-k|}{2}\right)^N \leq N!e^{\frac{\overline{\mu}|nx-k|}{2}}, \quad \text{or}$$
$$\left(\overline{\mu}\,|nx-k|\right)^N \leq 2^N N!e^{\frac{\overline{\mu}|nx-k|}{2}}, \quad \text{or} \tag{38}$$
$$|nx-k|^N \leq \frac{2^N}{\overline{\mu}^N} N!e^{\frac{\overline{\mu}|nx-k|}{2}}.$$

Hence it holds

$$(*) \leq \frac{\mu}{\overline{\mu}^N} e^{\overline{\mu}} 2^N N! \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{cases}}^{\infty} e^{-\overline{\mu}|nx-k|} e^{\frac{\overline{\mu}|nx-k|}{2}} =$$

$$\frac{\mu}{\overline{\mu}^N} e^{\overline{\mu}} 2^N N! \left( \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{cases}}^{\infty} e^{-\frac{\overline{\mu}}{2}|nx-k|} \right) \leq$$

$$\frac{\mu}{\overline{\mu}^N} 2e^{\overline{\mu}} 2^N N! \left( \int_{n^{1-\beta}-1}^{\infty} e^{-\frac{\overline{\mu}}{2}x} dx \right) =$$

$$\frac{\mu}{\overline{\mu}^N} \frac{2}{\overline{\mu}} 2e^{\overline{\mu}} 2^N N! \left( \int_{n^{1-\beta}-1}^{\infty} e^{-\frac{\overline{\mu}}{2}x} d\frac{\overline{\mu}x}{2} \right) \overset{(y:=\frac{\overline{\mu}x}{2})}{=} \tag{39}$$

$$\frac{\mu}{\overline{\mu}^{N+1}} 2^{N+2} e^{\overline{\mu}} N! \left( \int_{n^{1-\beta}-1}^{\infty} e^{-y} dy \right) = \frac{\mu}{\overline{\mu}^{N+1}} 2^{N+2} e^{\overline{\mu}} N! \left( e^{-y}\Big|_{\infty}^{n^{1-\beta}-1} \right) =$$

$$\frac{\mu}{\overline{\mu}^{N+1}} 2^{N+2} e^{\overline{\mu}} N! e^{-\frac{\overline{\mu}}{2}\left(n^{1-\beta}-1\right)} = \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Thus we have

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\beta} \end{cases}}^{\infty} \Phi\left(|nx-k|\right) |nx-k|^N \leq \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}. \tag{40}$$

And, it is

$$|R|\,|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq \left( \frac{2\left\|f^{(N)}\right\|_\infty}{N! n^N} \right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)} \tag{41}$$
$$= \frac{4\left\|f^{(N)}\right\|_\infty}{n^N \lambda^N} \left(q+\frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

We proved that

$$|R|\,|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq \frac{4\left\|f^{(N)}\right\|_\infty}{n^N \lambda^N} \left(q+\frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}. \tag{42}$$

Finally, we derive that

$$|R| \leq |R|\,|_{\left|\frac{k}{n}-x\right| < \frac{1}{n^\beta}} + |R|\,|_{\left|\frac{k}{n}-x\right| \geq \frac{1}{n^\beta}} \leq \tag{43}$$

$$\frac{\omega_1\left(f^{(N)}, \frac{1}{n^\beta}\right)}{n^{\beta N} N!} + \frac{4\left\|f^{(N)}\right\|_\infty}{n^N \lambda^N}\left(q + \frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Finally we estimate

$$\left|B_n\left((\cdot - x)^j\right)(x)\right| \leq \sum_{k=-\infty}^{\infty} \Phi(nx-k)\left|\frac{k}{n} - x\right|^j =$$

$$\sum_{\begin{cases} k=-\infty \\ : \left|x-\frac{k}{n}\right| < \frac{1}{n^\beta}\end{cases}}^{\infty} \Phi(nx-k)\left|\frac{k}{n}-x\right|^j + \sum_{\begin{cases} k=-\infty \\ : \left|x-\frac{k}{n}\right| \geq \frac{1}{n^\beta}\end{cases}}^{\infty} \Phi(nx-k)\left|\frac{k}{n}-x\right|^j \leq$$

$$\frac{1}{n^{\beta j}} + \frac{1}{n^j} \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\beta}\end{cases}}^{\infty} \Phi(|nx-k|)\,|nx-k|^j \leq \tag{44}$$

$$\frac{1}{n^{\beta j}} + \frac{1}{n^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

The theorem is proved. □

Next comes

**Theorem 4.** *Here* $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $N \in \mathbb{N}$, $f \in C^N(\mathbb{R})$, *with* $f^{(i)} \in C_B(\mathbb{R})$, $i = 0, 1, ..., N$; $x \in \mathbb{R}$. *Then*

*(i)*

$$\left|C_n(f,x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} C_n\left((\cdot - x)^j\right)(x)\right| \leq$$

$$\omega_1\left(f^{(N)}, \frac{1}{n} + \frac{1}{n^\beta}\right)\frac{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^N}{N!} +$$

$$\frac{2^N\left\|f^{(N)}\right\|_\infty}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}\right] =: \Psi^s, \tag{45}$$

*(ii) assume all* $f^{(j)}(x) = 0$, $j = 1, ..., N$, *then*

$$|C_n(f,x) - f(x)| \leq \Psi^s, \tag{46}$$

*at high speed* $n^{-\beta(N+1)}$,

*(iii)*

$$|C_n(f,x) - f(x)| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}$$

$$\left\{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^j + \frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\} + \Psi^s, \tag{47}$$

*and*
*(iv)*

$$\|C_n(f)-f\|_\infty \leq \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!}$$

$$\left\{\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^j + \frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\} + \Psi^s. \tag{48}$$

*Proof.* One can write

$$C_n(f,x) = \sum_{k=-\infty}^{\infty}\left(n\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right) G(nx-k). \tag{49}$$

Let now $f \in C^N(\mathbb{R})$ with $f^{(i)} \in C_B(\mathbb{R})$, $i = 0, 1, ..., N \in \mathbb{N}$.

We have that

$$f\left(t+\frac{k}{n}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!}\left(t+\frac{k}{n}-x\right)^j + \tag{50}$$

$$\int_x^{t+\frac{k}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!} ds,$$

and

$$n\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} n\int_0^{\frac{1}{n}}\left(t+\frac{k}{n}-x\right)^j dt +$$

$$n\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{k}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!} ds\right) dt. \tag{51}$$

Hence

$$C_n(f,x) = \sum_{k=-\infty}^{\infty}\left(n\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right)\Phi(nx-k) =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} C_n\left((\cdot-x)^j\right)(x) + \tag{52}$$

$$\sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(n\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{k}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!} ds\right) dt\right).$$

Therefore we can write

$$C_n(f,x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} C_n\left((\cdot-x)^j\right)(x) = R, \tag{53}$$

where

$$R := \sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(n\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{k}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!} ds\right) dt\right). \tag{54}$$

Call

$$\lambda(k) := n\int_0^{\frac{1}{n}}\left(\int_x^{t+\frac{k}{n}} \left(f^{(N)}(s) - f^{(N)}(x)\right) \frac{\left(t+\frac{k}{n}-s\right)^{N-1}}{(N-1)!} ds\right) dt, \tag{55}$$

where $k \in \mathbb{Z}$.

I) Let $\left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}$ $(0 < \beta < 1)$.

i) if $t + \frac{k}{n} \geq x$, then

$$|\lambda(k)| \leq n \int_0^{\frac{1}{n}} \left( \int_x^{t+\frac{k}{n}} \left| f^{(N)}(s) - f^{(N)}(x) \right| \frac{\left(t + \frac{k}{n} - s\right)^{N-1}}{(N-1)!} ds \right) dt$$

$$\leq n \int_0^{\frac{1}{n}} \left( \omega_1 \left( f^{(N)}, \left| t + \frac{k}{n} - x \right| \right) \frac{\left(t + \frac{k}{n} - x\right)^N}{N!} \right) dt \tag{56}$$

$$\leq n \int_0^{\frac{1}{n}} \omega_1 \left( f^{(N)}, |t| + \left| \frac{k}{n} - x \right| \right) \frac{\left(|t| + \left|\frac{k}{n} - x\right|\right)^N}{N!} dt$$

$$\leq \omega_1 \left( f^{(N)}, \frac{1}{n} + \frac{1}{n^\beta} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^N}{N!}.$$

(ii) if $t + \frac{k}{n} < x$, then

$$|\lambda(k)| \leq n \int_0^{\frac{1}{n}} \left| \int_{t+\frac{k}{n}}^x \left( f^{(N)}(s) - f^{(N)}(x) \right) \frac{\left(t + \frac{k}{n} - s\right)^{N-1}}{(N-1)!} ds \right| dt$$

$$\leq n \int_0^{\frac{1}{n}} \left( \int_{t+\frac{k}{n}}^x \left| f^{(N)}(x) - f^{(N)}(s) \right| \frac{\left(s - \left(t + \frac{k}{n}\right)\right)^{N-1}}{(N-1)!} ds \right) dt$$

$$\leq n \int_0^{\frac{1}{n}} \omega_1 \left( f^{(N)}, \left( x - \frac{k}{n} - t \right) \right) \frac{\left(x - t - \frac{k}{n}\right)^N}{N!} dt \tag{57}$$

$$\leq n \int_0^{\frac{1}{n}} \omega_1 \left( f^{(N)}, \left| \frac{k}{n} - x \right| + |t| \right) \frac{\left(\left|x - \frac{k}{n}\right| + |t|\right)^N}{N!} dt$$

$$\leq \omega_1 \left( f^{(N)}, \frac{1}{n^\beta} + \frac{1}{n} \right) \frac{\left(\frac{1}{n^\beta} + \frac{1}{n}\right)^N}{N!}.$$

Therefore, when $\left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}$, then

$$|\lambda(k)| \leq \omega_1 \left( f^{(N)}, \frac{1}{n^\beta} + \frac{1}{n} \right) \frac{\left(\frac{1}{n^\beta} + \frac{1}{n}\right)^N}{N!}. \tag{58}$$

Clearly now it holds

$$|R| \big|_{\left|\frac{k}{n} - x\right| < \frac{1}{n^\beta}} \leq \omega_1 \left( f^{(N)}, \frac{1}{n} + \frac{1}{n^\beta} \right) \frac{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^N}{N!}. \tag{59}$$

II) Let $\left|\frac{k}{n} - x\right| \geq \frac{1}{n^\beta}$.

i) if $t + \frac{k}{n} \geq x$, then

$$|\lambda(k)| \leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(t + \frac{k}{n} - x\right)^N}{N!} dt \tag{60}$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty n \int_0^{\frac{1}{n}} \frac{\left(\left|\frac{k}{n} - x\right| + |t|\right)^N}{N!} dt$$

$$\leq 2 \left\| f^{(N)} \right\|_\infty \frac{\left(\left|\frac{k}{n} - x\right| + \frac{1}{n}\right)^N}{N!}.$$

ii) if $t+\frac{k}{n}<x$, then

$$\left|\lambda\left(k\right)\right|\leq n\int_{0}^{\frac{1}{n}}\left(\int_{t+\frac{k}{n}}^{x}\left|f^{(N)}\left(x\right)-f^{(N)}\left(s\right)\right|\frac{\left(s-\left(t+\frac{k}{n}\right)\right)^{N-1}}{(N-1)!}ds\right)dt$$

$$\leq 2\left\|f^{(N)}\right\|_{\infty}n\int_{0}^{\frac{1}{n}}\frac{\left(x-t-\frac{k}{n}\right)^{N}}{N!}dt \tag{61}$$

$$\leq 2\left\|f^{(N)}\right\|_{\infty}n\int_{0}^{\frac{1}{n}}\frac{\left(\left|\frac{k}{n}-x\right|+|t|\right)^{N}}{N!}dt\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)^{N}}{N!}.$$

Hence when $\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}$, then

$$\left|\lambda\left(k\right)\right|\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left(|nx-k|+1\right)^{N} \tag{62}$$

$$\leq\frac{2^{N}\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left(1+|nx-k|^{N}\right),\ \ \forall\, k\in\mathbb{Z}.$$

Clearly, then

$$|R|\,|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}}\leq\left(\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\beta}\end{cases}}^{\infty}\Phi\left(nx-k\right)\left|\lambda\left(k\right)\right|\right)\leq$$

$$\frac{2^{N}\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left(\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\beta}\end{cases}}^{\infty}\Phi\left(|nx-k|\right)\left(1+|nx-k|^{N}\right)\right)=$$

$$\frac{2^{N}\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left[\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\beta}\end{cases}}^{\infty}\Phi\left(|nx-k|\right)+\right.$$

$$\left.\left(\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\beta}\end{cases}}^{\infty}\Phi\left(|nx-k|\right)|nx-k|^{N}\right)\right]\leq \tag{63}$$

(by (14), (40))

$$\frac{2^{N}\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right].$$

We have found that

$$|R|\,|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}}\leq\frac{2^{N}\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]. \tag{64}$$

Therefore, it holds

$$|R|\leq|R|\,|_{\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}}+|R|\,|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}}\leq$$

$$\omega_{1}\left(f^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)\frac{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{N}}{N!}+ \tag{65}$$

$$\frac{2^{N}\left\|f^{(N)}\right\|_{\infty}}{n^{N}N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right].$$

Finally we estimate

$$\left|C_{n}\left((\cdot-x)^{j}\right)(x)\right|\leq\sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(n\int_{0}^{\frac{1}{n}}\left|t+\frac{k}{n}-x\right|^{j}dt\right)\leq$$

$$\sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(n\int_{0}^{\frac{1}{n}}\left(\left|\frac{k}{n}-x\right|+|t|\right)^{j}dt\right)\leq$$

$$\sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)^{j}=$$

$$\sum_{\substack{k=-\infty\\ :\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}}}^{\infty}\Phi(nx-k)\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)^{j}+$$

$$\sum_{\substack{k=-\infty\\ :\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}}}^{\infty}\Phi(nx-k)\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)^{j}\leq \tag{66}$$

$$\left(\frac{1}{n^{\beta}}+\frac{1}{n}\right)^{j}+\frac{1}{n^{j}}\sum_{\substack{k=-\infty\\ :|nx-k|\geq n^{1-\beta}}}^{\infty}\Phi(|nx-k|)\left(|nx-k|+1\right)^{j}\leq$$

$$\left(\frac{1}{n^{\beta}}+\frac{1}{n}\right)^{j}+\frac{2^{j-1}}{n^{j}}\left(\sum_{\substack{k=-\infty\\ :|nx-k|\geq n^{1-\beta}}}^{\infty}\Phi(|nx-k|)\left(1+|nx-k|^{j}\right)\right)$$

(... as earlier)

$$\leq\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{j}+\frac{2^{j-1}}{n^{j}}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^{j}}2e^{2\lambda}j!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]. \tag{67}$$

That is

$$\left|C_n\left((\cdot - x)^j\right)(x)\right| \leq \left(\frac{1}{n} + \frac{1}{n^\beta}\right)^j +$$

$$\frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right]. \tag{68}$$

The theorem is proved. □

It follows

**Theorem 5.** *Here* $0 < \beta < 1$*,* $n \in \mathbb{N} : n^{1-\beta} > 2$*,* $N \in \mathbb{N}$*,* $f \in C^N(\mathbb{R})$*, with* $f^{(i)} \in C_B(\mathbb{R})$*,* $i = 0, 1, ..., N$*;* $x \in \mathbb{R}$*. Then*

*(i)*

$$\left|D_n(f, x) - f(x) - \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} D_n\left((\cdot - x)^j\right)(x)\right| \leq$$

$$\omega_1\left(f^{(N)}, \frac{1}{n} + \frac{1}{n^\beta}\right) \frac{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^N}{N!} + \tag{69}$$

$$\frac{2^N \left\|f^{(N)}\right\|_\infty}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}\right] =: \Psi^s,$$

*(ii) assume all* $f^{(j)}(x) = 0$*,* $j = 1, ..., N$*, then*

$$|D_n(f, x) - f(x)| \leq \Psi^s, \tag{70}$$

*at high speed* $n^{-\beta(N+1)}$*,*

*(iii)*

$$|D_n(f, x) - f(x)| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}$$

$$\left\{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^j + \frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\} + \Psi^s, \tag{71}$$

*(iv)*

$$\|D_n(f) - f\|_\infty \leq \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!}$$

$$\left\{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^j + \frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\} + \Psi^s. \tag{72}$$

*Proof.* We have that

$$f\left(\frac{k}{n} + \frac{i}{nr}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!}\left(\frac{k}{n} + \frac{i}{nr} - x\right)^j + \tag{73}$$

$$\int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt,$$

and

$$\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\sum_{i=1}^{r} w_i\left(\frac{k}{n}+\frac{i}{nr}-x\right)^{j}+ \tag{74}$$

$$\sum_{i=1}^{r} w_i\int_{x}^{\frac{k}{n}+\frac{i}{nr}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt.$$

Furthermore it holds

$$D_n(f,x)-f(x)-\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}D_n\left((\cdot-x)^{j}\right)(x)=R(x), \tag{75}$$

where

$$R(x)=\sum_{k=-\infty}^{\infty}\Phi(nx-k)\left(\sum_{i=1}^{r} w_i\int_{x}^{\frac{k}{n}+\frac{i}{nr}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt\right). \tag{76}$$

Call

$$\gamma(k):=\sum_{i=1}^{r} w_i\int_{x}^{\frac{k}{n}+\frac{i}{nr}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt. \tag{77}$$

I) Let $\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}$.
i) if $\frac{k}{n}+\frac{i}{nr}\geq x$, then

$$|\gamma(k)|\leq\sum_{i=1}^{r} w_i\omega_1\left(f^{(N)},\left|x-\frac{k}{n}\right|+\frac{i}{nr}\right)\frac{\left(\left|\frac{k}{n}-x\right|+\frac{i}{nr}\right)^{N}}{N!}$$

$$\leq\omega_1\left(f^{(N)},\frac{1}{n^{\beta}}+\frac{1}{n}\right)\frac{\left(\frac{1}{n^{\beta}}+\frac{1}{n}\right)^{N}}{N!}. \tag{78}$$

ii) if $\frac{k}{n}+\frac{i}{nr}<x$, then

$$|\gamma(k)|\leq\sum_{i=1}^{r} w_i\left|\int_{x}^{\frac{k}{n}+\frac{i}{nr}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt\right|$$

$$\leq\sum_{i=1}^{r} w_i\int_{\frac{k}{n}+\frac{i}{nr}}^{x}\left|\left(f^{(N)}(x)-f^{(N)}(t)\right)\right|\frac{\left(t-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}dt$$

$$\leq\sum_{i=1}^{r} w_i\omega_1\left(f^{(N)},\left|x-\frac{k}{n}\right|+\frac{i}{nr}\right)\frac{\left(x-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^{N}}{N!}$$

$$\leq\omega_1\left(f^{(N)},\frac{1}{n^{\beta}}+\frac{1}{n}\right)\frac{\left(\frac{1}{n^{\beta}}+\frac{1}{n}\right)^{N}}{N!}. \tag{79}$$

Therefore, when $\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}$, then

$$|\gamma(k)|\leq\omega_1\left(f^{(N)},\frac{1}{n^{\beta}}+\frac{1}{n}\right)\frac{\left(\frac{1}{n^{\beta}}+\frac{1}{n}\right)^{N}}{N!}. \tag{80}$$

Clearly now it holds

$$|R|\,|_{\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}}\leq\omega_1\left(f^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)\frac{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{N}}{N!}. \tag{81}$$

II) Let $\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}$.
i) if $\frac{k}{n}+\frac{i}{nr}\geq x$, then

$$|\gamma(k)|\leq\sum_{i=1}^{r}w_i 2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\frac{k}{n}+\frac{i}{nr}-x\right)^N}{N!}$$

$$\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\left|\frac{k}{n}-x\right|+\frac{1}{n}\right)^N}{N!}. \tag{82}$$

ii) if $\frac{k}{n}+\frac{i}{nr}<x$, then

$$|\gamma(k)|\leq\sum_{i=1}^{r}w_i\int_{\frac{k}{n}+\frac{i}{nr}}^{x}\left|f^{(N)}(x)-f^{(N)}(t)\right|\frac{\left(t-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}dt$$

$$\leq 2\left\|f^{(N)}\right\|_{\infty}\sum_{i=1}^{r}w_i\frac{\left(\left|x-\frac{k}{n}\right|+\frac{i}{nr}\right)^N}{N!} \tag{83}$$

$$\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right)^N}{N!}.$$

So, in general we obtain

$$|\gamma(k)|\leq 2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\left|x-\frac{k}{n}\right|+\frac{1}{n}\right)^N}{N!}$$

$$=\frac{2\left\|f^{(N)}\right\|_{\infty}}{n^N}\frac{(|nx-k|+1)^N}{N!}\leq$$

$$\frac{2^N\left\|f^{(N)}\right\|_{\infty}}{n^N N!}\left(1+|nx-k|^N\right),\ \ \forall\, k\in\mathbb{Z}. \tag{84}$$

Clearly, then

$$|R(x)|\,|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}}\leq\left(\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\beta}\end{cases}}^{\infty}\Phi(nx-k)\,|\gamma(k)|\right)\leq \tag{85}$$

$$\frac{2^N\left\|f^{(N)}\right\|_{\infty}}{n^N N!}\left(\sum_{\begin{cases}k=-\infty\\ :|nx-k|\geq n^{1-\beta}\end{cases}}^{\infty}\Phi(|nx-k|)\left(1+|nx-k|^N\right)\right)$$

$$\leq \text{(as earlier)}$$

$$\leq\frac{2^N\left\|f^{(N)}\right\|_{\infty}}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]. \tag{86}$$

Therefore, it holds

$$|R(x)|\leq|R(x)|\,|_{\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}}+|R(x)|\,|_{\left|\frac{k}{n}-x\right|\geq\frac{1}{n^{\beta}}}\leq$$

$$\omega_1\left(f^{(N)}, \frac{1}{n} + \frac{1}{n^{\beta}}\right) \frac{\left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^N}{N!} + \tag{87}$$
$$\frac{2^N \left\|f^{(N)}\right\|_{\infty}}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}\right].$$

Next, we estimate

$$\left|D_n\left((\cdot - x)^j\right)(x)\right| \leq \sum_{k=-\infty}^{\infty} \Phi(nx-k)\left(\sum_{i=1}^{r} w_i\left(\left|\frac{k}{n} - x\right| + \frac{i}{nr}\right)^j\right) \leq$$
$$\sum_{k=-\infty}^{\infty} \Phi(nx-k)\left(\left|\frac{k}{n} - x\right| + \frac{1}{n}\right)^j \leq \text{(as earlier)} \tag{88}$$
$$\leq \left(\frac{1}{n} + \frac{1}{n^{\beta}}\right)^j + \frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right].$$

The theorem is proved. □

We need,

**Definition 6.** *A function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous over $\mathbb{R}$, iff $f|_{[a,b]}$ is absolutely continuous, for every $[a, b] \subset \mathbb{R}$. We write $f \in AC^n(\mathbb{R})$, iff $f^{(n-1)} \in AC(R)$ (absolutely continuous functions over $\mathbb{R}$), $n \in \mathbb{N}$.*

**Definition 7.** *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n(\mathbb{R})$. The left Caputo fractional derivative (*[17], [21], [15]*, pp. 49-52) of the function f is defined by*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{89}$$

*$\forall\, x \in [a, \infty)$, $a \in \mathbb{R}$, where $\Gamma$ is the gamma function.*

*Notice $D_{*a}^{\nu} f \in L_1([a, b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a, b]$, $\forall\, [a, b] \subset \mathbb{R}$.*

*We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a, \infty)$.*

We need

**Lemma 8.** *(see also* [4]*) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}(\mathbb{R})$ and $f^{(n)} \in L_{\infty}(\mathbb{R})$. Then $D_{*a}^{\nu} f(a) = 0$ for any $a \in \mathbb{R}$.*

**Definition 9.** *(see also* [3, 16, 17]*) Let $f \in AC^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \tag{90}$$

*$\forall\, x \in (-\infty, b]$, $b \in \mathbb{R}$. We set $D_{b-}^{0} f(x) = f(x)$.*

*Notice $D_{b-}^{\alpha} f \in L_1([a, b])$ and $D_{b-}^{\alpha} f$ exists a.e. on $[a, b]$, $\forall\, [a, b] \subset \mathbb{R}$.*

**Lemma 10.** *(see also* [4]*) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_{\infty}(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$, for any $b \in \mathbb{R}$.*

**Convention 11.** *We assume that*

$$\begin{aligned} &D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \\ &\text{and} \\ &D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0. \end{aligned} \tag{91}$$

We mention

**Proposition 12.** *(see also* [4]*) Let $f \in C^n(\mathbb{R})$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D^{\nu}_{*a} f(x)$ is continuous in $x \in [a, \infty)$, $a \in \mathbb{R}$.*

Also we have

**Proposition 13.** *(see also* [4]*) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D^{\alpha}_{b-} f(x)$ is continuous in $x \in (-\infty, b]$, $b \in \mathbb{R}$.*

We further mention

**Proposition 14.** *(see also* [4]*) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and let $x, x_0 \in \mathbb{R} : x \geq x_0$. Then $D^{\alpha}_{*x_0} f(x)$ is continuous in $x_0$.*

**Proposition 15.** *(see also* [4]*) Let $f \in C^{m-1}(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and let $x, x_0 \in \mathbb{R} : x \leq x_0$. Then $D^{\alpha}_{x_0-} f(x)$ is continuous in $x_0$.*

**Proposition 16.** *(see also* [4]*) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$; $x, x_0 \in \mathbb{R}$. Then $D^{\alpha}_{*x_0} f(x)$, $D^{\alpha}_{x_0-} f(x)$ are jointly continuous functions in $(x, x_0)$ from $R^2 \to \mathbb{R}$.*

Fractional results follow.

**Theorem 17.** *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in AC^N(\mathbb{R})$, $f^{(N)} \in L_\infty(\mathbb{R})$, $0 < \beta < 1$, $x \in \mathbb{R}$, $n \in \mathbb{N} : n^{1-\beta} \geq 3$. Assume also that both $\sup_{x \in \mathbb{R}} \|D^{\alpha}_{*x} f\|_\infty, \sup_{x \in \mathbb{R}} \|D^{\alpha}_{x-} f\|_\infty < \infty$. Then*

*(I)*

$$\left| B_n(f, x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} B_n\left( (\cdot - x)^j \right)(x) \right| \leq$$

$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)} \left[ \omega_1\left( D^{\alpha}_{x-} f, \frac{1}{n^\beta} \right)_{(-\infty, x]} + \omega_1\left( D^{\alpha}_{*x} f, \frac{1}{n^\beta} \right)_{[x, \infty)} \right] +$$

$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)} \frac{\left( q + \frac{1}{q} \right)}{\lambda^N} 2e^{2\lambda} N! e^{-\lambda\left( n^{1-\beta} - 1 \right)} \left[ \left\| D^{\alpha}_{x-} f \right\|_{\infty, (-\infty, x]} + \left\| D^{\alpha}_{*x} f \right\|_{\infty, [x, \infty)} \right] =: \theta^s, \tag{92}$$

*(II) given $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, we have*

$$|B_n(f, x) - f(x)| \leq \theta^s, \tag{93}$$

*(III)*

$$|B_n(f, x) - f(x)| \leq \sum_{j=1}^{N-1} \frac{\left| f^{(j)}(x) \right|}{j!}$$

$$\left\{ \frac{1}{n^{\beta j}} + \frac{1}{n^j} \frac{\left( q + \frac{1}{q} \right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left( n^{1-\beta} - 1 \right)} \right\} + \theta^s, \tag{94}$$

*and*

*(IV) adding $\left\| f^{(j)} \right\|_\infty < \infty$, $j = 1, ..., N-1$, we get*

$$\|B_n(f) - f\|_\infty \leq \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + \frac{1}{n^j} \frac{\left( q + \frac{1}{q} \right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left( n^{1-\beta} - 1 \right)} \right\} +$$

$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\sup_{x\in\mathbb{R}}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\sup_{x\in\mathbb{R}}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,\infty)}\right]+$$

$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)}\frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}$$

$$\left[\sup_{x\in\mathbb{R}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]}+\sup_{x\in\mathbb{R}}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}\right]. \quad (95)$$

*Above, when $N=1$, the sum* $\sum_{j=1}^{N-1}\cdot=0$.

*As we see here we obtain fractional pointwise and uniform convergence with rates of $B_n \to I$ the unit operator, as $n\to\infty$.*

*Proof.* Let $x\in\mathbb{R}$. We have that $D_{x-}^{\alpha}f(x)=D_{*x}^{\alpha}f(x)=0$.

From [15], p. 54, we get the left Caputo fractional Taylor's formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \quad (96)$$

$$\frac{1}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}<\infty$.

Also from [3], using the right Caputo fractional Taylor's formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \quad (97)$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $-\infty<\frac{k}{n}\leq x$.

Hence it holds

$$f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \quad (98)$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}<\infty$, iff $\lceil nx\rceil\leq k<\infty$,

and

$$f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \quad (99)$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $-\infty<\frac{k}{n}\leq x$, iff $-\infty<k\leq\lfloor nx\rfloor$.

We have that $\lceil nx\rceil\leq\lfloor nx\rfloor+1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\infty} f\left(\frac{k}{n}\right)\Phi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\infty}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j} + \tag{100}$$

$$\frac{\sum_{k=\lfloor nx\rfloor+1}^{\infty}\Phi(nx-k)}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

and

$$\sum_{k=-\infty}^{\lfloor nx\rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=-\infty}^{\lfloor nx\rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j} + \tag{101}$$

$$\frac{\sum_{k=-\infty}^{\lfloor nx\rfloor}\Phi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ.$$

Adding the last two equalities (100), (101) we obtain

$$B_n(f,x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}B_n\left((\cdot-x)^{j}\right)(x)=R_n(x), \tag{102}$$

where

$$R_n(x):=R_{1n}(x)+R_{2n}(x), \tag{103}$$

with

$$R_{1n}(x):=\frac{\sum_{k=\lfloor nx\rfloor+1}^{\infty}\Phi(nx-k)}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ, \tag{104}$$

and

$$R_{2n}(x):=\frac{\sum_{k=-\infty}^{\lfloor nx\rfloor}\Phi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ. \tag{105}$$

Furthermore, let

$$\delta_{1n}(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ, \tag{106}$$

for $k=\lfloor nx\rfloor+1,...,\infty$,

and

$$\delta_{2n}(x):=\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ, \tag{107}$$

for $k=-\infty,...,\lfloor nx\rfloor$.

Let $\left|\frac{k}{n}-x\right|<\frac{1}{n^{\beta}}$, we derive that

$$|\delta_{1n}(x)|\leq\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,\infty)}\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}, \tag{108}$$

$k=\lfloor nx\rfloor+1,...,\infty$,

and

$$\left|\delta_{2n}(x)\right| \leq \omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{(-\infty,x]} \frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}, \tag{109}$$

$k = -\infty, ..., \lfloor nx \rfloor$.

Also we obtain that

$$\left|\delta_{1n}(x)\right| \leq \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)} \frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\Gamma(\alpha+1)}, \quad k = \lfloor nx \rfloor + 1, ..., \infty, \tag{110}$$

and

$$\left|\delta_{2n}(x)\right| \leq \left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]} \frac{\left(x-\frac{k}{n}\right)^{\alpha}}{\Gamma(\alpha+1)}, \quad k = -\infty, ..., \lfloor nx \rfloor. \tag{111}$$

Therefore, it holds

$$\left|R_{1n}(x)\right| |_{\left|x-\frac{k}{n}\right|<\frac{1}{n^{\beta}}} \leq \frac{\omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,\infty)}}{n^{\alpha\beta}\Gamma(\alpha+1)}, \tag{112}$$

$k = \lfloor nx \rfloor + 1, ..., \infty$,

and

$$\left|R_{2n}(x)\right| |_{\left|x-\frac{k}{n}\right|<\frac{1}{n^{\beta}}} \leq \frac{\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{(-\infty,x]}}{n^{\alpha\beta}\Gamma(\alpha+1)}, \tag{113}$$

$k = -\infty, ..., \lfloor nx \rfloor$.

Next, we estimate

$$\left|R_{1n}(x)\right| |_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

$$\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}}{\Gamma(\alpha+1)} \sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left|x-\frac{k}{n}\right| \geq \frac{1}{n^{\beta}} \end{cases}}^{\infty} \Phi(nx-k)\left(\frac{k}{n}-x\right)^{\alpha} \leq$$

(by $N = \lceil \alpha \rceil$)

$$\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)} \sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\beta} \end{cases}}^{\infty} \Phi(|nx-k|)\,|nx-k|^{N} \leq \tag{114}$$

$$\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\left(q+\frac{1}{q}\right)\lambda e^{2\lambda} \sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\beta} \end{cases}}^{\infty} e^{-2\lambda|nx-k|}\,|nx-k|^{N} \leq \text{(as earlier)}$$

$$\leq \left(\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}. \tag{115}$$

Hence, it holds

$$\left|R_{1n}(x)\right| |_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq \tag{116}$$

$$\left(\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}}{n^{\alpha}\Gamma(\alpha+1)}\right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}.$$

Furthermore, we get that

$$\left|R_{2n}(x)\right| |_{\left|x-\frac{k}{n}\right|\geq\frac{1}{n^{\beta}}} \leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{\Gamma(\alpha+1)} \sum_{\left\{\begin{array}{c} k=-\infty \\ :\left|x-\frac{k}{n}\right| \geq \frac{1}{n^{\beta}} \end{array}\right.}^{\lfloor nx \rfloor} \Phi(nx-k)\left(x-\frac{k}{n}\right)^{\alpha} \leq$$

$$\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)} \sum_{\left\{\begin{array}{c} k=-\infty \\ :|nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(|nx-k|)|nx-k|^{\alpha} \leq$$

$$\left(\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)}\right) \sum_{\left\{\begin{array}{c} k=-\infty \\ :|nx-k| \geq n^{1-\beta} \end{array}\right.}^{\infty} \Phi(|nx-k|)|nx-k|^{N} \leq \text{(as earlier)}$$

$$\leq \left(\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)}\right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}. \tag{117}$$

That is

$$|R_{2n}(x)|\,|_{\left|x-\frac{k}{n}\right| \geq \frac{1}{n^{\beta}}} \leq$$
$$\left(\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x])}}{n^{\alpha}\Gamma(\alpha+1)}\right) \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}. \tag{118}$$

We have proved that

$$|R_n(x)| \leq \frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{(-\infty,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,\infty)}\right] + \tag{119}$$
$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)} \frac{\left(q+\frac{1}{q}\right)}{\lambda^{N}} 2e^{2\lambda} N! e^{-\lambda\left(n^{1-\beta}-1\right)}\left[\left\|D_{x-}^{\alpha}f\right\|_{\infty,(-\infty,x]} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,\infty)}\right].$$

As earlier, we have that

$$\left|B_n\left((\cdot-x)^{j}\right)(x)\right| \leq \frac{1}{n^{\beta j}} + \frac{1}{n^{j}} \frac{\left(q+\frac{1}{q}\right)}{\lambda^{j}} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}. \tag{120}$$

We have that

$$\left(D_{*x}^{\alpha}f\right)(s) = \frac{1}{\Gamma(N-\alpha)} \int_{x}^{s} (s-t)^{N-\alpha-1} f^{(N)}(t)\,dt, \ \ (s \geq x) \tag{121}$$

and

$$\left(D_{x-}^{\alpha}f\right)(s) = \frac{(-1)^{N}}{\Gamma(N-\alpha)} \int_{s}^{x} (t-s)^{N-\alpha-1} f^{(N)}(t)\,dt, \ \ (s \leq x) \tag{122}$$

$\forall\, x, s \in \mathbb{R}$.

Therefore it holds

$$\left|\left(D_{*x}^{\alpha}f\right)(s)\right| \leq \frac{1}{\Gamma(N-\alpha)}\left\|f^{(N)}\right\|_{\infty} \frac{(s-x)^{N-\alpha}}{N-\alpha} = \frac{\left\|f^{(N)}\right\|_{\infty}(s-x)^{N-\alpha}}{\Gamma(N-\alpha+1)}, \tag{123}$$

$s \geq x$,

and

$$\left|\left(D_{x-}^{\alpha}f\right)(s)\right| \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha)}\frac{(x-s)^{N-\alpha}}{N-\alpha} = \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(x-s)^{N-\alpha}, \tag{124}$$

$s \leq x$.

Thus, it is reasonable to assume that $\sup_{x\in\mathbb{R}}\left\|D_{*x}^{\alpha}f\right\|_{\infty}, \sup_{x\in\mathbb{R}}\left\|D_{x-}^{\alpha}f\right\|_{\infty} < \infty$.

Consequently it holds

$$\sup_{x\in\mathbb{R}}\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{(-\infty,x]}, \sup_{x\in\mathbb{R}}\omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,\infty)} < \infty. \tag{125}$$

The theorem now it is proved. □

## 4. Illustrations for $N = 1$

We obtain the following results:

**Corollary 18.** *Here $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $f \in C^1(\mathbb{R})$, with $f, f' \in C_B(\mathbb{R})$; $x \in \mathbb{R}$. Then*

*I)*

$$\left|B_n(f,x) - f(x) - f'(x)B_n((.-x))(x)\right| \leq \tag{126}$$
$$\frac{\omega_1\left(f', \frac{1}{n^{\beta}}\right)}{n^{\beta}} + \frac{4\left\|f'\right\|_{\infty}}{n\lambda}\left(q+\frac{1}{q}\right)e^{2\lambda}e^{-\lambda\left(n^{1-\beta}-1\right)} =: M_1^s,$$

*II) assume $f'(x) = 0$, we have that*

$$\left|B_n(f,x) - f(x)\right| \leq M_1^s, \tag{127}$$

*at high speed $n^{-2\beta}$,*

*III)*

$$\left|B_n(f,x) - f(x)\right| \leq \left|f'(x)\right|\left[\frac{1}{n^{\beta}} + \frac{1}{n}\frac{\left(q+\frac{1}{q}\right)}{\lambda}2e^{2\lambda}e^{-\lambda\left(n^{1-\beta}-1\right)}\right] + M_1^s, \tag{128}$$

*IV)*

$$\left\|B_n(f) - f\right\|_{\infty} \leq \left\|f'\right\|_{\infty}\left[\frac{1}{n^{\beta}} + \frac{1}{n}\frac{\left(q+\frac{1}{q}\right)}{\lambda}2e^{2\lambda}e^{-\lambda\left(n^{1-\beta}-1\right)}\right] + M_1^s. \tag{129}$$

*Proof.* By Theorem 3 for $N = 1$. □

**Corollary 19.** *Here $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $f \in C^1(\mathbb{R})$, with $f, f' \in C_B(\mathbb{R})$; $x \in \mathbb{R}$. Then*

*I)*

$$\left\{\begin{array}{l} \left|C_n(f,x) - f(x) - f'(x)C_n((\cdot-x))(x)\right|, \\ \left|D_n(f,x) - f(x) - f'(x)D_n((\cdot-x))(x)\right| \end{array}\right\} \leq$$
$$\omega_1\left(f', \frac{1}{n} + \frac{1}{n^{\beta}}\right)\left(\frac{1}{n} + \frac{1}{n^{\beta}}\right) + \tag{130}$$
$$\frac{2\left\|f'\right\|_{\infty}}{n}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda}2e^{2\lambda}e^{-\lambda\left(n^{1-\beta}-1\right)}\right] =: \psi_1^s,$$

*II) assume $f'(x)=0$, we have that*

$$\left\{ \begin{array}{c} |C_n(f,x)-f(x)|, \\ |D_n(f,x)-f(x)| \end{array} \right\} \leq \psi_1^s, \tag{131}$$

*at high speed $n^{-2\beta}$,*

*III)*

$$\left\{ \begin{array}{c} |C_n(f,x)-f(x)|, \\ |D_n(f,x)-f(x)| \end{array} \right\} \leq$$

$$|f'(x)| \left\{ \left( \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{1}{n} \left[ \frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda} 2e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)} \right] \right\} + \psi_1^s, \tag{132}$$

*and*

*IV)*

$$\left\{ \begin{array}{c} \|C_n(f)-f\|_\infty, \\ \|D_n(f)-f\|_\infty \end{array} \right\} \leq$$

$$\|f'\|_\infty \left\{ \left( \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{1}{n} \left[ \frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q+\frac{1}{q}\right)}{\lambda} 2e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)} \right] \right\} + \psi_1^s. \tag{133}$$

*Proof.* By Theorems 4, 5 for $N=1$. □

**Corollary 20.** *Let $0<\alpha<1$, $f\in AC^1(\mathbb{R})$, $f'\in L_\infty(\mathbb{R})$, $0<\beta<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}: n^{1-\beta}\geq 3$. Assume also that $\sup_{x\in\mathbb{R}} \|D_{*x}^\alpha f\|_\infty, \sup_{x\in\mathbb{R}} \|D_{x-}^\alpha f\|_\infty < \infty$. Then*

*I)*

$$|B_n(f,x)-f(x)| \leq$$

$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)} \left[ \omega_1 \left( D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{(-\infty,x]} + \omega_1 \left( D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,\infty)} \right] +$$

$$\frac{1}{n^\alpha \Gamma(\alpha+1)} \frac{\left(q+\frac{1}{q}\right)}{\lambda} 2e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)} \left[ \|D_{x-}^\alpha f\|_{\infty,(-\infty,x]} + \|D_{*x}^\alpha f\|_{\infty,[x,\infty)} \right], \tag{134}$$

*and*

*II)*

$$\|B_n(f)-f\|_\infty \leq$$

$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)} \left[ \sup_{x\in\mathbb{R}} \omega_1 \left( D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{(-\infty,x]} + \sup_{x\in\mathbb{R}} \omega_1 \left( D_{*x}^\alpha f, \frac{1}{n^\beta} \right)_{[x,\infty)} \right] +$$

$$\frac{1}{n^\alpha \Gamma(\alpha+1)} \frac{\left(q+\frac{1}{q}\right)}{\lambda} 2e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)} \left[ \sup_{x\in\mathbb{R}} \|D_{x-}^\alpha f\|_{\infty,(-\infty,x]} + \sup_{x\in\mathbb{R}} \|D_{*x}^\alpha f\|_{\infty,[x,\infty)} \right]. \tag{135}$$

*Proof.* By Theorem 17 for $N=1$. □

Next, it is the case of $\alpha=\frac{1}{2}$.

**Corollary 21.** *Let $f\in AC^1(\mathbb{R})$, $f'\in L_\infty(\mathbb{R})$, $0<\beta<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}: n^{1-\beta}\geq 3$. Assume that $\sup_{x\in\mathbb{R}} \left\|D_{*x}^{\frac{1}{2}} f\right\|_\infty, \sup_{x\in\mathbb{R}} \left\|D_{x-}^{\frac{1}{2}} f\right\|_\infty < \infty$. Then*

*I)*

$$|B_n(f,x)-f(x)| \leq$$

$$\frac{2}{n^{\frac{\beta}{2}}\sqrt{\pi}}\left[\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{(-\infty,x]}+\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,\infty)}\right]+$$

$$\frac{1}{\sqrt{\pi n}}\frac{\left(q+\frac{1}{q}\right)}{\lambda}4e^{2\lambda}e^{-\lambda\left(n^{1-\beta}-1\right)}\left[\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,(-\infty,x]}+\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,\infty)}\right], \quad (136)$$

*and*

*II)*

$$\left\|B_n(f)-f\right\|_\infty \leq$$

$$\frac{2}{n^{\frac{\beta}{2}}\sqrt{\pi}}\left[\sup_{x\in\mathbb{R}}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{(-\infty,x]}+\sup_{x\in\mathbb{R}}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,\infty)}\right]+$$

$$\frac{1}{\sqrt{\pi n}}\frac{\left(q+\frac{1}{q}\right)}{\lambda}4e^{2\lambda}e^{-\lambda\left(n^{1-\beta}-1\right)}\left[\sup_{x\in\mathbb{R}}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,(-\infty,x]}+\sup_{x\in\mathbb{R}}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,\infty)}\right]. \quad (137)$$

*Proof.* By Corollary 20. □

## 5. Complex Neural Network Approximation

We make

**Remark 22.** *Let $f:\mathbb{R}\to\mathbb{C}$, with real and imaginary parts $f_1,f_2: f=f_1+if_2$, $i=\sqrt{-1}$. Clearly $f$ is continuous iff $f_1$ and $f_2$ are continuous.*

*Also it holds*

$$f^{(j)}(x)=f_1^{(j)}(x)+if_2^{(j)}(x), \quad (138)$$

*for all $j=1,...,N$, given that $f_1,f_2\in C^N(\mathbb{R})$, $N\in\mathbb{N}$.*

*Here define*

$$\begin{aligned} B_n(f,x) &:= B_n(f_1,x)+iB_n(f_2,x),\\ C_n(f,x) &:= C_n(f_1,x)+iC_n(f_2,x),\\ D_n(f,x) &:= D_n(f_1,x)+iD_n(f_2,x). \end{aligned} \quad (139)$$

*We observe here that*

$$|B_n(f,x)-f(x)|\leq|B_n(f_1,x)-f_1(x)|+|B_n(f_2,x)-f_2(x)|, \quad (140)$$

*and*

$$\|B_n(f)-f\|_\infty\leq\|B_n(f_1)-f_1\|_\infty+\|B_n(f_2)-f_2\|_\infty; \quad (141)$$

$$|C_n(f,x)-f(x)|\leq|C_n(f_1,x)-f_1(x)|+|C_n(f_2,x)-f_2(x)|, \quad (142)$$

$$\|C_n(f)-f\|_\infty\leq\|C_n(f_1)-f_1\|_\infty+\|C_n(f_2)-f_2\|_\infty; \quad (143)$$

*and*

$$|D_n(f,x)-f(x)|\leq|D_n(f_1,x)-f_1(x)|+|D_n(f_2,x)-f_2(x)|, \quad (144)$$

$$\|D_n(f)-f\|_\infty\leq\|D_n(f_1)-f_1\|_\infty+\|D_n(f_2)-f_2\|_\infty. \quad (145)$$

We denote by $C_B(\mathbb{R},\mathbb{C})$ the space of continuous and bounded functions $f:\mathbb{R}\to\mathbb{C}$. Clearly $f$ is bounded, iff both $f_1,f_2$ are bounded from $\mathbb{R}$ into $\mathbb{R}$, where $f=f_1+if_2$.

We give

**Theorem 23.** *Let $f:\mathbb{R}\to\mathbb{C}$, such that $f=f_1+if_2$. Assume $f_1,f_2\in C^N(\mathbb{R})$, $N\in\mathbb{R}$, with $f_1^{(i)},f_2^{(i)}\in C_B(\mathbb{R})$, $i=0,1,...,N$; $x\in\mathbb{R}$. Here $0<\beta<1$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

*(I)*

$$|B_n(f,x) - f(x)| \leq \sum_{j=1}^{N} \frac{\left(\left|f_1^{(j)}(x)\right| + \left|f_2^{(j)}(x)\right|\right)}{j!}$$

$$\left[\frac{1}{n^{\beta j}} + \frac{1}{n^j} \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right] + \tag{146}$$

$$\left[\frac{\left(\omega_1\left(f_1^{(N)}, \frac{1}{n^\beta}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\beta}\right)\right)}{n^{\beta N} N!} + \right.$$

$$\left.\frac{4\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)\left(q + \frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}}{n^N \lambda^N}\right],$$

*(II) given that* $f_1^{(j)}(x) = f_2^{(j)}(x) = 0$, $j = 1, ..., N$, *we have*

$$|B_n(f,x) - f(x)| \leq \left[\frac{\left(\omega_1\left(f_1^{(N)}, \frac{1}{n^\beta}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\beta}\right)\right)}{n^{\beta N} N!} + \right. \tag{147}$$

$$\left.\frac{4\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)\left(q + \frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}}{n^N \lambda^N}\right],$$

*(III)*

$$\|B_n(f) - f\|_\infty \leq \sum_{j=1}^{N} \frac{\left(\left\|f_1^{(j)}\right\|_\infty + \left\|f_2^{(j)}\right\|_\infty\right)}{j!}$$

$$\left[\frac{1}{n^{\beta j}} + \frac{1}{n^j} \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right] + \tag{148}$$

$$\left[\frac{\left(\omega_1\left(f_1^{(N)}, \frac{1}{n^\beta}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\beta}\right)\right)}{n^{\beta N} N!} + \right.$$

$$\left.\frac{4\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)\left(q + \frac{1}{q}\right) e^{2\lambda} e^{-\lambda\left(n^{1-\beta}-1\right)}}{n^N \lambda^N}\right].$$

*Proof.* By Theorem 3. □

We continue with

**Theorem 24.** *All as in Theorem 23. Then*

*(I)*

$$\left\{\begin{array}{l} |C_n(f,x) - f(x)|, \\ |D_n(f,x) - f(x)| \end{array}\right\} \leq \sum_{j=1}^{N} \frac{\left(\left|f_1^{(j)}(x)\right| + \left|f_2^{(j)}(x)\right|\right)}{j!}$$

$$\left\{\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^j + \frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}} + \frac{\left(q + \frac{1}{q}\right)}{\lambda^j} 2e^{2\lambda} j! e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\} +$$

$$\left\{\left(\omega_1\left(f_1^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)\right)\frac{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^N}{N!}+\right.$$
$$\left.\frac{2^N\left(\left\|f_1^{(N)}\right\|_\infty+\left\|f_2^{(N)}\right\|_\infty\right)}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\}, \quad (149)$$

*(II) given that* $f_1^{(j)}(x)=f_2^{(j)}(x)=0$, $j=1,...,N$, *we have*

$$\left\{\begin{array}{c}|C_n(f,x)-f(x)|,\\ |D_n(f,x)-f(x)|\end{array}\right\}\leq$$
$$\left(\omega_1\left(f_1^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)\right)\frac{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^N}{N!}+ \quad (150)$$
$$\frac{2^N\left(\left\|f_1^{(N)}\right\|_\infty+\left\|f_2^{(N)}\right\|_\infty\right)}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right],$$

*(III)*

$$\left\{\begin{array}{c}\|C_n(f)-f\|_\infty,\\ \|D_n(f)-f\|_\infty\end{array}\right\}\leq\sum_{j=1}^{N}\frac{\left(\left\|f_1^{(j)}\right\|_\infty+\left\|f_2^{(j)}\right\|_\infty\right)}{j!}$$
$$\left\{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^j+\frac{2^{j-1}}{n^j}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^j}2e^{2\lambda}j!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\}+$$
$$\left\{\left(\omega_1\left(f_1^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\omega_1\left(f_2^{(N)},\frac{1}{n}+\frac{1}{n^{\beta}}\right)\right)\frac{\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^N}{N!}+\right.$$
$$\left.\frac{2^N\left(\left\|f_1^{(N)}\right\|_\infty+\left\|f_2^{(N)}\right\|_\infty\right)}{n^N N!}\left[\frac{T}{e^{2\lambda n^{1-\beta}}}+\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}\right]\right\}. \quad (151)$$

*Proof.* By Theorems 4, 5. □

We finish with the following fractional result.

**Theorem 25.** *Let* $f:\mathbb{R}\to\mathbb{C}$, *such that* $f=f_1+if_2$. *Assume* $f_1,f_2\in AC^N(\mathbb{R})$, $f_1^{(N)},f_2^{(N)}\in L_\infty(\mathbb{R})$, $N\in\mathbb{N}$. *Here* $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $0<\beta<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}:n^{1-\beta}\geq 3$. *Suppose also that* $\sup_{x\in\mathbb{R}}\|D_{*x}^{\alpha}f_1\|_\infty$, $\sup_{x\in\mathbb{R}}\|D_{*x}^{\alpha}f_2\|_\infty$, $\sup_{x\in\mathbb{R}}\|D_{x-}^{\alpha}f_1\|_\infty$, $\sup_{x\in\mathbb{R}}\|D_{x-}^{\alpha}f_2\|_\infty<\infty$. *Then*

*(I)*

$$|B_n(f,x)-f(x)|\leq\sum_{j=1}^{N-1}\frac{\left(\left|f_1^{(j)}(x)\right|+\left|f_2^{(j)}(x)\right|\right)}{j!}$$
$$\left\{\frac{1}{n^{\beta j}}+\frac{1}{n^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j}2e^{2\lambda}j!e^{-\lambda\left(n^{1-\beta}-1\right)}\right\}+ \quad (152)$$
$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\omega_1\left(D_{x-}^{\alpha}f_1,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\omega_1\left(D_{x-}^{\alpha}f_2,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\right.$$

$$\omega_1\left(D^{\alpha}_{*x}f_1,\frac{1}{n^{\beta}}\right)_{[x,\infty)}+\omega_1\left(D^{\alpha}_{*x}f_2,\frac{1}{n^{\beta}}\right)_{[x,\infty)}\Bigg]+$$

$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)}\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}$$

$$\left[\left\|D^{\alpha}_{x-}f_1\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{x-}f_2\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{*x}f_1\right\|_{\infty,[x,\infty)}+\left\|D^{\alpha}_{*x}f_2\right\|_{\infty,[x,\infty)}\right],$$

*(II) given $f_1^{(j)}(x)=f_2^{(j)}(x)=0$, $j=1,...,N-1$, we have*

$$|B_n(f,x)-f(x)|\leq \tag{153}$$

$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\omega_1\left(D^{\alpha}_{x-}f_1,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\omega_1\left(D^{\alpha}_{x-}f_2,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\right.$$

$$\left.\omega_1\left(D^{\alpha}_{*x}f_1,\frac{1}{n^{\beta}}\right)_{[x,\infty)}+\omega_1\left(D^{\alpha}_{*x}f_2,\frac{1}{n^{\beta}}\right)_{[x,\infty)}\right]+$$

$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)}\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)}$$

$$\left[\left\|D^{\alpha}_{x-}f_1\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{x-}f_2\right\|_{\infty,(-\infty,x]}+\left\|D^{\alpha}_{*x}f_1\right\|_{\infty,[x,\infty)}+\left\|D^{\alpha}_{*x}f_2\right\|_{\infty,[x,\infty)}\right],$$

*and*

*(III)*

$$\|B_n(f)-f\|_{\infty}\leq$$

$$\sum_{j=1}^{N-1}\frac{\left(\left\|f_1^{(j)}\right\|_{\infty}+\left\|f_2^{(j)}\right\|_{\infty}\right)}{j!}\left\{\frac{1}{n^{\beta j}}+\frac{1}{n^j}\frac{\left(q+\frac{1}{q}\right)}{\lambda^j}2e^{2\lambda}j!e^{-\lambda\left(n^{1-\beta}-1\right)}\right\}+$$

$$\frac{1}{n^{\alpha\beta}\Gamma(\alpha+1)}\left[\sup_{x\in\mathbb{R}}\omega_1\left(D^{\alpha}_{x-}f_1,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\sup_{x\in\mathbb{R}}\omega_1\left(D^{\alpha}_{x-}f_2,\frac{1}{n^{\beta}}\right)_{(-\infty,x]}+\right.$$

$$\left.\sup_{x\in\mathbb{R}}\omega_1\left(D^{\alpha}_{*x}f_1,\frac{1}{n^{\beta}}\right)_{[x,\infty)}+\sup_{x\in\mathbb{R}}\omega_1\left(D^{\alpha}_{*x}f_2,\frac{1}{n^{\beta}}\right)_{[x,\infty)}\right]+$$

$$\frac{1}{n^{\alpha}\Gamma(\alpha+1)}\frac{\left(q+\frac{1}{q}\right)}{\lambda^N}2e^{2\lambda}N!e^{-\lambda\left(n^{1-\beta}-1\right)} \tag{154}$$

$$\left[\sup_{x\in\mathbb{R}}\left\|D^{\alpha}_{x-}f_1\right\|_{\infty,(-\infty,x]}+\sup_{x\in\mathbb{R}}\left\|D^{\alpha}_{x-}f_2\right\|_{\infty,(-\infty,x]}+\right.$$

$$\left.\sup_{x\in\mathbb{R}}\left\|D^{\alpha}_{*x}f_1\right\|_{\infty,[x,\infty)}+\sup_{x\in\mathbb{R}}\left\|D^{\alpha}_{*x}f_2\right\|_{\infty,[x,\infty)}\right].$$

*Proof.* By Theorem 17. □

**Conclusion:** Here we presented univariate neural network approximation over infinite domains under differentiation of functions. The activation function was the symmetrized and perturbed hyperbolic tangent function. The rates of convergence were higher and the data needed to feed the neural network were half due to symmetry.

**Conflict of interest:** None.

**Data use:** None, it is a theoretical article.

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GEORGE A. ANASTASSIOU
DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MEMPHIS, MEMPHIS, TN 38152, U.S.A.
*Email address*: ganastss@memphis.edu