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FUZZY MULTIPLICATION Γ-SEMIGROUPS

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ABSTRACT. The concept of fuzzy right multiplication Γ-semigroup is introduced by means of fuzzy right Γ-ideals, and some properties are investigated with the help of some classes of Γ-semigroups. It is shown that for any fuzzy right Γ-ideal ϑ of X, $\vartheta \subseteq X \circ_{\Gamma} \vartheta$. Moreover, if a Γ-semigroup X is simple with $a \in a\Gamma X$ for every $a \in X$, then X is a fuzzy right multiplication Γ-semigroup.

1. INTRODUCTION

Nobusawa [11] initiated a new type of algebraic system known as Γ-ring. Γ-ring is an algebraic extension of ring theory. This has given rise to a large body of new results by several researchers in literature as seen in [2, 3, 7, 8]. Motivated by this generalisation of a ring, a similar research line of thought began in the field of semigroups, and the theory of Γ-semigroups was initiated by Sen [14]. Later, Sen collaborated with Saha in [15] to slightly weaken the defining conditions and redefine Γ-semigroup. The development of Γ-semigroups hinges on the fact that subsets of a semigroup naturally inherit associativity but are not necessarily closed. As a result of this, various generalisations and similarities of corresponding results in semigroup theory have been obtained based on the modified definition (see [5, 12, 16, 17, 18]).

Continuing to broaden the theoretical aspect of Γ-semigroup theory, Awolola and Ibrahim [1] were inspired by the research of [9], and the concept of multiplication Γ-semigoups in the non-commutative case was introduced. A non-commutative Γ-semigroup X is said to be a right multiplication Γ-semigroup if there exist any two right Γ-ideals L, M of X with $L \subseteq M$; there is a right Γ-ideal N of X such that $L = N\Gamma M$. Dually, left multiplication Γ-semigroups can be defined. In [1], some properties of right multiplication Γ-semigroup were investigated with the help of some classes of Γ-semigroups.

The algebraic extension of Γ-semigroups using fuzzy ideals was first introduced by Mustafa *et al*. [10]. This idea was developed further by Dutta *et al.* [4], Sardar and Majumder [13], Shashikumar and Reddy [19] and Williams *et al.* [21]. Related to this concept, Subha and Dhanalakshmi [20] introduced the notions of interval rough fuzzy Γ-ideals, bi-Γ-ideals, prime-Γ-ideals, and prime-bi-Γ-ideals in Γ-semigroups and established some properties of these structures.

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In this class of algebraic structures, we feel it is felicitous to initiate the study of fuzzy multiplication Γ-semigroups induced by multiplication Γ-semigroups. The motivation behind this study is the enhancement of Γ-ideals and their applications in multiplication Γ-semigroups.

The present paper introduces the concept of fuzzy multiplication Γ-semigroup and the validity of some results analogous to those obtained by Awolola and Ibrahim [1] in the case of multiplication Γ-semigroups is examined via fuzzy subsets.

2. Preliminaries

We state some definitions and results that will be useful for this paper in the next section.

Definition 2.1. [15] Let X and Γ be two non-empty sets. Then X is said to be a Γ-semigroup if there is a mapping $X \times \Gamma \times X \longrightarrow X \mid (x, \gamma, y) \longrightarrow x\gamma y \in X$ which satisfies the property $(x\gamma y)\delta z = x\gamma(y\delta z)\ \forall\ x, y, z \in X$ and $\gamma, \delta \in \Gamma$.

The following examples show that Γ-semigroups generalise semigroups.

Example 2.2. Let X be a semigroup and $\Gamma = \{\delta\}$. Define a mapping $X \times \Gamma \times X \longrightarrow X$ by $x\delta y = xy\ \forall\ x, y \in X$ and $\delta \in \Gamma$. Then X is a Γ-semigroup.

Example 2.3. Let $X = \{-i, i, 0\}$ and $\Gamma = X$. Then X is a Γ-semigroup with respect to multiplication of complex numbers, whereas X does not reduce to a semigroup with respect to multiplication of complex numbers.

A Γ-semigroup X is said to be regular (right regular) if for every $x \in X$, there exist $y \in X$ and $\gamma, \delta \in \Gamma$ such that $x = x\gamma y\delta x$ ($x = x\gamma x\delta y$). An element x of a Γ-semigroup X is called a δ-idempotent if there exists $\delta \in \Gamma$ such that $x\delta x = x$.

A Γ-semigroup X is said to be left (right) cancellative if $a\gamma x = a\gamma y \Longrightarrow x = y$ ($x\delta b = y\delta b \Longrightarrow x = y$) $\forall\ a, b, x, y \in X$ and $\gamma \in \Gamma$. It is cancellative if X is both left and right Γ-cancellative. A Γ-semigroup X is called left (right) simple if it has no proper left (right) Γ-ideal, i.e., for each $a \in X,\ X\Gamma a = X$ ($a\Gamma X = X$).

Definition 2.4. [6] Let ϑ and ι be any two fuzzy subsets of X. We define the product of ϑ and ι as follows: for every $a \in X$, if there exist $x, y \in X$ and $\delta \in \Gamma$ such that $a = x\delta y$, then

$$\vartheta \circ_\Gamma \iota = \max_{a=x\delta y} \min\{\vartheta(x), \iota(y)\}$$

and otherwise, $\vartheta \circ_\Gamma \iota = 0$

Definition 2.5. [6] A fuzzy subset ϑ of a Γ-semigroup X is called a fuzzy left Γ-ideal and a fuzzy right Γ-ideal of X if $\vartheta(x\delta y) \geq \vartheta(y)$ and $\vartheta(x\delta y) \geq \vartheta(x)\ \forall\ x, y \in X$ and $\delta \in \Gamma$ respectively. The fuzzy subset ϑ is called a fuzzy Γ-ideal of X if $\vartheta(x\delta y) \geq \max\{\vartheta(x), \vartheta(y)\}\ \forall\ x, y \in X$ and $\delta \in \Gamma$.

The following results can be found in [1].

Proposition 2.1. *Let X be a right multiplication Γ-semigroup. Then $X\Gamma X = X$.*

Proposition 2.2. *Let X be a right multiplication Γ-semigroup. Then $A \subseteq X\Gamma A$ for any right Γ-ideal.*

Proposition 2.3. *Let X be a Γ-semigroup. If $a \in (a\Gamma X)\Gamma(a\Gamma X)$ for every $a \in X$, then X is a right multiplication Γ-semigroup.*

Proposition 2.4. *Let X be a Γ-semigroup. If X is regular (right regular), then X is a right multiplication Γ-semigroup.*

Proposition 2.5. *Let X be a Γ-semigroup. If X is simple with $a \in a\Gamma X$ for every $a \in X$, then X is a right multiplication Γ-semigroup.*

Proposition 2.6. *Let X be a right multiplication Γ-semigroup. If every left Γ-ideal is Γ-ideal, then X is left regular.*

Proposition 2.7. *Let X be a right multiplication Γ-semigroup. If X contains a left canellative element, then X contains a β-idempotent which is a left identity.*

3. Fuzzy multiplication Γ-semigroups

Definition 3.1. Let X be a Γ-semigroup. By a fuzzy right multiplication Γ-semigroup we mean if exist any two fuzzy right Γ-ideals ϑ and ι of X with $\vartheta \subseteq \iota$, there is a fuzzy right Γ-ideal η of X such that $\vartheta = \eta \circ_\Gamma \iota$. Dually, left fuzzy multiplication Γ-semigroup can be defined.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a semigroup under the operation given by the table below:

.	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	2	2	0
3	3	3	3	3

Let $\Gamma = \{\alpha\}$. Define $a\delta b = ab$. Clearly, X is a Γ-semigroup. Moreover, the fuzzy subsets ϑ, ι and η of X defined as follows:

$$\vartheta(0) = 0.6,\ \vartheta(1) = \vartheta(2) = \vartheta(3) = 0$$
$$\iota(0) = 0.6,\ \iota(1) = \iota(2) = 0,\ \iota(3) = 0.4$$
$$\eta(1) = 0.6,\ \eta(2) = 0.3,\ \eta(1) = \eta(3) = 0$$

are all fuzzy right Γ-ideals of the Γ-semigroup X. It is not difficult to verify that X is a fuzzy right multiplication Γ-semigroup.

Remark. *Let $X \times Y$ be the direct product of two Γ-semigroups X and Y . The fuzzy right Γ-ideals ϑ, ι of $X \times Y$ such that $\vartheta \subseteq \iota$ need not be a fuzzy right multiplication Γ-semigroup for some fuzzy right Γ-ideal η of $X \times Y$, where*

$$X \times Y = (a_1, b_1)\delta(a_2, b_2) = (a_1\delta a_2, b_1\delta b_2) \text{ for all } a_1, a_2 \in X,\ b_1, b_2 \in Y \text{ and } \delta \in \Gamma.$$

Example 3.3. Let $X = \{0, 1\}$ and $Y = \{0, 1, 2, 3\}$ be Γ-semigroups with $\Gamma = \{\alpha\}$ as shown in the tables below:

α	0	1
0	0	0
1	1	1

α	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	2	2	0
3	3	3	3	3

Clearly, X and Y are right multiplication Γ-semigroups. Now,

$$X \times Y = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)\}$$

and we define the fuzzy subsets ϑ, ι and η of $X \times Y$ as follows:

$$\vartheta(0,0) = 0.8, \;\; \vartheta(0,1) = \vartheta(0,2) = \vartheta(0,3) = \vartheta(1,0) = \vartheta(1,1) = \vartheta(1,2) = \vartheta(1,3) = 0$$
$$\iota(0,0) = 0.8, \;\; \iota(0,2) = 0.5, \;\; \iota(0,1) = \iota(0,3) = \iota(1,0) = \iota(1,1) = \iota(1,2) = \iota(1,3) = 0$$
$$\eta(0,0) = 0.8, \;\; \eta(0,3) = 0.2, \;\; \eta(0,1) = \eta(0,2) = \eta(1,0) = \eta(1,1) = \eta(1,2) = \eta(1,3) = 0.$$

By the Definition 2.5, it is obvious that ϑ, ι, and η are fuzzy right Γ-ideals of $X \times Y$. From Definition 2.4, let

$$\eta \circ_\Gamma \iota(a,b) = \max_{(a,b)=(x_1,y_1)\delta(x_2,y_2)} \min\{\eta(x_1,y_1), \iota(x_2,y_2)\}$$

and by routine calculation process, we have

$$\eta \circ_\Gamma \iota(0,0) = 0.8 \qquad \eta \circ_\Gamma \iota(0,1) = 0$$
$$\eta \circ_\Gamma \iota(0,2) = 0 \qquad \eta \circ_\Gamma \iota(0,3) = 0.2$$
$$\eta \circ_\Gamma \iota(1,0) = 0 \qquad \eta \circ_\Gamma \iota(1,1) = 0$$
$$\eta \circ_\Gamma \iota(1,2) = 0 \qquad \eta \circ_\Gamma \iota(1,3) = 0$$

Since $\eta \circ_\Gamma \iota(0,3) = 0.2 \neq \vartheta(0,3)$, this means that $\vartheta \neq \eta \circ_\Gamma \iota$, and thus $X \times Y$ is not a fuzzy right multiplication Γ-semigroup.

Definition 3.4. A fuzzy Γ-ideal is called idempotent if $\vartheta = \vartheta \circ_\Gamma \vartheta$.

Definition 3.5. A fuzzy Γ-ideal η of X is said to be maximal if it has no proper fuzzy Γ-ideal ϑ of X such that $\eta \subset \vartheta$.

Proposition 3.1. *In a fuzzy right multiplication Γ-semigroup X, the following assertions hold.*

(i) $\vartheta = \vartheta \circ_\Gamma \vartheta$.
(ii) For any fuzzy right Γ-ideal ϑ of X, $\vartheta \subseteq X \circ_\Gamma \vartheta$.
(iii) If ϑ is a maximal fuzzy right Γ-ideal containing every proper fuzzy right Γ-ideal η of X, then $\vartheta = \vartheta \circ_\Gamma \vartheta$.

Proof. (i) Let ϑ be a fuzzy right Γ-ideal of X. Since $\vartheta \subseteq \vartheta$, there exist a fuzzy right Γ-ideal η of X such that $\vartheta = \eta \circ_\Gamma \vartheta$ and so $\vartheta = \vartheta \circ_\Gamma \vartheta$.
(ii) Let ϑ be any fuzzy right Γ-ideal of X. Since $\vartheta \subseteq \vartheta$, there exist a fuzzy right Γ-ideal η of X such that $\vartheta = \eta \circ_\Gamma \vartheta$ and thus $\vartheta \subseteq X \circ_\Gamma \vartheta$.
(iii) Let ϑ be a maximal right Γ-ideal of X. Since $\vartheta \subseteq \vartheta$, we have $\vartheta = \eta \circ_\Gamma \vartheta$ for some fuzzy right Γ-ideal η of X. If $\eta \subseteq \vartheta$, then $\eta \circ_\Gamma \vartheta \subseteq \vartheta \circ_\Gamma \vartheta$. Hence, $\vartheta = \vartheta \circ_\Gamma \vartheta$. □

Theorem 3.2. *Let X be a Γ-semigroup. If every fuzzy right Γ-ideal ϑ of X is idempotent, then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Let ϑ and ι be two fuzzy right Γ-ideals of X such that $\vartheta \subseteq \iota$. For every fuzzy right Γ-right ϑ of X, we have $\vartheta \subseteq \vartheta \circ_\Gamma \vartheta \subseteq \vartheta \circ_\Gamma \iota \subseteq \vartheta$. Thus, $\vartheta \subseteq \vartheta \circ_\Gamma \iota \subseteq \vartheta$ and so $\vartheta = \vartheta \circ_\Gamma \iota$. This proves that X is a fuzzy right multiplication Γ-semigroup. □

Theorem 3.3. *Let X be a Γ-semigroup. If X is right regular, then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Let X be right regular. Let ϑ and ι be any two fuzzy right Γ-ideals of X such that $\vartheta \subseteq \iota$. Then $\vartheta \circ_\Gamma \iota \subseteq \vartheta$. Next, let $a \in X$. Then there exists $x \in X$ and $\gamma, \delta \in \Gamma$ such that

$a = a\gamma a\delta x$. Thus

$$\begin{aligned} \vartheta \circ_\Gamma \iota(a) &= \max_{a=a\gamma a\delta x}\min\{\vartheta(a), \iota(a\delta x)\} \\ &\geq \min\{\vartheta(a), \iota(a\delta x)\} \\ &\geq \min\{\vartheta(a), \iota(a)\} \ \ (\because\ \iota(a\delta x) \geq \iota(a)) \\ &= \vartheta(a) \end{aligned}$$

Hence, $\vartheta \subseteq \vartheta \circ_\Gamma \iota$ and thus $\vartheta \circ_\Gamma \iota = \vartheta$. Therefore, X is a fuzzy right multiplication Γ-semigroup. □

Analogously, we have the following theorem and corollary.

Theorem 3.4. *If X is regular, then X is a fuzzy right multiplication Γ-semigroup.*

Corollary 3.5. *If every γ-idempotent element of X is regular, then X is a fuzzy right multiplication Γ-semigroup.*

Theorem 3.6. *Let X be a left cancellative Γ-semigroup. If X is left simple and regular (right regular), then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Let ϑ and ι be any two fuzzy right Γ-ideals of X such that $\vartheta \subseteq \iota$. Then $\vartheta \circ_\Gamma \iota \subseteq \vartheta$. Let $a \in X$. Since X is left simple, $x = x\gamma a$ for some $x \in X$ and $\gamma \in \Gamma$. Again, $x\gamma a = x\gamma a\gamma x\delta a$. Therefore, by left cancellation of X we have $a = a\gamma x\delta a$. Now, the completion of this proof follows from Theorem 3.3. Similar argument holds if X is left simple and right regular. □

By Theorem 3.6, we have the subsequent theorem.

Theorem 3.7. *Let X be a right cancellative Γ-semigroup. If X is right simple and regular (right regular), then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Same as the proof of Theorem 3.6. □

Theorem 3.8. *Let X be a left cancellative Γ-semigroup. If X is left simple and idempotent, then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Let ϑ and ι be any two fuzzy right Γ-ideals of X such that $\vartheta \subseteq \iota$. Then $\vartheta \circ_\Gamma \iota \subseteq \vartheta$. Let $a \in X$. Since X is left simple, we have $x = x\gamma a$ for some $x \in X$ and $\gamma \in \Gamma$. Clearly, $x\gamma a = x\gamma a\gamma a$ and thus $a = a\gamma a$ since X is left cancellative. Therefore, $\vartheta(a) = \min\{\vartheta(a), \iota(a)\} \leq \max_{a=a\gamma a}\min\{\vartheta(a), \iota(a)\} = \vartheta \circ_\Gamma \iota(a)$. Thus, $\vartheta = \vartheta \circ_\Gamma \iota$ and so X is a fuzzy multiplication Γ-semigroup. □

Theorem 3.9. *Let X be a right cancellative Γ-semigroup. If X is right simple and idempotent, then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Same as the proof of Theorem 3.8. □

The next theorem gives a condition for a simple Γ-semigroup X to be a fuzzy right multiplication Γ-semigroup.

Theorem 3.10. *Let X be a Γ-semigroup. If X is simple with $a \in a\Gamma X$ for every $a \in X$, then X is a fuzzy right multiplication Γ-semigroup.*

Proof. Let ϑ and ι be any two fuzzy right Γ-ideals of X such that $\vartheta \subseteq \iota$. Then $\vartheta \circ_\Gamma \iota \subseteq \vartheta$. Let Let X be a simple Γ-semigroup with $a \in a\Gamma X$. Then $a = a\delta x$ for some $x \in X$ and $\delta \in \Gamma$. Clearly, $x = x\gamma a\delta x$ for some $\gamma \in \Gamma$. Hence, $a = a\delta x\gamma a\delta x$. Now,

$$\begin{aligned} \vartheta \circ_\Gamma \iota(a) &= \max_{a=a\delta x\gamma a\delta x}\min\{\vartheta(a\delta x), \iota(a\delta x)\} \\ &\geq \min\{\vartheta(a\delta x), \iota(a\delta x)\} \\ &\geq \min\{\vartheta(a), \iota(a)\} \ \ (\because\ \vartheta(a\delta x) \geq \vartheta(a) \text{ and } \iota(a\delta x) \geq \iota(a)) \\ &= \vartheta(a) \end{aligned}$$

Thus, $\vartheta \subseteq \vartheta \circ_\Gamma \iota$ and hence $\vartheta \circ_\Gamma \iota = \vartheta$. Therefore, X is a fuzzy right multiplication Γ-semigroup. □

Remark. *Similar characterisation can be proved for fuzzy left multiplication Γ-semigroups.*

4. Conclusion

We have proven in this paper that in a fuzzy right multiplication Γ-semigroup, some fuzzy right Γ-ideal theoretic properties are true. Some classes of Γ-semigroup such as regular Γ-semigroups, right regular Γ-semigroups, simple Γ-semigroups, cancellative Γ-semigroups, and idempotent Γ-semigroups, form subclasses of fuzzy right multiplication Γ-semigroups. For future research, one can study the condition for a semisimple Γ-semigroup to be a fuzzy right multiplication Γ-semigroup. Also, the sufficient condition for a fuzzy right multiplication Γ-semigroup to be a regular Γ-semigroup.

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