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ON IYENGAR'S AND OSTROWSKI'S INTEGRAL MEANS

JITEN AHUJA* AND RICARDO ESTRADA

ABSTRACT. Frullani's Integral Formula is an old formula that was known to hold under strict conditions. Iyengar, and later Ostrowski, provided necessary and sufficient conditions for the existence of the Frullani Integral Formula. Their conditions were different but equivalent. In this article, we identify other conditions that are equivalent. We show that these conditions are, in fact, solutions to a family of linear differential equations of the first order. We study the limiting behavior of these solutions at zero and infinity, and in doing so, arrive at a new proof of the equivalence of Iyengar's and Ostrowski's conditions. Lastly, we provide applications of our results.

1. INTRODUCTION

The following formula [3, 5]

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} \,\mathrm{d}x = C \ln\left(\frac{a}{b}\right) , \tag{1.1}$$

for $a > 0$, $b > 0$, and some constant C is known as Frullani's Integral formula [7]. The initial literature assumed strong conditions on $f(x)$ to ensure the validity of (1.1). For example, [3] considers the formula when an integrable function, $f(x)$, meets certain growth conditions, whereas [5] assumes finiteness, integrability, and differentiability of $f(x)$ and more. We refer interested readers to [11] for the history of this formula.

Recent records indicate that Iyengar is the first to investigate conditions on $f(x)$ that would allow (1.1) to hold. Assuming that $f : (0, \infty) \to \mathbb{R}$ is locally Lebesgue integrable in $(0, \infty)$, he [9] proved that the existence of the Frullani integral for all $a, b > 0$ is equivalent to the existence of the following four expressions:

$$\int_{0^+}^1 f(x)\,\mathrm{d}x = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\,\mathrm{d}x , \qquad m_* = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{0^+}^\varepsilon f(x)\,\mathrm{d}x , \tag{1.2}$$

$$\int_1^\infty \frac{f(x)}{x^2}\,\mathrm{d}x = \lim_{\lambda \to \infty} \int_1^\lambda \frac{f(x)}{x^2}\,\mathrm{d}x , \qquad M_* = \lim_{\lambda \to \infty} \lambda \int_\lambda^\infty \frac{f(x)}{x^2}\,\mathrm{d}x . \tag{1.3}$$

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*Corresponding author.

We shall, henceforth, refer to the aforementioned four expressions as Iyengar's integral means. In this case, equation (1.1) holds with

$$C = M_* - m_* \ . \tag{1.4}$$

Iyengar's proof has some problems; a correct proof was later provided by Agnew [1, 2] and then by Ostrowski [10, 11], who showed that the improper Frullani integrals exist for all $a, b > 0$ if and only if the following two mean values exist:

$$m = \lim_{\varepsilon \to 0^+} \varepsilon \int_{\varepsilon}^{1} \frac{f(x)}{x^2} \, \mathrm{d}x \ , \tag{1.5}$$

$$M = \lim_{\lambda \to \infty} \frac{1}{\lambda} \int_{1}^{\lambda} f(x) \, \mathrm{d}x \ , \tag{1.6}$$

in which case, we have

$$m_* = m \quad \text{and} \quad M_* = M \ . \tag{1.7}$$

We would refer to equations (1.5) and (1.6) as Ostrowski's integral means.

In this article, we give a new proof of the equivalence of Iyengar's and Ostrowski's integral means. Our proof applies not only to the Lebesgue integral but also to the more general Denjoy-Perron-Henstock-Kurzweil integral[1].

The outline of this article is as follows: In section 2, we consider a family of differential equations, $M_\alpha(F) = f$, for each $\alpha \in \mathbb{C}$ and $\Re e \ \alpha \neq 0$ and prove how the existence of the limit at zero of one solution ensures the existence of the limit at zero for a subclass of other solutions. In section 3, we prove the same but at infinity, thereby furnishing the proof of the equivalence of Iyengar's and Ostrowski's integral means. Lastly, in section 4, we provide some applications to study the Cesàro behavior of order 1 of sequences of partial sums of a series.

2. Equivalence of means

Consider the differential operator

$$M_\alpha(F) = -\alpha F + xF' = x^{\alpha+1} \left(x^{-\alpha} F(x) \right)' . \tag{2.1}$$

Any Euler operator of order 1 is a multiple of one of the M_α. Suppose f is a locally Denjoy-Perron-Henstock-Kurzweil integrable function in $(0, \infty)$. The equation

$$M_\alpha(F) = f \ , \tag{2.2}$$

has the solution

$$F(x) = x^\alpha \left[\int_{x_0}^{x} \frac{f(t)}{t^{\alpha+1}} \, \mathrm{d}t + C \right] \ , \tag{2.3}$$

where x_0 is fixed and C is an arbitrary constant. The fact that (2.3) is a solution is obvious and may be readily verified. We note that the solutions to this family of differential equations, when they exist, could be interpreted as the various definitions of integral means. We also note that for $\alpha = -1$, we obtain Iyengar's integral mean at zero, and for $\alpha = 1$, we obtain Ostrowski's integral mean at zero. If $\Re e \, \alpha > 0$, then if one solution has a limit γ_α at $x = 0^+$ then all the others do. If $\alpha = -\beta$, $\Re e \, \beta > 0$, then there is at most one constant C for which the limit exists, and for this constant,

$$\lim_{x \to 0^+} \int_{x_0}^{x} t^{\beta-1} f(t) \, \mathrm{d}t = -C \ , \tag{2.4}$$

[1] Details on the Denjoy-Perron-Henstock-Kurzweil integral can be seen in [8].

so that the integral

$$\int_0^{x_0} t^{\beta-1} f(t)\, \mathrm{d}t \quad \text{exists}\,, \tag{2.5}$$

and

$$\lim_{x\to 0^+} \frac{1}{x^\beta} \int_0^x t^{\beta-1} f(t)\, \mathrm{d}t = \gamma_{-\beta} \quad \text{exists}\,. \tag{2.6}$$

We have the following result.

Lemma 2.1. *Let f be a Denjoy-Perron-Henstock-Kurzweil integrable function in $(0,\infty)$. Suppose that the limit*

$$\lim_{x\to 0^+} f(x) = \gamma\,, \tag{2.7}$$

exists. If $\Re e\,\alpha > 0$ then all solutions of $M_\alpha(F) = f$ have a limit as $x \to 0^+$. If $\Re e\,\alpha < 0$ then there is one solution with a limit as $x \to 0^+$. In either case,

$$\lim_{x\to 0^+} F(x) = -\frac{\gamma}{\alpha}\,. \tag{2.8}$$

Proof. Let $\varepsilon > 0$. There exists x_0 such that $|f(x) - \gamma| < \varepsilon$ for $x \le x_0$. When $\Re e\,\alpha > 0$ we have that for $x \le x_0$ and F given by (2.3),

$$F(x) + \frac{\gamma}{\alpha} = x^\alpha \left\{ \int_x^{x_0} \frac{\gamma - f(t)}{t^{\alpha+1}}\, \mathrm{d}t + \widetilde{C} \right\}\,, \tag{2.9}$$

where

$$\widetilde{C} = C + \frac{\gamma}{\alpha} \int_{x_0}^\infty \frac{\mathrm{d}t}{t^{\alpha+1}}\,, \tag{2.10}$$

so that

$$\begin{aligned} \left| F(x) + \frac{\gamma}{\alpha} \right| &\le x^{\Re e\,\alpha} \left\{ \int_x^{x_0} \frac{|\gamma - f(t)|}{t^{\Re e\,\alpha+1}}\, \mathrm{d}t + \left|\widetilde{C}\right| \right\} \\ &\le \frac{\varepsilon}{\Re e\,\alpha} + x^{\Re e\,\alpha} \left|\widetilde{C}\right|\,. \end{aligned}$$

Hence $\limsup_{x\to 0^+} |F(x) + \gamma/\alpha| \le \varepsilon/\Re e\,\alpha$ and consequently $\lim_{x\to 0^+} F(x) = -\gamma/\alpha$.

If $\Re e\,\beta > 0$, $\beta = -\alpha$, then the integral (2.5) exists since f has a limit at 0^+. Also,

$$\begin{aligned} \left| \frac{1}{x^\beta} \int_0^x f(t)\, t^{\beta-1}\, \mathrm{d}t - \frac{\gamma}{\beta} \right| &= \left| \frac{1}{x^\beta} \int_0^x (f(t) - \gamma)\, t^{\beta-1} \mathrm{d}t \right| \\ &\le \frac{1}{x^{\Re e\,\beta}} \int_0^x |f(t) - \gamma|\, t^{\Re e\,\beta-1} \mathrm{d}t \\ &\le \frac{\varepsilon}{\Re e\,\beta}\,, \end{aligned}$$

and it follows that $\lim_{x\to 0^+} F(x) = \gamma/\beta$. □

The result of this lemma does not hold when $\Re e\ \alpha = 0$ since we can find functions f that have a limit at $x = 0^+$ but such that no solution of $M_\alpha(F) = f$ has a limit at $x = 0^+$. Indeed, suppose that for η small, $\eta < e^{-1}$,

$$f(x) = \frac{x^\alpha}{\ln x}\,, \quad x < \eta\,. \tag{2.11}$$

Then $F(x) = -x^\alpha \int_x^\eta f(t)/t^{\alpha+1}\, \mathrm{d}t + Cx^\alpha$ is the sum of $-x^\alpha \int_x^\eta f(t)\, \mathrm{d}t/t^{\alpha+1}$, which satisfies

$$\lim_{x\to 0^+} \left| x^\alpha \int_x^\eta \frac{f(t)}{t^{\alpha+1}}\, \mathrm{d}t \right| = \infty\,, \tag{2.12}$$

since the integral $\int_0^\eta f(t)\ \mathrm{d}t/t^{\alpha+1} = \int_0^\eta \mathrm{d}t/t \ln t$ diverges, and an oscillatory term, Cx^α, that remains bounded and does not have a limit at 0^+.

We can now give our equivalence result.

Theorem 2.2. *If f is locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,\infty)$ and for some value $\Re e\ \alpha_0 > 0$ the limit*

$$\lim_{x\to 0^+} x^\alpha \int_1^x \frac{f(t)}{t^{\alpha+1}}\ \mathrm{d}t = \gamma_\alpha\ , \tag{2.13}$$

exists, then the limit will exist for all α with $\Re e\ \alpha > 0$. Also for each α with $\Re e\ \alpha < 0$ we have that

$$\int_0^1 t^{-\alpha-1} f(t)\ \mathrm{d}t \ \textit{ exists and } \lim_{x\to 0^+} \frac{1}{x^{-\alpha}} \int_0^x t^{-\alpha-1} f(t)\ \mathrm{d}t = \gamma_\alpha \quad \textit{exists}\ . \tag{2.14}$$

On the other hand, if (2.14) holds for some α_0 with $\Re e\ \alpha_0 < 0$, then it will hold for all α with $\Re e\ \alpha < 0$, and (2.13) holds for all α with $\Re e\ \alpha > 0$. In those cases, we have that

$$\frac{\gamma_\alpha}{\alpha} = \frac{\gamma_\beta}{\beta}\ , \tag{2.15}$$

for all α and β that are not purely imaginary; ergo, $\frac{\gamma_\alpha}{\alpha}$ is a constant whenever $\Re e\ \alpha \neq 0$.

Proof. We need to prove that if for some α with $\Im m\,\alpha \neq 0$ there is a solution F of the equation $M_\alpha(F) = f$ with a limit as $x = 0^+$ then the same holds for all ρ with $\Im m\,\rho \neq 0$. But the Lemma 2.1 tell us that we can find a solution G of $M_\rho(G) = F$ with a limit at 0^+. Thus

$$f = M_\alpha(F) = M_\rho(F) + (\rho-\alpha) F = M_\rho(F + (\rho-\alpha) G)\ ,$$

and $F + (\rho-\alpha) G$ has a limit at 0^+. □

In particular, we obtain the equivalence of the Iyengar condition $(\alpha = -1)$ [9] and the Ostrowski condition $(\alpha = 1)$ [10, 11].

Corollary 2.3. *Let f be locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,\infty)$. Then the Ostrowski mean*

$$\lim_{x\to 0^+} x \int_x^1 \frac{f(t)}{t^2}\ \mathrm{d}t = \gamma\ , \tag{2.16}$$

exists if and only if $\int_0^1 f(t)\ \mathrm{d}t$ exists and the Iyengar mean

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon f(x)\ \mathrm{d}x = \gamma_*\ , \tag{2.17}$$

exists. We then have $\gamma = \gamma_*$.

It is convenient to give the following definition at this point.

Definition 2.1. Let f be locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,a)$ for some $a > 0$. We say that f has the limit γ in the average sense[2] as $x \to 0^+$, and write $f(0^+) = \gamma$ (av) or

$$\lim_{x\to 0^+} f(x) = \gamma\,(\mathrm{av})\ , \tag{2.18}$$

if any of the conditions of the Corollary 2.3 are satisfied.

[2]Other names, as for example a Cesàro limit of order 1, or a (C,1) limit, could be employed. We prefer to reserve that term for limits at infinity.

Lemma 2.1 states that if $f(0^+) = \gamma$, then $f(0^+) = \gamma\ (\mathrm{av})$. The converse is of course not true, as functions like $\cos x^{-1}$ or $\sin x^{-1}$ show.

Theorem 2.2 gives other equivalent ways to show that $f(0^+) = \gamma\ (\mathrm{av})$. This theorem also shows that *some* changes of variables are valid for average limits.

Proposition 2.4. *Let f be locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,\infty)$. Suppose that $f(0^+) = \gamma\ (av)$. Let $\alpha > 0$ and put $g(x) = f(x^\alpha)$. Then $g(0^+) = \gamma\ (av)$, that is,*

$$\lim_{x\to 0^+} f(x^\alpha) = \gamma\ (av)\ . \tag{2.19}$$

Proof. In order to prove (2.19) we need to show that

$$\lim_{x\to 0^+} x \int_x^1 \frac{f(t^\alpha)}{t^2}\,\mathrm{d}t = \gamma\ .$$

However, a change of variables and (2.15) yield

$$\begin{aligned}\lim_{x\to 0^+} x \int_x^1 \frac{f(t^\alpha)}{t^2}\,\mathrm{d}t &= \lim_{x\to 0^+} \frac{x}{\alpha} \int_{x^\alpha}^1 \frac{f(u)}{u^{1+1/\alpha}}\,\mathrm{d}t \\ &= \lim_{z\to 0^+} \frac{1}{\alpha} z^{1/\alpha} \int_z^1 \frac{f(u)}{u^{1+1/\alpha}}\,\mathrm{d}t \\ &= \frac{1}{\alpha}\frac{\gamma}{(1/\alpha)} = \gamma\ ,\end{aligned}$$

as required. ☐

Notice that many changes of variable, like $t \to 1/\ln t$, do not preserve average limits. In fact, for $f(x) = \cos x^{-1}$ we have $f(0^+) = 0\ (\mathrm{av})$, but $f(1/\ln t) = \cos \ln t$ does not have an average limit as $x \to 0^+$ [4].

3. Limits at Infinity

The results of Section 2 are also valid at other points, as a change of variables shows. More importantly, the change $u = 1/t$ immediately yields that they also hold at infinity. We have the ensuing equivalence of means at infinity.

Theorem 3.1. *If f is locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,\infty)$ and for some value $\Re e\ \rho_0 < 0$ the limit*

$$\lim_{x\to\infty} x^\rho \int_1^x \frac{f(t)}{t^{\rho+1}}\,\mathrm{d}t = \mu_\rho\ , \tag{3.1}$$

exists, then the limit will exist for all ρ with $\Re e\ \rho < 0$. Also for all ρ with $\Re e\ \rho > 0$ we have that

$$\int_1^\infty \frac{f(t)}{t^{\rho+1}}\,\mathrm{d}t \ \textit{ exists and } \lim_{x\to\infty} x^\rho \int_x^\infty \frac{f(t)}{t^{\rho+1}}\,\mathrm{d}t = -\mu_\rho \quad \textit{exists}\ . \tag{3.2}$$

On the other hand, if (3.2) holds for some ρ_0 with $\Re e\ \rho_0 > 0$, then it will hold for all ρ with $\Re e\ \rho > 0$ and (3.1) holds for all ρ with $\Re e\ \rho < 0$. In those cases, we have that

$$\frac{\mu_\rho}{\rho} = \frac{\mu_\sigma}{\sigma}\ , \tag{3.3}$$

for all ρ and σ that are not purely imaginary; ergo, $\frac{\mu_\rho}{\rho}$ is a constant whenever $\Re e\ \rho \neq 0$.

Proof. It follows from Theorem 2.2 if we use the change $u = 1/t$. ☐

In particular, as with limits at the origin, the Iyengar and Ostrowski means are equivalent.

Corollary 3.2. *Let f be locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,\infty)$. Then the Ostrowski mean*

$$\lim_{x\to\infty}\frac{1}{x}\int_1^x f(t)\ \mathrm{d}t=\mu\ , \tag{3.4}$$

exists if and only if $\int_1^\infty f(t)\ \mathrm{d}t/t^2$ exists and the Iyengar mean

$$\lim_{x\to\infty}x\int_x^\infty\frac{f(t)}{t^2}\ \mathrm{d}t=\mu_*\ , \tag{3.5}$$

also exists. We then have $\mu=\mu_*$.

Thus, we introduce the following terminology.

Definition 3.1. If f is locally Denjoy-Perron-Henstock-Kurzweil integrable in (a,∞) for some $a>0$ and it satisfies the conditions of Corollary 3.2 we say that the limit of $f(x)$ as $x\to\infty$ in the Cesàro sense of order 1 exists and equals μ, and write

$$f(\infty)=\lim_{x\to\infty}f(x)=\mu\quad(\mathrm{C},1)\ . \tag{3.6}$$

We also have that if the ordinary limit $\lim_{x\to\infty}f(x)=\mu$ exists, then $\lim_{x\to\infty}f(x)=\mu\ (\mathrm{C},1)$, but not conversely. Furthermore, as with limits at the origin, we obtain the following.

Proposition 3.3. *Let f be locally Denjoy-Perron-Henstock-Kurzweil integrable in $(0,\infty)$. Suppose that $f(\infty)=\mu\,(C,1)$. Let $\rho>0$ and put $g(x)=f(x^\rho)$. Then $g(\infty)=\mu\,(C,1)$, that is,*

$$\lim_{x\to\infty}f(x^\rho)=\mu\ (C,1)\ . \tag{3.7}$$

4. Applications

We now consider applications of our results. Let $\{a_n\}_{n=1}^\infty$ be a sequence. Let us put

$$f(x)=a_n\ ,\quad n\le x<n+1\ . \tag{4.1}$$

Observe that $\lim_{x\to\infty}f(x)=L$ (C,1) if and only if $\lim_{n\to\infty}a_n=L$ (C,1), that is, if and only if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N a_n=L\ , \tag{4.2}$$

the usual definition of the limit in the Cesàro sense of order 1 of a sequence [6, Chp. 6].

Using the equivalence of the Iyengar, and the Ostrowski means at infinity, we thus obtain the ensuing equivalence.

Theorem 4.1. *If $\{a_n\}_{n=1}^\infty$ is a sequence of real or complex numbers, then*

$$\lim_{n\to\infty}a_n=L\qquad(C,1)\ , \tag{4.3}$$

if and only if the series

$$\sum_{n=1}^\infty\frac{a_n}{n(n+1)}\ , \tag{4.4}$$

converges and the tails of this series have the asymptotic behavior

$$\sum_{n=N}^{\infty} \frac{a_n}{n(n+1)} \sim \frac{L}{N} , \tag{4.5}$$

that is,

$$\lim_{N\to\infty} N \sum_{n=N}^{\infty} \frac{a_n}{n(n+1)} = L . \tag{4.6}$$

We can obtain a more general equivalence if we start with a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of different positive numbers that increases to infinity and then for a sequence $\{a_n\}_{n=1}^{\infty}$ put

$$f(x) = \frac{a_n}{\lambda_{n+1} - \lambda_n} , \quad \lambda_n \le x < \lambda_{n+1} . \tag{4.7}$$

We have that if $\lambda_N \le \lambda < \lambda_{N+1}$ then

$$\int_0^{\lambda} f(t)\ \mathrm{d}t = \sum_{n=1}^{N-1} a_n + a_N \frac{\lambda - \lambda_N}{\lambda_{N+1} - \lambda_N} . \tag{4.8}$$

Suppose now that $\lambda_n/\lambda_{n+1} \to 1$ as $n \to \infty$, which is satisfied, for instance, if $\lambda_n = n^{\alpha}$ for some $\alpha > 0$. If $a_n = o(\lambda_n)$ then

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \int_0^{\lambda} f(t)\ \mathrm{d}t = L , \tag{4.9}$$

if and only if

$$\lim_{N\to\infty} \frac{1}{\lambda_N} \sum_{n=1}^{N} a_n = L . \tag{4.10}$$

The equivalence of Iyengar and Ostrowski means yields that (4.10) is equivalent to the convergence of the series

$$\sum_{n=1}^{\infty} \frac{a_n}{\lambda_n \lambda_{n+1}} , \tag{4.11}$$

and the following asymptotic behavior of the tails,

$$\lim_{N\to\infty} \lambda_N \sum_{n=N+1}^{\infty} \frac{a_n}{\lambda_n \lambda_{n+1}} = L . \tag{4.12}$$

5. Conclusion

Ostrowski [11] proved the equivalence of Iyengar's Means (1.2), (1.3) and Ostrowski's Means (1.5), (1.6). However, that proof is long and limited to the framework of Lebesgue integration. The authors in this paper provide a new proof using the theory of differential operators of first order. This proof is arguably shorter, provides a range of other equivalent conditions, and is applicable in the more general framework of the Denjoy-Perron-Henstock-Kurzweil integral. The authors note that each of these equivalent conditions may be recognized as a type of integral mean, two of which are the aforementioned Iyengar's and Ostrowski's means.

The authors expect this paper to inspire more results involving some higher-order differential operators related to 2.2.

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REFERENCES

[1] R. P. Agnew, Limits of integrals, *Duke Math. J.***9** (1942), 10–19.

[2] R. P. Agnew, Mean values and Frullani integrals, *Proc. Amer. Math. Soc.* **22** (1951), 237–241.

[3] T.J.I.A. Bromwich, *An Introduction to the Theory of Infinite Series,* AMS Chelsea Publishing Series, Amer. Math. Soc., Providence, 2005.

[4] J. Campos Ferreira, *Introduction to the Theory of Distributions,* Longman, Essex, 1997.

[5] J. Edwards, *A Treatise on the Integral Calculus: With Applications, Examples and Problems, vol. 2* Chelsea, London, 1922.

[6] R. Estrada and R. P. Kanwal, *A Distributional Approach to Asymptotics. Theory and Applications,* Second edition, Birkhäuser, Boston, 2002.

[7] G. Frullani, Sopra Gli Integrali Definiti, *Memorie della Societa Italiana delle Socienze* **20** (1828), 448-467.

[8] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock,* Amer. Math. Soc., Providence, 1994.

[9] K.S.K. Iyengar, On Frullani's integrals, *J. Indian Math. Soc.* **4** (1940), 145-150, reprinted as On Frullani's integrals, *Proc. Cambridge Phil. Soc.* **37** (1941), 9-13.

[10] A. M. Ostrowski, On some generalizations of the Cauchy-Frullani integral, *Proc. Nat. Ac. Scs. U. S. A.* **35** (1949), 612–616.

[11] A. M. Ostrowski, On Cauchy-Frullani integrals,*Comm. Math. Helv.* **51** (1976), 57–91.

JITEN AHUJA

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA, 70803, USA

Email address: jahuja1@lsu.edu

RICARDO ESTRADA

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA, 70803, USA

Email address: restrada@math.lsu.edu