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REGULARITY BASED CHARACTERIZATIONS OF le-Γ-SEMIGROUPS

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ABSTRACT. This article provides an in-depth examination of (m, n, γ)-regular le-Γ-semigroups, focusing on the characterization and properties of various types of ideal elements within these structures. Specifically, the discussion encompasses (m, n, γ)-ideal elements, $(m, 0, \gamma)$-ideal elements, and $(0, n, \gamma)$-ideal elements, highlighting their significance and interrelationships. Furthermore, the article investigates the (m, n, α, β)-regularity associated with the set denoted as $I_{(m,n,\alpha,\beta)}$, which consists of all (m, n, α, β)-ideal elements. In conjunction, the study explores the set $Q_{(m,n,\alpha,\beta)}$, which comprises all (m, n, α, β)-quasi-ideal elements of le-Γ-semigroups, detailing the implications of these classifications on the structure and behavior of the semigroups. Additionally, the research delves into the concept of 0-minimality, particularly concerning $(0, m, \gamma)$-ideal elements in both poe-Γ-semigroups and le-Γ-semigroups. This aspect of the study aims to clarify the foundational properties of ideal elements and their roles in the broader context of semigroup theory. The findings presented in this article contribute to a deeper understanding of the algebraic properties of le-Γ-semigroups and their ideal elements, paving the way for future research in this area.

1. INTRODUCTION

Concept of (m, n)-ideals in semigroups was introduced by S. Lajos [9] and, thereafter, many other authors had studied (m, n)-ideals in various algebraic structures. In 1986, Sen and Saha introduced the notion of a Γ-semigroup as a generalization of the notion of a semigroup and that of a ternary semigroup. Later on, in 1996, the notion of an ordered Γ-semigroup was introduced by Kwon and Lee [8]. In recent years, the study of le-Γ-semigroups has gained significant attention due to their applications in various mathematical contexts. Specifically, Hila has provided a comprehensive characterization of regular le-Γ-semigroups, elucidating their structural properties and implications for algebraic systems [1]. Further contributions include the exploration of specific classes of these semigroups, where Hila offers new insights into their algebraic characteristics and interrelations [2]. Moreover, the investigation into generalized ideal elements within le-Γ-semigroups has been notably advanced through collaborative work by Hila and Pisha, which addresses the complexities of these structures [3]. Notably, the groundwork laid by

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Kehayopulu has also been instrumental in this area, as the author introduces the concept of generalized ideal elements within the broader context of poe-semigroups, asserting their relevance to our current understanding of semigroup theory [4]. Furthermore, Kehayopulu expands on le-Γ-semigroups, contributing valuable findings that align with and enhance the works of Hila [5]. Also, Kehayopulu [6], studied several properties of (m, n, α, β)-ideal elements in le-Γ-semigroups and also characterized (m, n, α, β)-regular le-Γ-semigroups in terms of its (m, n, α, β)-ideal elements. In addition to Hila and Kehayopulu's work, Sen's studies on Γ-semigroups have further enriched the field, particularly through foundational examinations of their algebraic structures and properties [14, 15]. Siripitukdet and Iampan also contribute to this discourse by investigating the *Green-Kehayopulu* relations in le-Γ-semigroups, providing deeper insights into their ideal elements and related frameworks [16]. More concepts related to this study have been studied in [10, 11, 12, 13].

In this paper, we first show that an le-Γ-semigroup S is (m, n, γ)-regular if and only if $a \wedge q = q^m_\gamma \gamma a \gamma q^n_\gamma$ for each (m, n, γ)-quasi-ideal element q and for each γ-ideal element a of S and, then, prove that an le-Γ-semigroup S is (m, n, γ)-regular if and only if $a \wedge b = a^m_\gamma \gamma b^n_\gamma$ for each $(m, 0, \gamma)$-ideal element a and for each $(0, n, \gamma)$-ideal element b of S. We also obtain some characterizations of (m, n, γ)-regular le-Γ-semigroups and discuss (m, n, α, β)-regularity of the set $I_{(m,n,\alpha,\beta)}$ of all (m, n, α, β)-ideal elements and the set $Q_{(m,n,\alpha,\beta)}$ of all (m, n, α, β)-quasi-ideal elements of le-Γ-semigroups. Finally $(0, m, \gamma)$-ideal elements and 0-minimal $(0, m, \gamma)$-ideal elements in poe-Γ-semigroups as well as in le-Γ-semigroups are characterized.

2. Preliminaries

Let S and Γ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \to a\alpha b$ and satisfy $(a\gamma b)\mu c = a\gamma(b\mu c)$ for each $a, b, c \in S$ and $\gamma, \mu \in \Gamma$. The triplet $(S, \Gamma, \leq)$ is called a po-Γ-semigroup if S is a Γ-semigroup and $(S, \leq)$ is a partially ordered set such that, for each $a, b, c \in S$ and $\gamma \in \Gamma$, $a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$. A po-Γ-semigroup with a greatest element "e" (i.e. for each $a \in S, a \leq e$) is said to be a poe-Γ-semigroup. A poe-Γ-semigroup S is said to be $\vee e$-Γ-semigroup if it is an upper semilattice under $\vee$ and, for each $a, b, c \in S$ and $\gamma \in \Gamma$, $c\gamma(a \vee b) = c\gamma a \vee c\gamma b$ and $(a \vee b)\gamma c = a\gamma c \vee b\gamma c$. A $\vee e$-Γ-semigroup which is also a lattice is said to be an le-Γ-semigroup.

Let S be a poe-Γ-semigroup and $\gamma \in \Gamma$. An element a is called a ***γ-subsemigroup element*** if $a\gamma a \leq a$ and a *γ-left (resp. γ-right)-ideal* element of S if $e\gamma a \leq a$ (resp. $a\gamma e \leq a$). It is called an ***γ-ideal-element*** of S if it is both a γ-left and a γ-right-ideal element of S. An element a is called a ***γ-bi-ideal element*** if $a\gamma e\gamma a \leq a$ and a ***γ-quasi-ideal element*** if $a\gamma e \wedge e\gamma a$ exists and $a\gamma e \wedge e\gamma a \leq a$. A poe-Γ-semigroup S is called ***γ-regular (γ-left-regular, γ-right-regular)*** if $a \leq a\gamma e\gamma a$ $(a \leq e\gamma a\gamma a, a \leq a\gamma a\gamma e)$ for each $a \in S$.

Let S be any le-Γ-semigroup. Then it is easy to verify that under the order relation $\leq$ on S

$$a \leq b \Leftrightarrow a \wedge b = a \text{ and } a \vee b = b$$

for each $a, b \in S$.

Remark. [6] *Let S be a Γ-semigroup, $a \in S$, $\gamma \in \Gamma$ and $m \in \mathbb{N}$. Throughout this paper, we shall be using the following conventions without further mention:*

(1) if $m = 1$, then we write $a^1_\gamma = a$;
(2) if $m \geq 2$, then we write $a^m_\gamma = a\gamma a\gamma ... \gamma a$ ((m – 1)-times γ and m-times a); and
(3) $a^m_\gamma \gamma a^n_\gamma = a^{m+n}_\gamma$.

Let S be a *poe*-Γ-semigroup and $m, n \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$. An element a of S is called an (m, n, α, β)-ideal element of S if $a^m_\alpha \alpha e \beta a^n_\beta \leq a$. When $\alpha = \beta$, the element a shall be called an (m, n, α)-ideal element. An element a of S is called an $(m, 0, \alpha)$-ideal $[(0, m, \alpha)$-ideal$]$ element if $a^m_\alpha \alpha e \leq a$ $[e\alpha a^m_\alpha \leq a]$ while an element q of S is called a (m, n, α, β)-quasi-ideal element of S if $q^m_\alpha \alpha e \wedge e\beta q^n_\beta$ exists and $q^m_\alpha \alpha e \wedge e\beta q^n_\beta \leq q$. Again, when $\alpha = \beta$, the element a shall be called a (m, n, α)-quasi-ideal element.

We denote, in the sequel, by $(a)_\gamma, < a >_{(m,0,\alpha)}, < a >_{(0,m,\alpha)}, < a >_{(m,n,\alpha,\beta)}$ and $(a)_{(m,n,\alpha,\beta)}$, the γ-ideal-element, the $(m, 0, \alpha)$-ideal element, the $(0, m, \alpha)$-ideal element, the (m, n, α, β)-ideal element and the (m, n, α, β)-quasi-ideal element of S generated by the element a of S respectively; i.e. the least γ-ideal-element, the least $(m, 0, \alpha)$-ideal element, the least $(0, m, \alpha)$-ideal element, the least (m, n, α, β)-ideal element and the least (m, n, α, β)-quasi-ideal element of S greater than the element a respectively and given by [6] as follows:

$$\begin{aligned}
(a) &= a \vee e\gamma a \vee a\gamma e \vee e\gamma a\gamma e \\
< a >_{(m,0,\alpha)} &= a \vee a^m_\alpha \alpha e \\
< a >_{(0,m,\alpha)} &= a \vee e\alpha a^m_\alpha \\
< a >_{(m,n,\alpha,\beta)} &= a \vee a^m_\alpha \alpha e \beta a^n_\beta \\
(a)_{(m,n,\alpha,\beta)} &= a \vee (a^m_\alpha \alpha e \wedge e\beta a^n_\beta).
\end{aligned}$$

Thus, an element a of S is an γ-ideal (resp. an $(m, 0, \alpha)$-ideal, an $(0, m, \alpha)$-ideal, an (m, n, α, β)-ideal, an (m, n, α, β) quasi-ideal) element if $(a)_\gamma = a$ (resp. $< a >_{(m,0,\alpha)}= a, < a >_{(0,m,\alpha)}= a, < a >_{(m,n,\alpha,\beta)}= a, (a)_{(m,n,\alpha,\beta)} = a$).

Lemma 2.1. [6] *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $a, b \in S, \gamma \in \Gamma$. Then we have*

(1) $(a \vee a^m_\gamma \gamma b)^m_\gamma \gamma e = a^m_\gamma \gamma e$;
(2) $e\gamma(a \vee b\gamma a^n_\gamma)^n_\gamma = e\gamma a^n_\gamma$.

Lemma 2.2. [6] *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $a, b \in S, \alpha, \beta \in \Gamma$. Then we have*

(1) $(a \vee a^m_\alpha \alpha e \beta a^n_\beta)^m_\alpha \alpha e = a^m_\alpha \alpha e$;
(2) $e\beta(a \vee a^m_\alpha \alpha e \beta a^n_\beta)^n_\beta = e\beta a^n_\beta$.

Lemma 2.3. [7] *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $a, b \in S, \gamma \in \Gamma$. Then*

(1) $(a \vee (a^m_\alpha \alpha e \wedge e\beta a^n_\beta))^m_\alpha \alpha e \leq a^m_\alpha \alpha e$;
(2) $e\beta(a \vee (a^m_\alpha \alpha e \wedge e\beta a^n_\beta))^n_\beta \leq e\beta a^n_\beta$.

3. (m,n,γ)-Regular Semigroups

Definition 3.1. Let S be a *poe*-Γ-semigroup $m, n \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$. An element a of S is said to be an (m, n, α, β)-regular element if $a \leq a^m_\alpha \alpha e \beta a^n_\beta$. Further S is said to be (m, n, α, β)-regular if each element of S is (m, n, α, β)-regular. For $\alpha = \beta$, we shall call S as (m, n, α)-regular.

It is clear from above definition that, for all $m, n \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$, each (m, n, α, β)-regular *poe*-Γ-semigroup is (r, s, α, β)-regular *poe*-Γ-semigroup ($r \leq m, s \leq n$). In particular, for all $m, n \in \mathbb{N}$ and $\alpha \in \Gamma$, each (m, n, α)-regular *poe*-Γ-semigroup is α-regular. Indeed $a \leq a^m_\alpha \alpha e \alpha a^n_\alpha \leq a\alpha e\alpha a$. On the other hand, for any $m \in \mathbb{N}$, $(m, 0, \alpha)$-regular *poe*-Γ-semigroups need not be α-regular *poe*-Γ-semigroups.

Example 3.2. Let $S = \{x, y, z\}$ and $\Gamma = \{\gamma\}$. Define $S \times \Gamma \times S \longrightarrow S$ and "$\leq$" on S as follows:

γ	x	y	z
x	z	x	z
y	x	y	z
z	z	z	z

$$\leq = \{(x,x), (y,y), (z,z), (z,x), (x,y), (z,y)\}.$$

Clearly S is a poe-Γ-semigroup with greatest element $e = y$. As $(x, x\gamma y\gamma x) \notin \leq$, S is not γ-regular. But, as for each $a \in S$, $a \leq a\gamma e$, S is $(1, 0, \gamma)$-regular.

Theorem 3.1. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $\gamma \in \Gamma$. Then S is (m, n, γ)-regular if and only if $a \wedge q = q^m_\gamma \gamma a \gamma q^n_\gamma$ for each (m, n, γ)-quasi-ideal element q and for each γ-ideal element a of S.*

Proof. Let S be an (m, n, γ)-regular le-Γ-semigroup. Now take any (m, n, γ)-quasi-ideal element q and any γ-ideal element a of S. Since $q^m_\gamma \gamma a \gamma q^n_\gamma \leq q^m_\gamma \gamma e$ and $q^m_\gamma \gamma a \gamma q^n_\gamma \leq e\gamma q^n_\gamma$. Therefore $q^m_\gamma \gamma a \gamma q^n_\gamma \leq q^m_\gamma \gamma e \wedge e\gamma q^n_\gamma$. As q is (m, n, γ)-quasi-ideal element of S, $q^m_\gamma e \wedge e\gamma q^n_\gamma \leq q$. So $q^m_\gamma \gamma a \gamma q^n_\gamma \leq q$. Since a is γ-ideal-element of S, $q^m_\gamma \gamma a \gamma q^n_\gamma \leq e\gamma a \gamma e \leq a\gamma e \leq a$. Therefore $q^m_\gamma \gamma a \gamma q^n_\gamma \leq q \wedge a$. As S is (m, n, γ)-regular, we have

$$\begin{aligned}
&(a \wedge q)\\
&\leq (a \wedge q)^m_\gamma \gamma e \gamma (a \wedge q)^n_\gamma\\
&\leq (a \wedge q)^m_\gamma \gamma e \gamma ((a \wedge q)^n_\gamma)^m_\gamma \gamma e \gamma ((a \wedge q)^n_\gamma)^n_\gamma\\
&= (a \wedge q)^m_\gamma \gamma e \gamma \underbrace{(a \wedge q)^n_\gamma \gamma (a \wedge q)^n_\gamma \gamma \dots \gamma (a \wedge q)^n_\gamma}_{m\text{-times}} \gamma e \gamma \underbrace{(a \wedge q)^n_\gamma \gamma (a \wedge q)^n_\gamma, \dots \gamma (a \wedge q)^n_\gamma}_{n\text{-times}}\\
&\leq q^m_\gamma \gamma e \gamma \underbrace{a^n_\gamma \gamma a^n_\gamma \dots a^n_\gamma}_{m\text{-times}} \gamma e \gamma \underbrace{a^n_\gamma \gamma a^n_\gamma \dots a^n_\gamma}_{n-1\text{-times}} \gamma q^n_\gamma\\
&= q^m_\gamma \gamma e \gamma a \gamma a^{n-1} \gamma \underbrace{a^n_\gamma \gamma a^n_\gamma \dots a^n_\gamma}_{m-1\text{-times}} \gamma e \gamma \underbrace{a^n_\gamma a^n_\gamma \dots a^n_\gamma}_{n-1\text{-times}} \gamma q^n_\gamma\\
&\leq q^m_\gamma \gamma e \gamma a \gamma (a^{n-1} \gamma \underbrace{a^n_\gamma \gamma a^n_\gamma \dots a^n_\gamma}_{m-1\text{-times}} \gamma e \gamma \underbrace{a^n_\gamma \gamma a^n_\gamma \dots a^n_\gamma}_{n-1\text{-times}}) \gamma q^n_\gamma\\
&\leq q^m_\gamma \gamma e \gamma a \gamma e \gamma q^n_\gamma\\
&\leq q^m_\gamma \gamma a \gamma q^n_\gamma
\end{aligned}$$

Hence $a \wedge q = q^m_\gamma \gamma a \gamma q^n_\gamma$.

Conversely assume that $a \wedge q = q^m_\gamma \gamma a \gamma q^n_\gamma$ for each (m, n, γ)-quasi-ideal element q and for each γ-ideal-element a of S. For any $b \in S$, as $(b)_{(m,n,\gamma)}$ and $(b)_\gamma$ are (m, n, γ)-quasi-ideal and γ-ideal element of S respectively, we have

$$\begin{aligned}
(b)_\gamma \wedge (b)_{(m,n,\gamma)} &= ((b)_{(m,n,\gamma)})^m_\gamma \gamma (b)_\gamma \gamma ((b)_{(m,n,\gamma)})^n_\gamma\\
&\leq ((b)_{(m,n,\gamma)})^m_\gamma \gamma e \gamma ((b)_{(m,n,\gamma)})^n_\gamma\\
&= (b \vee (b^m_\gamma \gamma e \wedge e\gamma b^n_\gamma))^m_\gamma \gamma e \gamma (b \vee (b^m_\gamma \gamma e \wedge e\gamma b^n_\gamma))^n_\gamma\\
&\leq b^m_\gamma \gamma e \gamma (b \vee (b^m_\gamma \gamma e \wedge e\gamma b^n_\gamma))^n \text{ (by Lemma 1.4)}\\
&\leq b^m_\gamma \gamma e \gamma b^n_\gamma \text{ (by Lemma 1.4)}.
\end{aligned}$$

As $b \leq (b)_\gamma \wedge (b)_{(m,n,\gamma)}$, we have $b \leq b^m_\gamma \gamma e \gamma b^n_\gamma$. Hence S is (m, n, γ)-regular. □

Theorem 3.2. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $\gamma \in \Gamma$. Then S is (m, n, γ)-regular if and only if $a \wedge b = a^m_\gamma \gamma b^n_\gamma$ for each $(m, 0, \gamma)$-ideal element a and for each $(0, n, \gamma)$-ideal element b of S.*

Proof. Let a be any $(m, 0, \gamma)$-ideal element and b be any $(0, n, \gamma)$-ideal element of S. Therefore $a^m_\gamma \gamma b^n_\gamma \leq a^m_\gamma \gamma e \leq a$ and $a^m_\gamma \gamma b^n_\gamma \leq e\gamma b^n_\gamma \leq b$. So $a^m_\gamma \gamma b^n_\gamma \leq a \wedge b$. As S is (m, n, γ)-regular, we have

$$
\begin{aligned}
(a \wedge b) &\leq (a \wedge b)^m_\gamma \gamma e \gamma (a \wedge b)^n_\gamma \\
&\leq a^m_\gamma \gamma e \gamma b^n_\gamma \\
&\leq a^m_\gamma \gamma e \gamma b^{n-1}_\gamma \gamma (b^m_\gamma \gamma e \gamma b^n_\gamma) \\
&\leq a^m_\gamma \gamma e \gamma b^{n-1}_\gamma \gamma b^{m-1}_\gamma \gamma (b^m_\gamma \gamma e \gamma b^n_\gamma) \gamma e \gamma b^n_\gamma \\
&\leq a^m_\gamma \gamma e \gamma b^{n-1}_\gamma \gamma b^{m-1}_\gamma \gamma b^{m-1}_\gamma \gamma (b^m_\gamma \gamma e \gamma b^n_\gamma) \gamma e \gamma b^n_\gamma \gamma e \gamma b^n_\gamma \\
&\leq a^m_\gamma \gamma e \gamma b^{n-1}_\gamma \, \underbrace{b^{m-1}_\gamma \gamma b^{m-1}_\gamma \ldots b^{m-1}_\gamma}_{n-1\text{-times}} (b^m_\gamma \gamma e \gamma b^n_\gamma) \, \underbrace{e \gamma b^n_\gamma \gamma e \gamma b^n_\gamma \ldots e \gamma b^n_\gamma}_{n-1\text{-times}} \\
&= a^m_\gamma \gamma e \gamma b^{n-1}_\gamma \gamma (b^{m-1}_\gamma)^{n-1}_\gamma \gamma b^m_\gamma \, \underbrace{e \gamma b^n_\gamma \gamma e \gamma b^n_\gamma \ldots e \gamma b^n_\gamma}_{n\text{-times}} \\
&= a^m_\gamma \gamma e \gamma b^{n-1}_\gamma \gamma b^{mn-m-n+1}_\gamma \gamma b^m_\gamma \gamma \, \underbrace{e \gamma b^n_\gamma \gamma e \gamma b^n_\gamma \ldots e \gamma b^n_\gamma}_{n\text{-times}} \\
&= a^m_\gamma \gamma (e \gamma b^{mn}_\gamma \gamma e) \gamma b^n_\gamma \gamma \, \underbrace{e \gamma b^n_\gamma \gamma e \gamma b^n_\gamma \ldots e \gamma b^n_\gamma}_{n-1\text{-times}} \\
&= a^m_\gamma \gamma e \gamma b^n_\gamma \gamma \, \underbrace{e \gamma b^n_\gamma \gamma e \gamma b^n_\gamma \ldots e \gamma b^n_\gamma}_{n-1\text{-times}} \\
&\leq a^m_\gamma \gamma (e \gamma b^n_\gamma)^n_\gamma \\
&\leq a^m_\gamma \gamma b^n_\gamma.
\end{aligned}
$$

Therefore $a \wedge b = a^m_\gamma \gamma b^n_\gamma$.

Conversely assume that $a \wedge b = a^m_\gamma \gamma b^n_\gamma$ for each $(m, 0, \gamma)$-ideal element a and for each $(0, n, \gamma)$-ideal element b of S. For any $a \in S$, as $< a >_{(m,0,\gamma)}$ is $(m, 0, \gamma)$-ideal element and e is a $(0, n, \gamma)$-ideal element of S, we have

$$
\begin{aligned}
< a >_{(m,0,\gamma)} &= < a >_{(m,0,\gamma)} \wedge e = (< a >_{(m,0,\gamma)})^m_\gamma \gamma e^n_\gamma \\
&\leq (< a >_{(m,0,\gamma)})^m_\gamma \gamma e = a^m_\gamma \gamma e \text{ (by Lemma 1.2)}.
\end{aligned}
$$

Similarly $< a >_{(0,n,\gamma)} \leq e\gamma a^n_\gamma$. As $a^m_\gamma \gamma e$ is an $(m, 0, \gamma)$-ideal element and $e\gamma a^n_\gamma$ is a $(0, n, \gamma)$-ideal element of S, by hypothesis

$$
\begin{aligned}
a &\leq < a >_{(m,0,\gamma)} \wedge < a >_{(0,n,\gamma)} \leq a^m_\gamma \gamma e \wedge e \gamma a^n_\gamma \\
&= (a^m_\gamma \gamma e)^m_\gamma \gamma (e_\gamma a^n_\gamma)^n_\gamma \text{ (by hypothesis)} \\
&\leq a^m_\gamma \gamma e \gamma a^n_\gamma.
\end{aligned}
$$

Hence S is (m, n, γ)-regular. □

Corollary 3.3. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $\gamma \in \Gamma$. If S is (m, n, γ)-regular, then $x^2_\gamma = x, y^2_\gamma = y$ and $x\gamma y \in Q_{(m,n,\gamma)}$ for all $x \in I_{(m,0,\gamma)}$ and $y \in I_{(0,n,\gamma)}$.*

Proof. Since $x \in I_{(m,0,\gamma)}, y \in I_{(0,n,\gamma)}$, $x_\gamma^m \gamma e \leq x,\ e\gamma y_\gamma^n \leq y$. As S is (m,n,γ)-regular, we have

$$\begin{aligned} x &\leq x_\gamma^m \gamma e \gamma x_\gamma^n = x_\gamma^m \gamma e \gamma x_\gamma^{n-1} \gamma x \\ &\leq x_\gamma^m \gamma e \gamma x_\gamma^{n-1} \gamma x_\gamma^m \gamma e \gamma x_\gamma^n \leq x_\gamma^m \gamma e \gamma x_\gamma^m \gamma e \\ &\leq x \gamma x = x_\gamma^2 \end{aligned}$$

and

$$x_\gamma^2 \leq x_\gamma^m \gamma e \gamma x_\gamma^n \gamma x_\gamma^m \gamma e \gamma x_\gamma^n \leq x_\gamma^m \gamma e \leq x.$$

Therefore $x_\gamma^2 = x$. Similarly $y_\gamma^2 = y$. As $(x \wedge y)_\gamma^m \gamma e \wedge e\gamma (x \wedge y)_\gamma^n \leq x_\gamma^m \gamma e \wedge e \gamma y_\gamma^n \leq x \wedge y$, $x \wedge y$ is a (m,n,γ)-quasi-ideal. Therefore, by Theorem 2.3, $x_\gamma^m \gamma y_\gamma^n \in Q_{(m,n,\gamma)}$. As $x_\gamma^2 = x$ and $y_\gamma^2 = y$, we have $x\gamma y \in Q_{(m,n,\gamma)}$. □

Theorem 3.4. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ such that either $m \geq 2$ or $n \geq 2$ and $\gamma \in \Gamma$. Then the following are equivalent:*

(1) Each (m,n,γ)-ideal element of S is a γ-idempotent;
(2) For each (m,n,γ)-ideal elements a, b of S, $a \wedge b \leq a_\gamma^m \gamma b_\gamma^n$;
(3) $< a >_{(m,n,\gamma)} \wedge < b >_{(m,n,\gamma)} \leq (< a >_{(m,n,\gamma)})_\gamma^m \gamma (< b >_{(m,n,\gamma)})_\gamma^n$ $(\forall a, b \in S)$;
(4) $< a >_{(m,n,\gamma)} \leq (< a >_{(m,n,\gamma)})_\gamma^m \gamma (< a >_{(m,n,\gamma)})_\gamma^n$ $(\forall a \in S)$;
(5) S is (m,n,γ)-regular.

Proof. $(1) \Rightarrow (2)$ Assume that each (m,n,γ)-ideal element of S is γ-idempotent. Now, take any (m,n,γ)-ideal elements a and b of S. As $a \wedge b$ is an (m,n,γ)-ideal element, we have

$$\begin{aligned} (a \wedge b) &= (a \wedge b)_\gamma^2 = (a \wedge b)_\gamma^3 = ... = (a \wedge b)_\gamma^{m+n} \\ &= (a \wedge b)_\gamma^m \gamma (a \wedge b)_\gamma^n \leq a_\gamma^m \gamma b^n, \end{aligned}$$

as required.
$(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$. Obvious.
$(4) \Rightarrow (5)$. Take any $a \in S$. Then, by (4), we have

$$\begin{aligned} & < a >_{(m,n,\gamma)} \\ \leq\ & (< a >_{(m,n,\gamma)})_\gamma^m \gamma (< a >_{<m,n>})_\gamma^n \\ \leq\ & (< a >_{(m,n,\gamma)})_\gamma^m \gamma (< a >_{(m,n,\gamma)})_\gamma^{n-1} \gamma (< a >_{<m,n>})_\gamma^m \gamma (< a >_{(m,n,\gamma)})_\gamma^n \\ \leq\ & (< a >_{(m,n,\gamma)})_\gamma^m \gamma e \gamma (< a >_{<m,n>})_\gamma^n \\ =\ & a_\gamma^m \gamma e \gamma a_\gamma^n \text{ (by Lemma 1.3).} \end{aligned}$$

As $a \leq < a >_{(m,n,\gamma)},\ a \leq a_\gamma^m \gamma e \gamma a_\gamma^n$. Hence S is (m,n,γ)-regular.
$(5) \Rightarrow (1)$. Take any (m,n,γ)-ideal element a of S. As S is (m,n,γ)-regular and a is an (m,n,γ)-ideal element, $a = a_\gamma^m \gamma e \gamma a_\gamma^n$. Now

$$a_\gamma^2 = (a_\gamma^m \gamma e \gamma a_\gamma^n) \gamma (a_\gamma^m \gamma e \gamma a_\gamma^n) \leq (a_\gamma^m \gamma e \gamma a_\gamma^n) = a$$

and

$$\begin{aligned} a &= a_\gamma^m \gamma e \gamma a_\gamma^n = (a_\gamma^m \gamma e \gamma a_\gamma^n)^m \gamma e \gamma a_\gamma^n \\ &= \underbrace{(a_\gamma^m \gamma e \gamma a_\gamma^n) \gamma \dots (a_\gamma^m \gamma e \gamma a_\gamma^n)}_{m\text{-times}} \gamma e \gamma a_\gamma^n \\ &\leq (a_\gamma^m \gamma e \gamma a_\gamma^n) \gamma (a_\gamma^m \gamma e \gamma a_\gamma^n) = a \gamma a. \end{aligned}$$

Therefore $a = a_\gamma^2$. Hence each (m, n, γ)-ideal element of S is a γ-idempotent, as required. $\square$

Lemma 3.5. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $\gamma \in \Gamma$. Then S is $(m, 0, \gamma)$-regular ($(0, n, \gamma)$-regular) if and only if $I_{(m,0,\gamma)}$ ($I_{(0,n,\gamma)}$), the set of all $(m, 0, \gamma)$-ideal elements of S (the set of all $(0, n, \gamma)$-ideal elements of S), is $(m, 0, \gamma)$-regular ($(0, n, \gamma)$-regular).*

Proof. Let $a \in I_{(m,0,\gamma)}$. Then $a_\gamma^m \gamma e \leq a$. As S is $(m, 0, \gamma)$-regular, $a \leq a_\gamma^m \gamma e$. Thus $a = a_\gamma^m \gamma e$. Since $e \in I_{(m,0,\gamma)}$, a is $(m, 0, \gamma)$-regular element of $I_{(m,0,\gamma)}$. Hence $I_{(m,0,\gamma)}$ is $(m, 0, \gamma)$-regular.

Conversely assume that $I_{(m,0,\gamma)}$ is $(m, 0, \gamma)$-regular. Now take any $a \in S$. As $< a >_{(m,0,\gamma)} \in I_{(m,0,\gamma)}$ and $I_{(m,0,\gamma)}$ is $(m, 0, \gamma)$-regular, there exists $b \in I_{(m,0,\gamma)}$ such that $< a >_{(m,0,\gamma)} = (< a >_{(m,0,\gamma)})^m \gamma b \leq (< a >_{(m,0,\gamma)})^m \gamma e$. So, by Lemma 1.2, $(< a >_{(m,0,\gamma)})^m \gamma e = a_\gamma^m \gamma e$. As $a \leq < a >_{(m,0,\gamma)}$, we have $a \leq a_\gamma^m \gamma e$. Hence S is $(m, 0, \gamma)$-regular. $\square$

Lemma 3.6. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$. Then S is (m, n, α, β)-regular if and only if $I_{(m,n,\alpha,\beta)}$, the set of all (m, n, α, β)-ideal elements of S is (m, n, α, β)-regular.*

Proof. Let $a \in I_{(m,n,\alpha,\beta)}$. Therefore $a_\alpha^m \alpha e \beta a_\beta^n \leq a$. As S is (m, n, α, β)-regular, $a \leq a_\alpha^m \alpha e \beta a_\beta^n$. Thus $a = a_\alpha^m \alpha e \beta a_\beta^n$. As $e \in I_{(m,n,\alpha,\beta)}$, a is (m, n, α, β)-regular element of $I_{(m,n,\alpha,\beta)}$. Hence $I_{(m,n,\alpha,\beta)}$ is (m, n, α, β)-regular.

Conversely assume that $I_{(m,n,\alpha,\beta)}$ is (m, n, α, β)-regular and $a \in S$. As $< a >_{(m,n,\alpha,\beta)}$ is in $I_{(m,n,\alpha,\beta)}$ and $I_{(m,n,\alpha,\beta)}$ is (m, n, α, β)-regular, there exists $b \in I_{(m,n,\alpha,\beta)}$ such that

$$\begin{aligned} < a >_{(m,n,\alpha,\beta)} &= (< a >_{(m,n,\alpha,\beta)})_\alpha^m \alpha b \beta (< a >_{(m,n,\alpha,\beta)})_\beta^n \\ &\leq (< a >_{(m,n,\alpha,\beta)})_\alpha^m \alpha e \beta (< a >_{(m,n,\alpha,\beta)})_\beta^n \\ &= a_\alpha^m \alpha e \beta a_\beta^n \text{ (by Lemma 1.3).} \end{aligned}$$

As $a \leq < a >_{(m,n,\alpha,\beta)}$, $a \leq a_\alpha^m \alpha e \beta a_\beta^n$. This implies that a is an (m, n, α, β)-regular element of S. Hence S is (m, n, α, β)-regular. $\square$

Lemma 3.7. *Let S be an le-Γ-semigroup, $m, n \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$. Then S is (m, n, α, β)-regular if and only if $Q_{(m,n,\alpha,\beta)}$, the set of all (m, n, α, β)-quasi-ideal elements of S, is (m, n, α, β)-regular.*

Proof. Take any $a \in Q_{(m,n,\alpha,\beta)}$. Then $a_\alpha^m \alpha e \beta a_\beta^n \leq a_\alpha^m \alpha e \wedge e \beta a_\beta^n \leq a$. As S is (m, n, α, β)-regular, $a \leq a_\alpha^m \alpha e \beta a_\beta^n$. Thus $a = a_\alpha^m \alpha e \beta a_\beta^n$. Since $e \in Q_{(m,n,\alpha,\beta)}$, a is (m, n, α, β)-regular element of $Q_{(m,n,\alpha,\beta)}$. Hence $Q_{(m,n,\alpha,\beta)}$ is (m, n, α, β)-regular.

Conversely assume that $Q_{(m,n,\alpha,\beta)}$ is (m, n, α, β)-regular and let a be any element of S. Then $(a)_{(m,n,\alpha,\beta)} \in Q_{(m,n,\alpha,\beta)}$. Therefore there exists $b \in Q_{(m,n,\alpha,\beta)}$ such that

$$\begin{aligned} (a)_{(m,n,\alpha,\beta)} &= ((a)_{(m,n,\alpha,\beta)})_\alpha^m \alpha b \beta ((a)_{(m,n,\alpha,\beta)})_\beta^n \\ &\leq ((a)_{(m,n,\alpha,\beta)})_\alpha^m \alpha e \beta ((a)_{(m,n,\alpha,\beta)})_\beta^n \\ &= a_\alpha^m \alpha e \beta a_\beta^n \text{ (by Lemma 1.4).} \end{aligned}$$

As $a \leq (a)_{(m,n,\alpha,\beta)}$, $a \leq a_\alpha^m \alpha e \beta a_\beta^n$. Therefore a is (m, n, α, β)-regular and, hence, S is (m, n, α, β)-regular. $\square$

4. Minimality of (0,m,γ)-Ideal Elements

Lemma 4.1. *Let S be an le-Γ-semigroup, $m \in \mathbb{N}, \gamma \in \Gamma$ and a be a γ-subsemigroup element of S. Then a is an $(1, m, \gamma)$-ideal element of S if and only if there exists a $(0, m, \gamma)$-ideal element c and a γ-right-ideal element b of S such that $b\gamma c^m_\gamma \leq a \leq b \wedge c$.*

Proof. Let a be an $(1, m, \gamma)$-ideal element of S. Then $a \vee e\gamma a^m_\gamma$ and $a \vee a\gamma e$ are $(0, m, \gamma)$-ideal element and γ-right-ideal element of S respectively. Let $b = a \vee a\gamma e$ and $c = a \vee e\gamma a^m_\gamma$. Then

$$\begin{aligned}
b\gamma c^m_\gamma &= (a \vee a\gamma e)\gamma(a \vee e\gamma a^m)^m_\gamma \\
&= a\gamma(a \vee ea^m)^m_\gamma \vee a\gamma e\gamma(a \vee e\gamma a^m_\gamma)^m_\gamma \\
&= (a^2_\gamma \vee a\gamma e\gamma a^m_\gamma)\gamma(a \vee e\gamma a^m_\gamma)^{m-1}_\gamma \vee a\gamma e\gamma a^m_\gamma \text{ (by Lemma 1.2)} \\
&\leq (a^2_\gamma \vee a)\gamma(a \vee e\gamma a^m_\gamma)^{m-1}_\gamma \vee a \text{ (since } a\gamma e\gamma a^m_\gamma \leq a) \\
&\leq a\gamma(a \vee e\gamma a^m_\gamma)^{m-1}_\gamma \vee a \text{ (as } a^2_\gamma \leq a) \\
&\vdots \\
&\leq a\gamma(a \vee e\gamma a^m_\gamma)^{m-(m-1)}_\gamma \vee a \\
&= a\gamma(a \vee e\gamma a^m_\gamma) \vee a \\
&= (a^2_\gamma \vee a\gamma e\gamma a^m_\gamma) \vee a \\
&= (a^2_\gamma \vee a) \vee a \\
&= a \vee a \\
&= a.
\end{aligned}$$

As $a \leq b \wedge c$, we have $b\gamma c^m_\gamma \leq a \leq b \wedge c$.
Conversely assume that b is a γ-right-ideal element and c is a $(0, m, \gamma)$-ideal element of S such that $b\gamma c^m_\gamma \leq a \leq b \wedge c$. As $a \leq b \wedge c$, b is a γ-right-ideal element and $b\gamma c^m_\gamma \leq a$, we have $a\gamma e\gamma a^m_\gamma \leq (b \wedge c)\gamma e\gamma(b \wedge c)^m_\gamma \leq b\gamma e\gamma c^m_\gamma \leq b\gamma c^m_\gamma \leq a$. Therefore a is $(1, m, \gamma)$ ideal element of S. □

Definition 4.1. Let S be a poe-Γ-semigroup, $\gamma \in \Gamma$ and let a be any γ-left-ideal (γ-right-ideal) element of S. Then a is said to be a minimal γ-left-ideal (γ-right-ideal) element of S if for every γ-left-ideal (γ-right-ideal) element b of S, $b \leq a$ implies $b = a$. Further any non-zero γ-left-ideal (γ-right-ideal) element a of a poe-Γ-semigroup S with 0 is said to be 0-minimal if for each γ-left-ideal (γ-right-ideal) element b of S, $b \leq a$ implies $b = 0$ or $b = a$.

Similarly we may define a minimal and a 0-minimal (m, n, γ)-ideal element for each $m, n \in \mathbb{N}$.

Lemma 4.2. *Let S be a poe-Γ-semigroup with 0, $m \in \mathbb{N}, \gamma \in \Gamma$ and a be a 0-minimal γ-left-ideal element of S. Then a γ-subsemigroup element b of S is a $(0, m, \gamma)$-ideal element of S smaller than a if and only if either $b^m_\gamma = 0$ or $b = a$.*

Proof. Let S be a poe-Γ-semigroup with 0 and let b be a γ-subsemigroup element and an $(0, m, \gamma)$-ideal element of S smaller than a 0-minimal γ-left-ideal element a of S. As $e\gamma b^m_\gamma$ is a γ-left-ideal element of S and $e\gamma b^m_\gamma \leq b \leq a$, so by minimality of the 0-minimal γ-left-ideal element a of S, either $e\gamma b^m_\gamma = 0$ or $e\gamma b^m_\gamma = a$. If $e\gamma b^m_\gamma = a$, then $a = e\gamma b^m_\gamma \leq b$. Therefore $b = a$. In the other case when $e\gamma b^m_\gamma = 0$, as $e\gamma b^m_\gamma = 0 \leq b^m_\gamma$, b^m_γ is γ-left-ideal

element of S smaller then a (as b is a γ-subsemigroup element, so $b^m_\gamma \leq b$). Now, by minimality of the 0-minimal γ-left-ideal element a of S, either $b^m_\gamma = 0$ or $b^m_\gamma = a$. Since b is a γ-subsemigroup element, $b^m_\gamma \leq b$. Therefore $a = b^m_\gamma \leq b$, as required.
Converse is obvious. □

Lemma 4.3. *Let S be a poe-Γ-semigroup with 0, $m \in \mathbb{N}, \gamma \in \Gamma$ and let a be any γ-subsemigroup element of S. If a is a 0-minimal $(0, m, \gamma)$-ideal element of S, then either $a^m_\gamma = 0$ or a is a 0-minimal γ-left-ideal element of S.*

Proof. Let S be a poe-Γ-semigroup with 0 and let a be any γ-subsemigroup 0-minimal $(0, m, \gamma)$-ideal element of S. Therefore $a^m_\gamma \leq a$ and $e\gamma(a^m_\gamma)^m = e\gamma \underbrace{a^m_\gamma \gamma a^m_\gamma \gamma \ldots a^m_\gamma}_{m\text{-times}} \leq$

$a \underbrace{a^m_\gamma \gamma a^m_\gamma \gamma \ldots a^m_\gamma}_{m-1\text{-times}} \leq \underbrace{a\gamma a\gamma \ldots a}_{m\text{-times}} = a^m_\gamma$, it follows that a^m_γ is $(0, m, \gamma)$-ideal element of S.
As $a^m_\gamma \leq a$, by minimality of the 0-minimal $(0, m, \gamma)$-ideal element a of S, $a^m_\gamma = 0$ or $a^m_\gamma = a$. Suppose $a^m_\gamma = a$. Now $e\gamma a = e\gamma a^m_\gamma \leq a$ implies a is a γ-left-ideal element of S. To show next that a is 0-minimal γ-left-ideal element of S. So, take any γ-left-ideal element b of S such that $b \leq a$. As b is a $(0, m, \gamma)$-ideal element of S, a is a 0-minimal $(0, m, \gamma)$-ideal element of S and $b \leq a$, we have that either $b = 0$ or $b = a$. Hence a is a 0-minimal γ-left-ideal element of S. □

Lemma 4.4. *Let S be an le-Γ-semigroup, $m \in \mathbb{N}, \gamma \in \Gamma$ and let a be any γ-subsemigroup element of S. Then a is a $(0, m, \gamma)$-ideal element of S if and only if $b\gamma a \leq a$ for some $(0, m–1, \gamma)$-ideal element b $(a \leq b)$ of S.*

Proof. Suppose a is a $(0, m, \gamma)$-ideal element of S. Then, by Lemma 1.2, $e\gamma(a \vee e\gamma a^{m-1}_\gamma)^{m-1}_\gamma = e\gamma a^{m-1}_\gamma$. Thus $e\gamma(a \vee e\gamma a^{m-1}_\gamma)^{m-1}_\gamma \leq a \vee e\gamma a^{m-1}_\gamma$ i.e. $a \vee e\gamma a^{m-1}_\gamma$ is $(0, m–1, \gamma)$-ideal element of S. Let $b = a \vee e\gamma a^{m-1}_\gamma$. Then, as a is a γ-subsemigroup and a $(0, m, \gamma)$-ideal element of S, $b\gamma a = (a \vee e\gamma a^{m-1}_\gamma)\gamma a = a^2_\gamma \vee e\gamma a^m_\gamma \leq a^2_\gamma \vee a = a$, as required.

Conversely assume that b is a $(0, m–1, \gamma)$-ideal element of S and let a be any element of S such that $b\gamma a \leq a$ with $a \leq b$. Now $e\gamma a^m_\gamma \leq e\gamma b^{m-1}_\gamma \gamma a \leq b\gamma a \leq a$. Therefore a is a $(0, m, \gamma)$-ideal element of S. □

Lemma 4.5. *Let S be a poe-Γ-semigroup and let a be any element of S. Then a is a minimal $(m, m–1, \gamma)$-ideal element of S ($m \in \mathbb{N}, m \geq 2$) if and only if a is a minimal γ-bi-ideal element of S.*

Proof. Let S be a poe-Γ-semigroup and let a be a minimal $(m, m–1, \gamma)$-ideal element of S. As, by definition, $a^m_\gamma \gamma e\gamma a^{m-1}_\gamma \leq a$, we have $(a^m_\gamma \gamma e\gamma a^{m-1}_\gamma)^m_\gamma \gamma e\gamma$ $(a^m_\gamma \gamma e\gamma a^{m-1}_\gamma)^{m-1}_\gamma \leq a^m_\gamma \gamma e\gamma a^{m-1}_\gamma$. Therefore $a^m_\gamma \gamma e\gamma a^{m-1}_\gamma$ is a $(m, m–1, \gamma)$-ideal element of S such that $a^m_\gamma \gamma e\gamma a^{m-1}_\gamma \leq a$. So, by minimality of $(m, m–1, \gamma)$-ideal element a of $S, a^m_\gamma \gamma e\gamma a^{m-1}_\gamma = a$. Now

$$a\gamma e\gamma a = a^m_\gamma \gamma e\gamma a^{m-1}_\gamma \gamma e\gamma a^m_\gamma \gamma e\gamma a^{m-1}_\gamma \leq a^m_\gamma \gamma e\gamma a^{m-1}_\gamma \leq a.$$

So a is a γ-bi-ideal element of S. Next we show that a is a minimal γ-bi-ideal element of S. So take any γ-bi-ideal element b of S such that $b \leq a$. As $b^m_\gamma \gamma e\gamma b^{m-1}_\gamma \leq b$, b is $(m, m–1, \gamma)$-ideal element of S. Since a is a minimal $(m, m–1, \gamma)$-ideal element of S, $b = a$. Hence a is a minimal γ-bi-ideal element of S.

Conversely assume that a is a minimal γ-bi-ideal element of S. Therefore a is a $(m, m–1, \gamma)$-ideal element of S. To show that a is a minimal $(m, m–1, \gamma)$-ideal element of S, let b be any $(m, m-1, \gamma)$-ideal element of S such that $b \leq a$. As

$$(b_\gamma^m \gamma e \gamma b_\gamma^{m-1}) \gamma e \gamma (b_\gamma^m \gamma e \gamma b_\gamma^{m-1}) \leq b_\gamma^m \gamma e \gamma b_\gamma^{m-1},$$

$b_\gamma^m \gamma e \gamma b_\gamma^{m-1}$ is a γ-bi-ideal element of S. Since a is a minimal γ-bi-ideal element of S and $b_\gamma^m \gamma e \gamma b_\gamma^{m-1} \leq a$, we have $b_\gamma^m \gamma e \gamma b_\gamma^{m-1} = a$. As $b_\gamma^m \gamma e \gamma b_\gamma^{m-1} \leq b$, $a \leq b$. Therefore $b = a$. Hence a is a minimal $(m, m-1, \gamma)$-ideal element of S. □

Proposition 4.6. *Let S be a poe-Γ-semigroup, $m, n \in \mathbb{N}, \gamma \in \Gamma$ and let a be any γ-subsemigroup 0-minimal (m, n, γ)-ideal element of S. If $a_\gamma^2 \neq 0$, then a is a 0-minimal γ-bi-ideal element of S.*

Proof. Suppose there exists a γ-bi-ideal element $b \neq 0$ of S such that $b \leq a$. As b is a γ-bi-ideal element, b is an (m, n, γ)-ideal element. So, by minimality of the (m, n, γ)-ideal element a of S, we have $b = a$. Therefore a is a γ-bi-ideal element which is 0-minimal γ-bi-ideal element.

Now consider the case when there does not exists any γ-bi-ideal element $b \neq 0$ of S such that $b \leq a$. As a is an (m, n, γ)-ideal element, a_γ^2 is an (m, n, γ)-ideal element. Since $0 \neq a_\gamma^2 \leq a$ and a is a 0-minimal (m, n, γ)-ideal element, $a_\gamma^2 = a$. Therefore $a \gamma e \gamma a = a_\gamma^m \gamma e \gamma a_\gamma^n \leq a$ implies a is a γ-bi-ideal element of S. □

Theorem 4.7. *Let S be a poe-Γ-semigroup with 0, $m, n \in \mathbb{N}, \gamma \in \Gamma$. If S is (m, n, γ)-regular, x is a 0-minimal $(m, 0, \gamma)$-ideal element and y is a $(0, n, \gamma)$-ideal element of S such that $x \wedge y$ exists and $x \gamma y \leq x \wedge y$, then either $x \gamma y = 0$ or $x \gamma y$ is a 0-minimal (m, n, γ)-ideal element of S.*

Proof. Let x be a 0-minimal $(m, 0, \gamma)$-ideal element and y be a $(0, n, \gamma)$-ideal element of S such that $x \wedge y$ exists and $x \gamma y \leq x \wedge y$. Suppose that $x \gamma y \neq 0$. Then we show that $x \gamma y$ is a 0-minimal (m, n, γ)-ideal element of S. For this, we first show that $x \gamma y$ is an (m, n, γ)-ideal element of S. Now

$$\begin{aligned} (x\gamma y)_\gamma^m \gamma e \gamma (x\gamma y)_\gamma^n &\leq (x \wedge y)_\gamma^m \gamma e \gamma (x \wedge y)_\gamma^n \text{ (since } x\gamma y \leq x \wedge y) \\ &\leq x_\gamma^m \gamma e \gamma y_\gamma^n \\ &= x \gamma y_\gamma^n \text{ (as } x_\gamma^m \gamma e \leq x) \\ &= x \gamma y \text{ (by Corollary 2.5)}. \end{aligned}$$

So $x \gamma y$ is an (m, n, γ)-ideal element of S. Next we show that $x \gamma y$ is a 0-minimal (m, n, γ)-ideal element of S. Let z be any (m, n, γ)-ideal element of S such that $0 < z \leq x \gamma y$. Then $z \leq x$ and $z \leq y$. Since $z \leq z_\gamma^m \gamma e \gamma z_\gamma^n$, we have $z_\gamma^m \gamma e \neq 0$ and $e \gamma z_\gamma^n \neq 0$. As $z_\gamma^m \gamma e \leq x_\gamma^m \gamma e \leq x$, so by minimality of $(m, 0, \gamma)$-ideal element x, we have $z_\gamma^m \gamma e = x$. Similarly $e \gamma z_\gamma^n = y$. Therefore $z \leq x \gamma y = z_\gamma^m \gamma e \gamma e \gamma z_\gamma^n \leq z_\gamma^m \gamma e \gamma z_\gamma^n \leq z$. So $z = x \gamma y$. Hence $x \gamma y$ is a 0-minimal (m, n, γ)-ideal element of S. □

Theorem 4.8. *Let S be a poe-Γ-semigroup with 0 and $m, n \in \mathbb{N}, \gamma \in \Gamma$. If x is a 0-minimal $(m, 0, \gamma)$-ideal element and y is a $(0, n, \gamma)$-ideal element of S such that $x \wedge y$ exists and $x \wedge y \neq 0$, then $x \wedge y$ is a 0-minimal (m, n, γ)-ideal element of S.*

Proof. Let x be a 0-minimal $(m, 0, \gamma)$-ideal element and y be a $(0, n, \gamma)$-ideal element of S such that $x \wedge y$ exists and $x \wedge y \neq 0$. Then

$$(x \wedge y)_\gamma^m \gamma e \gamma (x \wedge y)_\gamma^n \leq x_\gamma^m \gamma e \gamma y_\gamma^n \leq x_\gamma^m \gamma y_\gamma^n \leq x_\gamma^m \gamma e \leq x.$$

Similarly $(x \wedge y)_\gamma^m \gamma e \gamma (x \wedge y)_\gamma^n \leq y$. Therefore $(x \wedge y)_\gamma^m \gamma e \gamma (x \wedge y)_\gamma^n \leq x \wedge y$ implies $x \wedge y$ is an (m, n, γ)-ideal element of S. The rest of the proof is similar to the proof of Theorem 3.7. □

5. CONCLUSIONS

In conclusion, this study provides a comprehensive examination of (m, n, γ)-regular le-Γ-semigroups through the lens of various types of ideal elements. The relationships between (m, n, γ)-ideal elements, $(m, 0, \gamma)$-ideal elements, and $(0, n, \gamma)$-ideal elements are clarified, contributing to a deeper understanding of their structures. The exploration of the (m, n, α, β)-regularity of the sets $I_{(m,n,\alpha,\beta)}$ and $Q_{(m,n,\alpha,\beta)}$ elucidates the characteristics of (m, n, α, β)-ideal and quasi-ideal elements, highlighting their significance in the theory of le-Γ-semigroups. Furthermore, the investigation into the 0-minimality of $(0, m, \gamma)$-ideal elements in both poe-Γ-semigroups and le-Γ-semigroups reveals important insights into their foundational properties. Overall, these findings pave the way for further research in the area of semigroup theory, offering potential avenues for exploring their applications in various mathematical contexts.

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