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A NOTE ON RELATIVE TRI-QUASI-Γ-HYPERIDEALS OF Γ-SEMIHYPERRING

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ABSTRACT. In this paper, we introduce the concept of tri-quasi hyperideal in Γ-semihyperring generalizing the classical ideal, left ideal, right ideal, bi-ideal, quasi ideal, interior ideal, bi-interior ideal, weak interior ideal, bi-quasi ideal, tri-ideal, quasi-interior ideal and bi-quasi-interior ideal of Γ-semihyperring and semiring. Furthermore, charecterizations of Γ-semihyperring, regular Γ-semihyperring and simple Γ-semihyperring with relative tri-quasi hyperideals are provided discussing the characteristics of Γ-semihyperring of relative tri-quasi hyperideals.

1. INTRODUCTION

Nobusawa [17] introduced the concept of Γ-rings as a generalization of ternary rings. Barens [6] weakened the conditions in Nobusawa's definition of Γ-ring. Satyanarayana [24] introduced the notion of Gamma near-ring, and thereafter several authors studied this algebraic structure. Also, Satyanarayana, Abbasi, Basar and Kuncham [18] introdced abstract affine Gamma-nearrings. An abstract affine gamma near-ring is an algebraic system that generalizes both the gamma ring and near-ring. Sen [23] introduced Γ-semigroup. The notion of semiring, that is, a universal algebra with two associative binary operations, where one of them distributes over the other was introduced by Vandiver [30]. The concept of Γ-semiring was introduced by Rao [20], generalizing both Γ-rings and semirings. Also, Rao [21] studied Γ-semiring with identity. Krishnamoorthy and Doss [9] introduced the notion of regular Γ-semiring.

The concept of hyperstructures was introduced by Marty [16] when he defined hypergroups using the notion of hyperoperation at the 8th Congress of Scandinavian mathematicians. But due to his untimely demise, Marty only published two papers related to his concept of hypergroups [11]. Corsini gave many applications of hyperstructures in several branches of pure and applied sciences [7], [8].

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The notion of bi-ideals in semigroups was given by Good and Hughes in [12]. The theory of bi-ideals in rings and semigroups were introduced by Lajos et al [25]. The concept of interior ideals was introduced in semigroup by Lajos [15]. Steinfeld [26], [27] introduced the notion of quasi ideals in semigroups and then for rings.

Iseki [14] investigated ideals for semirings. Shabir, Ali and Batool [5] characterized semirings by the properties of their quasi-ideals. Munir and Mustafa [13] characterized regular semirings, intra-regular semirings and weakly regular semirings by their quasi and bi-ideals along with their right and left-ideals. Rao [19] introduced the notion of tri-quasi ideals, studied various types of ideals and also characterised Γ-semiring through tri-quasi ideals. Davvaz et. al.[10] defined the notion of Γ-semihyperring as a generalization of semiring, semihyperring and Γ-semiring. Kumbharde, Pawar and Ansari[3] introduced bi-interior, quasi-interior and bi-quasi-interior Γ-hyperideal in Γ-semihyperring.

Wallace [31], [32] introduced relative ideals in semigroups. Khan and Firoj [2] introduced relative ideal in ordered semigroups. Basar et al [4], [22], [28], [29] studied relative results on ideals in different algebraic structures. Recently, Rao et al [1]studied tri-quasi-ideals and fuzzy tri-quasi-ideals in semigroups. Ideals play a fundamental role in ring theory and it is thus natural to consider them also in the context of Γ-semihyperring. Needless to say that their role is no less crucial for its worthy consideration to special generalised types of hyperideals. The aim of the present paper is to introduce the concept of relative tri-quasi hyperideal in Γ-semihyperring as a generalization of tri-quasi ideal, bi-ideal, quasi ideal, interior ideal, bi-interior ideal, tri-ideal, bi-quasi-interior ideal and bi-quasi ideal of semiring and study some of the characterising properties of relative tri-quasi hyperideals in the algebraic home called Γ-semihyperrings.

2. Preliminaries

In this section, we recollect basic concepts and also define fundamental notions that are necessary in the study of this paper.

Definition 2.1. [10] Let R be a commutative semihypergroup and Γ be a commutative group. Then R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \to \mathcal{P}^*(R)$ (images to be denoted by $a\alpha b$ for all $a, b \in R, \alpha \in \Gamma$) satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$
(3) $a(\alpha + \beta)c = a\alpha c + a\beta c$
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$; for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

A Γ-semihyperring R is said to be commutative if $a\gamma b = b\gamma a$ for all $a, b \in R$ and $\gamma \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring S and $x \in S$. Then

$$A + B = \{x \mid x \in a + b, a \in A, b \in B\} \text{ and}$$
$$A\Gamma B = \{x \mid x \in a\gamma b, a \in A, b \in B, \gamma \in \Gamma\}.$$

Definition 2.2. [10] A non-empty subset R_1 of a Γ-semihyperring R is called a Γ-subsemihyperring if it is closed with respect to the multiplication and addition that is, $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Definition 2.3. Let S be a Γ-semihyperring. An element $1 \in S$ is called unity if for each $x \in S$, there exists $\gamma \in \Gamma$ such that $x\gamma 1 = 1\gamma x = x$.

Definition 2.4. Let S be a Γ-semihyperring. An element $1 \in S$ is called unity if for each $s \in S$ there exists $\alpha \in \Gamma$ such that $s\alpha 1 = 1\alpha s = s$.

Definition 2.5. Suppose that S is a Γ-semihyperring. An element $a \in S$ is said to be invertible if there exist $b \in S, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.6. An element a in a non-empty subset A of Γ-semihyperring S is called relative idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

Definition 2.7. Let S be a Γ-semihyperring. An element $a \in A \subseteq S$ is called relative regular element of S if there exist $x \in A, \alpha, \beta \in \Gamma$ for $a = a\alpha x\beta a$.

Definition 2.8. Suppose that S is a Γ-semihyperring. If each element of S is a relative regular element of S, then S is called relative regular Γ-semihyperring.

Definition 2.9. Suppose that S is a Γ-semihyperring. If each element of S is a relative idempotent, then S is called relative idempotent Γ-semihyperring.

Definition 2.10. A Γ-semihyperring S is called a division Γ-semihyperring if each non-zero element of S has multiplicative inverse.

Definition 2.11. A non-empty subset I of a Γ-semihyperring S for $A \subseteq S$, is called

(1) A relative bi-interior hyperideal of S if I is a Γ-semihyperring of S and $A\Gamma I\Gamma A \cap I\Gamma A\Gamma I \subseteq I$.
(2) A left (right) relative bi-quasi hyperideal of S if I is a sub-Γ-semihypergroup of $(S, +)$ and $A\Gamma I \cap I\Gamma A\Gamma I \subseteq I(I\Gamma A \cap I\Gamma A\Gamma I \subseteq I)$.
(3) A left (right) relative weak-interior hyperideal of S if I is a Γ-subsemihyperring of S and $A\Gamma I\Gamma I \subseteq I(I\Gamma I\Gamma A \subseteq I)$.
(4) A left (right) relative quasi-interior hyperideal of S if I is a Γ-subsemihyperring of S and $A\Gamma I\Gamma A\Gamma I \subseteq I(I\Gamma A\Gamma I\Gamma A \subseteq I)$.
(5) A left (right) relative tri- hyperideal of S if I is a Γ-subsemihyperring of S and $I\Gamma A\Gamma I\Gamma I \subseteq I(I\Gamma I\Gamma A\Gamma I \subseteq I)$.
(6) A relative bi-quasi-interior hyperideal of S if I is a Γ-subsemihyperring of S and $I\Gamma A\Gamma I\Gamma A\Gamma I \subseteq I$.

3. Relative tri-quasi Γ-hyperideals of Γ-semihyperrings

In this section, we introduce the concept of relative tri-quasi Γ-hyperideal as a generalization of relative bi-Γ-hyperideal, relative quasi-Γ-hyperideal and relative interior Γ-hyperideal of Γ-semihyperring. We also study the properties of relative tri-quasi Γ-hyperideal of Γ-semihyperring.

Definition 3.1. A non-empty subset B of a Γ-semihyperring S is said to be a relative tri-quasi Γ-hyperideal of S if B is a Γ-subsemihyperring of S and $B\Gamma B\Gamma A\Gamma B\Gamma B \subseteq B$ for $A \subseteq S$.

Remark. *Each relative tri-quasi Γ-hyperideal of a Γ-semihyperring S need not be relative bi-Γ-hyperideal, relative quasi-Γ-hyperideal, relative interior Γ-hyperideal, relative bi-interior Γ-hyperideal and relative bi-quasi Γ-hyperideals of S.*

In the following theorem, we gather some important implications and skip its proofs since it is easy to prove.

Theorem 3.1. *Suppose that S is a Γ-semihyperring. Then the following stands true.*

(1) *Each relative left hyperideal is a relative tri-quasi hyperideal of S.*

(2) *Every relative right hyperideal is a relative tri-quasi hyperideal of S.*
(3) *Every relative quasi hyperideal is a relative tri-quasi hyperideal of S.*
(4) *Every relative hyperideal is a relative tri-quasi hyperideal of S.*
(5) *Intersection of a relative right hyperideal and a relative left hyperideal of S is a relative tri-quasi hyperideal of S.*
(6) *If L is a relative left hyperideal and R is a right hyperideal of S then $B = R\Gamma L$ is a relative tri-quasi hyperideal of S.*
(7) *Every relative bi-hyperideal of S is a relative tri-quasi hyperideal of S.*
(8) *Every relative interior hyperideal of S is a relative tri-quasi hyperideal of S.*
(9) *Let B be relative bi-hyperideal of S and I be relative interior hyperideal of S. Then $B \cap I$ is a relative tri-quasi hyperideal of S.*
(10) *If B is a relative bi-interior hyperideal of S, then B is a relative tri-quasi hyperideal of S.*
(11) *If B is a relative left bi-quasi hyperideal of M, then B is a relative tri-quasi hyperideal of S.*
(12) *If B is a relative right bi-quasi hyperideal of S, then B is a relative tri-quasi hyperideal of S.*
(13) *If B is a relative bi-quasi hyperideal of M, then B is a relative tri-quasi hyperideal of S.*
(14) *Let A and C be Γ-subsemihyperrings of S and $B = A\Gamma C$. If A is the relative left hyperideal, then B is a relative tri-quasi-interior hyperideal of S.*

Theorem 3.2. *The intersection of relative tri-quasi hyperideals $\{Q_\lambda | \lambda \in A,\ A\, is\, index\, set\}$ of a Γ-semihyperring S is a relative tri-quasi hyperideal of S.*

Proof. Let $Q = \bigcap_{\lambda \in A} Q_\lambda$ and $D \subseteq S$. Then Q is a Γ-subsemihyperring of S. Since Q_λ is a relative tri-quasi hyperideal of S, we receive the following:

$$Q_\lambda \Gamma Q_\lambda \Gamma D \Gamma Q_\lambda \Gamma Q_\lambda \subseteq Q_\lambda,\ for\ all\ \lambda \in A$$

This shows that $Q\Gamma Q\Gamma D\Gamma Q\Gamma Q \subseteq Q$. Hence Q is a relative tri-quasi hyperideal of S. □

Theorem 3.3. *Let S be a Γ-semihyperring. Then, Q is a relative tri-quasi ideal of S and $Q\Gamma Q = Q$ if and only if there exist a relative left hyperideal L and a relative right hyperideal R such that $R\Gamma L \subseteq Q \subseteq R \cap L$.*

Proof. Suppose that Q is a relative tri-quasi hyperideal of the Γ-semiring S and $A \subseteq S$. Then, we have $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$. Let $R = Q\Gamma A$ and $L = A\Gamma Q$. Then L and R are relative left and relative right hyperideals of S, respectively.Thus $R\Gamma L \subseteq Q \subseteq R \cap L$.

Conversely, suppose that there exist L and R as relative left and relative right hyperideals of S, respectively such that $R\Gamma L \subseteq Q \subseteq R \cap L$. Then, $Q\Gamma Q\Gamma A\Gamma B\Gamma B \subseteq (R \cap L)\Gamma(R \cap L)\Gamma A\Gamma(R \cap L)\Gamma(R \cap L) \subseteq (R)\Gamma R\Gamma A\Gamma L\Gamma(L) \subseteq R\Gamma L \subseteq Q$. Hence Q is a relative tri-quasi hyperideal of Γ-semihyperring. □

Theorem 3.4. *Let S be a Γ-semihyperring.Then Q is a relative tri-quasi hyperideal of a Γ-semihyperring if and only if Q is a relative left hyperideal of some relative right hyperideal of Γ-semihyperring .*

Proof. Let Q be a relative tri-quasi hyperideal of Γ-semihyperring S and $A \subseteq S$. Then $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$. Thus $Q\Gamma Q$ is a relative left hyperideal of relative right hyperideal

$Q\Gamma Q\Gamma A$ of Γ-semihyperring.

Conversely, suppose that Q is a relative left hyperideal of some relative right hyperideal R of Γ-semihyperring S. Then $R\Gamma Q \subseteq Q, R\Gamma A \subseteq R$. Therefore $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q\Gamma A\Gamma Q \subseteq R\Gamma A\Gamma Q \subseteq R\Gamma Q \subseteq Q$. Hence Q is a relative tri-quasi hyperideal of Γ-semihyperring. □

Corollary 3.5. *Suppose that S is Γ-semihyperring. Then Q is a relative tri-quasi hyperideal of S if and only if Q is a relative right hyperideal of some relative left hyperideal of a Γ-semihyperring.*

Theorem 3.6. *Suppose that S is a Γ-semihyperring and $A \subseteq S$. If $A = A\Gamma < s >$, for all $s \in A$ where $< s >$ is the smallest relative tri-quasi hyperideal generated by s. Then Q is a relative tri-quasi hyperideal of S if and only if Q is a relative quasi hyperideal of S.*

Proof. Let Q be a relative tri-quasi hyperideal of a Γ-semihyperring S and $a \in Q$, $A \subseteq S$. Then $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$, which shows that $A\Gamma < a > \subseteq A\Gamma Q$ which in turn shows that $A \subseteq A\Gamma Q \subseteq A$ which implies that $A\Gamma Q = A$ which yields $Q\Gamma A = Q\Gamma A\Gamma Q \subseteq Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$ which gives $A\Gamma Q \cap Q\Gamma A \subseteq A\Gamma A \cap Q\Gamma A \subseteq Q$. Hence Q is a relative quasi hyperideal of S.
The converse is straightforward. □

4. Characterisation of regular Γ-semihyperring and simple Γ-semihyperring through relative tri-quasi hyperideals

In this part of the paper, we develop the notion of a tri-quasi simple Γ-semihyperring and characterize tri-quasi simple Γ-semihyperring with relative tri-quasi hyperideals of Γ-semihyperring, and study the characteristic properties of relative minimal tri-quasi hyperideals of Γ-semihyperring. We also study characterization of regular Γ-semihyperring with relative tri-quasi hyperideals in Γ-semihyperring.

Definition 4.1. A Γ-semihyperring S is a left (right) simple Γ-semihyperring if S has no proper relative left (right) hyperideals of S.

Definition 4.2. A Γ-semihyperring S is called simple Γ-semihyperring if S has no proper hyperideals of S.

Definition 4.3. A Γ-semihyperring S is called relative tri-quasi simple Γ-semihyperring if S has no relative tri-quasi hyperideals except S itself.

Theorem 4.1. *If S is a division Γ-semihyperring, then S is a tri-quasi simple Γ-semihyperring.*

Proof. Let Q be a proper relative tri-quasi hyperideal of the division Γ-semihyperring S, $A \subseteq S$ and $0 \neq a \in Q$. Since S is a division Γ-semihyperring, there exist $b \in A, \alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma, x \in A$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. So $x \in Q\Gamma A$. Thus $A \subseteq Q\Gamma A$. We obtain $Q\Gamma A \subseteq A$. Thus $S = Q\Gamma S$. Likewise, one can prove that $S\Gamma Q = S$. Moreover, $S = S\Gamma Q = Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$. Also, $Q \subseteq S$ Therefore, $S = Q$. Hence division Γ-semihyperring S has no proper relative tri-quasi-interior hyperideals. □

Theorem 4.2. *Suppose that S is a left and a right simple Γ-semihyperring. Then S is a relative tri-quasi simple Γ-semihyperring.*

Proof. Suppose that S is a left and a right simple Γ-semihyperring, and $A \subseteq S$. Also, let Q be a relative tri-quasi hyperideal of S. Then, we have $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q\Gamma Q$ and $A\Gamma Q$ and $Q\Gamma A$ are relative left and right hyperideals of S. As S is a left and right simple Γ-semihyperring, we have $A\Gamma Q = A$. Also, $Q\Gamma A = A$. Thus, $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q\Gamma Q$. This shows that $Q\Gamma A\Gamma Q \subseteq Q$. Hence $S \subseteq Q$. □

Theorem 4.3. *Suppose that S is a Γ-semihyperring. Then, S is a tri-quasi simple Γ-semihyperring if and only if $< a >= S$ for all $a \in A \subseteq S$, where $< a >$ is the smallest relative tri-quasi hyperideal generated by a.*

Proof. Suppose that S is a tri-quasi simple Γ-semihyperring, $a \in A \subseteq S$ and $Q = A\Gamma a$. Then Q is a relative left hyperideal of S. Thus, by Theorem 3.1, Q is a relative tri-quasi hyperideal of S. Thus $Q = S$. So $A\Gamma a = S$ for all $a \in A$. Also, $A\Gamma a \subseteq < a > \subseteq S$. This implies that $S \subseteq < a > \subseteq S$. Therefore $S =< a >$. Suppose $< a >$ is the smallest relative tri-quasi hyperideal of S generated by a and $< a >= S$. Also, I is relative tri-quasi hyperideal and $a \in I$. Then, we have $< a > \subseteq I \subseteq S$. It then follows that $S \subseteq I \subseteq S$. Therefore $I = S$. Hence S is a tri-quasi simple Γ-semihyperring. □

Theorem 4.4. *Suppose that S is a Γ-semihyperring. Then S is a tri-quasi simple Γ-semihyperring if and only if $a\Gamma a\Gamma S\Gamma a\Gamma a = S$ for all $a \in S$.*

Proof. Suppose that S is left bi-quasi simple Γ-semihyperring and $a \in A \subseteq S$. Thus, $a\Gamma a\Gamma S\Gamma a\Gamma a = S$ is a relative tri-quasi hyperideal of S. Hence $a\Gamma S\Gamma a\Gamma S\Gamma a = S$ for all $a \in S$.

Conversely, suppose that $a\Gamma a\Gamma S\Gamma a\Gamma a = S$ for all $a \in A \subseteq S$. Let Q be a relative tri-quasi hyperideal of Γ-semihyperring S and $a \in Q$. Also, $S = a\Gamma a\Gamma A\Gamma a\Gamma a$, and $S = a\Gamma a\Gamma A\Gamma a\Gamma a \subseteq Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$. Thus, $Q = B\Gamma Q$. Hence S is a tri-quasi simple Γ-semihyperring. □

Theorem 4.5. *Let Q be a relative minimal tri-quasi hyperideal of a Γ-semihyperring S. Then any two non-zero elements of Q construct the same right relative hyperideal of S.*

Proof. Suppose that Q is a relative minimal tri-quasi hyperideal of a Γ-semihyperring S and $q \in Q$. Then, it follows that $(q)_R \cap Q$ is a relative tri-quasi hyperideal of S. Thus, $(q)_R \cap Q \subseteq Q$. As Q is a relative minimal tri-quasi hyperideal of S, we have $(q)_R \cap Q = Q$. This shows that $Q \subseteq (q)_R$. Moreover, suppose that $r \in Q$. Then $r \in (q)_R, (r)_R \subseteq (q)_R$. Hence, $(q)_R = (r)_R$. □

Corollary 4.6. *If Q is a relative minimal tri-quasi hyperideal of a Γ-semihyperring S, then any two non-zero elements of Q generates the same relative left hyperideal of S.*

Theorem 4.7. *Suppose that S is a Γ-semihyperring and Q is a relative tri-quasi hyperideal of S. Then Q is a relative minimal tri-quasi hyperideal of S if and only if Q is a relative tri-quasi simple sub-Γsemihyperring.*

Proof. Let Q be a relative minimal tri-quasi hyperideal of Γ-semiring S and I be a relative tri-quasi hyperideal of Q. Then $I\Gamma I\Gamma Q\Gamma I\Gamma I \subseteq I$. Therefore $I\Gamma I\Gamma Q\Gamma I\Gamma I$ is a relative tri-quasi hyperideal of S. Since I is a relative tri-quasi hyperideal of Q. $I\Gamma I\Gamma Q\Gamma I\Gamma I = Q$. This shows that $Q = I\Gamma I\Gamma Q\Gamma I\Gamma I \subseteq I$. So, $Q = I$.

Conversely, suppose that Q is a relative tri-quasi simple Γ-subsemihyperring of S. Let I be a relative tri-quasi hyperideal of S and $I \subseteq Q$. This implies that $I\Gamma C\Gamma Q\Gamma I\Gamma I = I$. It then follows that for $A \subseteq S$ $I\Gamma I\Gamma Q\Gamma I\Gamma I \subseteq I\Gamma I\Gamma Q\Gamma I\Gamma I \subseteq Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$. Thus, I is a relative tri-quasi hyperideal of Q. Therefore, $Q = I$, since Q is a relative tri-quasi simple sub-Γ-semihyperring. Hence Q is a relative minimal tri-quasi hyperideal of S. □

Theorem 4.8. *Suppose that S is a Γ-semihyperring and $Q = R\Gamma L$, where L and R are relative minimal left and relative minimal right hyperideals of S, repectively. Then, Q is a relative minimal tri-quasi hyperideal of S.*

Proof. As a matter of fact, we have $Q = R\Gamma L$ is a relative tri-quasi hyperideal of S. Let $A \subseteq S$ and I be a relative tri-quasi hyperideal of S such that $I \subseteq Q$. Then $A\Gamma I\Gamma I$ is a relative left hyperideal of S. This shows that $A\Gamma I\Gamma I\Gamma A\Gamma Q\Gamma Q = A\Gamma R\Gamma L\Gamma R\Gamma L \subseteq L$, since L is a relative left hyperideal of S. Similarly, one can prove $I\Gamma I\Gamma A\Gamma R$. Thus $A\Gamma I\Gamma I = L, I\Gamma I\Gamma A = R$. Therefore $Q = I\Gamma I\Gamma A\Gamma A\Gamma I\Gamma I \subseteq I\Gamma I\Gamma A\Gamma I\Gamma I \subseteq I$. So $I = Q$. Hence Q is a relative minimal tri-quasi hyperideal of S. □

Theorem 4.9. *Suppose that S is a relative regular idempotent Γ-semihyperring. Then Q is a relative tri-quasi hyperideal of S if and only if $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = Q$ for all relative tri-quasi hyperideals Q of S for $A \subseteq S$.*

Proof. Suppose that S is a relative regular Γ-semihyperring, Q is a relative tri-quasi hyperideal of S and $x \in Q$. Also, let $A \subseteq S$. Then $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$ and there exist $y \in A, \alpha, \beta, \delta \in \Gamma$, such that $x = x\alpha x\alpha y\beta x\delta x \subseteq Q\Gamma Q\Gamma A\Gamma Q\Gamma Q$. Thus $x \in Q\Gamma Q\Gamma A\Gamma Q\Gamma Q$. Hence $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = Q$.

Conversely, suppose that $A \subseteq S$. Then, we have $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = Q$ for all relative tri-quasi hyperideals Q of S. Let $Q = R \cap L$, where R is a relative right hyperideal and L is a relative left hyperideal of S. Then Q is a relative tri-quasi hyperideal of S. Thus, we have the following:

$$\begin{aligned}(R\cap L)\Gamma A\Gamma(R\cap L)\Gamma A\Gamma(R\cap L) &= R\cap L\\ &= (R\Gamma L)\Gamma(R\cap L)A\Gamma A\Gamma(R\cap L)\Gamma(R\cap L)\\ &\subseteq R\Gamma A\Gamma L\Gamma A\Gamma L\\ &\subseteq R\Gamma L\\ &\subseteq R\cap L(since\ R\Gamma L\subseteq L\ and\ R\Gamma L\subseteq R).\end{aligned}$$

Thus $R \cap L = R\Gamma L$. Hence S is a relative regular Γ-semihyperring. □

Theorem 4.10. *Suppose that S is a relative regular commutative Γ-semihyperring. Then every relative tri-quasi hyperideal of S is a relative hyperideal of S.*

Proof. Suppose that Q is a relative tri-quasi hyperideal of S, $A \subseteq S$ and $C = Q\Gamma Q\Gamma A\Gamma Q\Gamma Q$. Then $C = Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = Q$. This shows that $Q\Gamma A = C\Gamma A \subseteq C\Gamma A\Gamma C$, since S is relative regular. It hence follows that $Q\Gamma A \subseteq Q\Gamma Q\Gamma A\Gamma Q\Gamma Q\Gamma A\Gamma Q\Gamma Q\Gamma A\Gamma Q\Gamma Q \subseteq Q$. □

Theorem 4.11. *The non-empty set S is regular Γ-semihyperring if and only if $I\Gamma Q = I \cap Q$ for any relative right hyperideal I and relative left hyperideal Q of Γ-semihyperring.*

Proof. The proof of the Theorem is easy. □

Theorem 4.12. *Suppose that Q is sub-Γ-semihyperring of a regular idempotent Γ-semihyperring. Then, Q can be represented as $Q = R\Gamma L$, where R is a relative right hyperideal and L is a relative left hyperideal of S if and only if Q is a relative tri-quasi hyperideal of S.*

Proof. Suppose that $Q = R\Gamma L$, where R is relative right hyperideal of S and L is a relative left hyperideal of S.Also, $A \subseteq S$. Then, we have the following: $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = R\Gamma L\Gamma R\Gamma L\Gamma A\Gamma R\Gamma L\Gamma R\Gamma L \subseteq R\Gamma L = Q$. Hence Q is a relative tri-quasi hyperideal of Γ-semihyperring.

Conversely, suppose that Q is a relative tri-quasi hyperideal of regular idempotent Γ-semihyperring. Then $Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = Q$. Let $R = Q\Gamma A$ and $L = A\Gamma Q$. Then $R = Q\Gamma A$ is a relative right hyperideal of S and $L = A\Gamma Q$ is a relative left hyperideal of S. Thus, $Q\Gamma A \cap A\Gamma Q \subseteq Q\Gamma Q\Gamma A\Gamma Q\Gamma Q = Q$. This shows that $Q\Gamma A \cap A\Gamma Q \subseteq Q$. It follows that $R \cap L \subseteq Q$. We have $Q \subseteq Q\Gamma A = R$ and $Q \subseteq A\Gamma Q = L$. Therefore, $Q \subseteq R\cap L$. This implies that $Q = R\cap L = R\Gamma L$ as S is a regular Γ-semihyperring. Hence Q can be represented as $R\Gamma L$ where R is a relative right hyperideal and L is a relative left hyperideal of S. □

The following theorem explicitly provides condition in terms of tri-quasi ideal under which a Γ-semihyperring becomes regular.

Theorem 4.13. *The non-empty subset S is a regular Γ-semihyperring if and only if $Q \cap I \cap L \subseteq Q\Gamma I\Gamma L$ for any relative tri-quasi hyperideal Q, hyperideal I and left hyperideal L of S.*

Proof. Suppose that S is a regular Γ-semiring, Q, I and L are relative tri-quasi hyperideal, relative hyperideal and relative left hyperideal of S, respectively. Also, let $A \subseteq S$. Let $a \in Q \cap I \cap L$. Then $a \in a\Gamma A\Gamma a$, since S is relative regular. Thus, we have $a \in a\Gamma A\Gamma a \subseteq a\Gamma A\Gamma a\Gamma A\Gamma a \subseteq Q\Gamma I\Gamma L$. Hence $Q \cap I \cap L \subseteq Q\Gamma I\Gamma L$.

Conversely, suppose that $Q \cap I \cap L \subseteq Q\Gamma I\Gamma L$ for any relative tri-quasi hyperideal Q, relative hyperideal I and relative left hyperideal L of S. Let R be a relative right hyperideal and L be relative left hyperideal of S. Then by the hypothesis, $R \cap L = R \cap A \cap L \subseteq R\Gamma A\Gamma L \subseteq R\Gamma L$. We have $R\Gamma L \subseteq R, R\Gamma L \subseteq L$. Thus, $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$. Hence S is a regular Γ-semihyperring. □

5. Conclusions and/or Discussions

In this paper, as a generalization of hyperideals, we introduced the concept of relative tri-quasi hyperideal in Γ-semihyperring that in turn generalize ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of semiring and Γ-semiring. We studied some of their characteristics. We also introduced the concept of tri-quasi simple Γ-semihyperring and characterized relative tri quasi simple Γ-semihyperring, regular Γ-semihyperring through relative tri-quasi hyperideals of Γ-semihyperring. We then proved that each relative bi-quasi hyperideal and relative b-interior hyperideal in Γ-semihyperring are relative tri-quasi hyperideals, and studied some of the properties of relative tri-quasi hyperideals in Γ-semihyperrings. In future research direction, one may study prime tri-quasi hyperideals, maximal and minimal tri-quasi hyperideals as relative hyperideals or generalised hyperideals in ordered Γ-semihyperrings and in other explorable algebraic structures.

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REFERENCES

[1] M. M. K. Rao, R. K Kona and N. Rafi and V. Bolineni, Tri-quasi Ideals and Fuzzy Tri-quasi Ideals of Semigroups, Annals of Communications in Mathematics, , 7(3)(2024), 281-295.
[2] N. M. Khan and M. F. Ali, Relative bi-ideals and relative quasi ideals in ordered semigroups, Hacet. J. Math. Stat., 20(10) (2020), 1–12.
[3] C. B. Kumbharde, K. F. Pawar and S. J. Ansari, Bi-interior, quasi-interior and bi-quasi-interior Γ-hyperideal in Γ-semihyperring, Journal of Hyperstructures, 12 (1) (2023), 1-17.
[4] A. Basar, A. Ahmad, P. K. Sharma, and S. Bhavanari, On relative ordered Hoehnke Γ-hyperideal and Γ-hyperideals in chain ordered Γ-semihypegroups, GIS Science Journal, 9(7)(2022), 2793-2803.
[5] M. Shabir, A. Ali and S. Batool, A note on quasiideals in semirings, VSoutheast Asian Bulletin of Mathematics, 27(5)(2004), 923-928.
[6] W. E. Barenes, On the Γ-rings of Nobusawa, Pacific J. Math., 18(1966), 411–422.
[7] P. Corsini, Prolegomena of hypergroup theory, Aviani Editore, 1993.
[8] P. Corsini, V. Leoreanu, Applications of hyperstructure theory, Kluwr Academic Publishers, Dordrecht, 2003.
[9] S. Krishnamoorthy and R. A. Doss, Commuting regular Γ-semiring, Int. J. Math. Comput. Sci., 4(8)(2010), 376–378.
[10] S. Ostadhadi-Dehkordi and B. Davvaz, Ideal Theory in Γ-Semihyperrings, Iran.J. Sci. Technol. A., 37(3) (2013), 251-263.
[11] A. C. Golzio, A Brief Historical Survey on Hyperstructures in Algebra and Logic, South American Journal of Logic, 4(1), 2018.
[12] R.A. Good and D. R. Hughes, Associated groups for a semigroup, Bull. Amer. Math. Soc., (58)(1952), 624-625.
[13] M. Munir and M. Habib, Characterizing Semirings using Their Quasi and Bi-Ideals, Proceedings of the Pakistan Academy of Sciences:A. Physical and Computational Sciences, 53 (4)(2016), 469–475.
[14] K. Iseki, Ideals in semirings, Proc. Japan Acad., 34(1)(1958), 29-31 (1958). DOI: 10.3792/pja/1195524845
[15] S. Lajos, (m; k; n)-ideals in semigroups, in: Notes on Semigroups II, Karl Marx Univ. Econ., Dept. Math. (Budapest, 1976),1, 12–19.
[16] F. Marty, Surune generalization de la notion de groupe, 8eme Congres Math. Scandinaves, Stockholm, (1934), 45-49.
[17] N. Nobusawa, On generalization of the ring theory, Osaka J. Math., 1(1)(1964), 81– 89.
[18] B. Satyanarayana, M. Y. Abbasi, A. Basar and K. S. Prasad, Some Results on Abstract Affine Gamma Near-Rings, International Journal of Pure and Applied Mathematical Sciences, 7(1) (2014), 43-49.
[19] M. M. K. Rao, Tri-quasi ideals of Γ-semirings, Discussiones Mathematicae General Algebra and Applications, 41 (2021), 33–44. doi:10.7151/dmgaa.1360
[20] M. M. K. Rao, Γ-semirings. I, Southeast Asian Bull. Math., 19(1)(1995), 49–54.
[21] M. M. K. Rao, Γ-semiring with identity, Discuss. Math. Gen. Alg. and Appl., 37 (2017) 189–207. doi:10.7151/dmgaa.1276
[22] A. Basar, A. Ahmad, B. Satyanarayana, P. K. Sharma and M Sanjeel, On relative ordered Gamma-hyperideals and relative prime segments in chain ordered Gamma-semihypergroups, Journal of Xi'an University of Architecture & Technlogy, 14(8)(2022), 300-309.
[23] M. K. Sen, On Γ-semigroup, Proc. of International Conference of algebra and its application, Decker Publicaiton, New York, 1981, 301–308.
[24] B. Satyanarayana, A note on Γ-near-rings, Indian Journal of Mathematics, 41(3)(1999), 427-433.
[25] S. Lajos and F. A. Szasz, On the bi-ideals in associative ring, Proc. Japan Acad., 46 (1970), 505–507.
[26] O. Steinfeld, On ideal-quotients and prime ideals, Acta. Math. Acad. Sci. Hung., 4(1953), 289-298.
[27] O. Steinfeld, Quasi ideals in Rings and Semigroups, vol. 10 of Disquisitiones Mathematicae Hungaricae, Akademiai Kiado, Budapest, Hungary, 1978.
[28] A. Basar, S. Kumari and B. Satyanarayana, On Some Properties of Relative Hyperideals In Relative Regular Ordered Krasner Hyperrings, GIS Science Journal, 10(1)(2023), 306-328.
[29] A. Basar, Shaista and B. Satyanarayana, A note on completely regular 2-duo ordered Γ-semihypergroups by relative ordered quasi-Γ-hyperideals and relative ordered Γ-hyperideals, Alochana Journal, 13(1)(2024), 69-77.
[30] H. S. Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold, Bull. Amer. Math. Soc., 40 Ž(1934) , 916-920.
[31] A. D. Wallace, Relative Ideals in Semigroups I, Colloq. Math., (1962), 55-61.
[32] A. D. Wallace, Relative Ideals in Semigroups II, Acta Mathematica Hungarica, 14(1-2)(1963), 137-148.

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