ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 7, Number 4 (2024), 455-465
DOI: 10.62072/acm.2024.070412
ISSN: 2582-0818

DIRECT PRODUCTS AND CUTS OF SOFT MULTIGROUPS

J. A. OTUWE*, I. A. ONYEOZILI*, M. M. MOGBONJU* AND D. A. OLUYORI †

ABSTRACT. In this paper, the concept of the direct product of a group is studied from the classical settings. We then propose the notion of the direct product of a soft multigroup and investigate some of its structural properties. Finally, the concept of upper and lower cuts of soft multigroup is introduced, exemplified and some related results are established.

1. INTRODUCTION

The methods in classical mathematics are not always successful in solving complicated obstacles in engineering, economics, medical sciences, biological science, social sciences, etc due to the fact that various uncertainties are typical for these barriers. Some special tools such as fuzzy set theory [2] and rough set theory [3] have been designed in the literature to handle the various kind of uncertainties. Molodtsov in [5] proposed a completely new approach called soft set theory for modeling vagueness and uncertainty. Soft set theory has potential applications in many fields including the smoothness of functions, game theory, operation research, Riemann integration, probability theory and measurement theory. Most of these applications have already been explored in the work of Molodtsov in [1]. Presently, works on soft set theory is progressing rapidly. The application of soft set theory to decision making problem was established in the work of Maji et al. Roy and Maji [6] proposed the concept of fuzzy soft set and provided its basic properties as an application in decision making under an imprecise environment. Aktas and Cagman in [8], studied the basic concept of soft set theory and compared the theory of soft set to fuzzy set and rough sets and also present an example to clarify their differences. In addition, Aktas and Cagman in [8] defined the concept of soft group as a parameterized family of subgroups. The theory of multiset is an important generalization of classical sets that an element can belong to a set only once. Multiset are very useful structures arising in many areas of mathematics and computer science such as data queries. For more, see Yager [2], Miyamoto [7], blizzard [4]. The concept of group theory is a very significant algebraic structure in modern mathematics. Several authors have established the notion of group theory in soft set [8], fuzzy set [15] and soft multisets [9] and so, studying the notion of group structure in soft multiset is very natural. Here we have explored the idea of group theory in the frame work of soft multiset.

2020 *Mathematics Subject Classification.* 20M99, 68T30, 03B65.
Key words and phrases. Group; Soft set; Soft multigroup; Direct product.
Received: November 20, 2024. Accepted: December 21, 2024. Published: December 31, 2024.
*Corresponding author.

2. Preliminaries

In this section, we present some existing definitions that are useful in the subsequent sections

Definition 2.1. [5] (**Soft Set**). A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \to P(X)$ and $A \subseteq E$.

Definition 2.2. [5] (**Soft Subset**). Let (F, A) and (H, B) be two soft sets over a common universe U, then we said that (H, B) is a soft subset of (F, A) if

i $B \subseteq A$ and
ii $H(e) \subseteq F(e) \; \forall \; e \; \in \; B$

We write $(H, B) \;\tilde{\subset}\; (F, A)$, (H, B) is said to be a soft super set of (F, A), if (F, A) is a subset of (H, B) we denote it by $(H, B) \;\tilde{\supset}\; (F, A)$.

Definition 2.3. [11] (**Soft Group**). Let G be a group and E be a set of parameters. For $A \subseteq E$, the pair (F, A) is called a soft group over G if and only if $F(\alpha) \leq G$ for all $\alpha \in A$ where F is a mapping of A into the set of subset of G.

Definition 2.4. [8] (**Soft Subgroup**). Let (F, A) and (H, K) be two soft group over G. Then (H, K) is a soft subgroup of (F, A) written as $(H, K) \;\overline{<}\; (A, A)$ if

i. $K \subseteq A$,
ii. $H(x) \leq F(x) \; \forall \; x \; \in \; K$

Definition 2.5. [2] (**Multiset**). Let X be a set. A multiset A is characterized by a count function

$$C_A(x) : X \to \mathbb{N}.$$

Such that for $x \;\in\; Dom(A)$ implies $A(x) = C_A(x) > 0$, where $C_A(x)$ denotes the number of times an object x occurs in A. Whenever $C_A(x) = 0$ implies $x \notin Dom(A)$ the set of all multiset is denoted by $MS(X)$.

The root of a multiset A, denoted by A, is defined as $A = \{x \in A : A(x) > 0\}$.

Definition 2.6. [2] (**Submultiset**). A multiset A is called a submultiset or a msubset of a multiset B denoted by $A \subseteq B$, if $C_A(x) \leq C_B(x)$, for all x.

A is a proper submultiset of B $(A \subset A)$ if $C_A(x) \leq C_B(x)$ for all x and there exists at least one x such that $C_A(x) < C_B(x)$.

Definition 2.7. [1] (**Multigroup**). Let X be a group. A multiset G over X is said to be a multigroup over X if the count function G or C_G satisfies the following two conditions:

i. $C_G(xy) \geq [C_G(x) \bigwedge C_G(y)], \; \forall \; x, \; y \in \; X,$
ii. $C_G(x^{-1}) \geq C_G(x), \; \forall \; x \in \; X,$

i.e., a multiset G is called a multigroup over X if $C_G(xy^{-1}) \geq [C_G(x) \wedge C_G(y)], \; \forall \; x, \; y \in X$. The set of all multigroups defined over X is denoted by $MG(X)$.

Definition 2.8. [11] (**Intersection and Union of Multigroup**). Let $\{A_i\}_{i \,\in\, I}, I = 1, 2, \cdots, n$ be an arbitrary family of multigroups of a group X. Then

$$C_{\cap A_i}(x) = \bigwedge C_{A_i}(x) \; x \;\in\; X.$$

$$C_{\cup A_i}(x) = \bigvee C_{A_i}(x) \; x \;\in\; X.$$

Definition 2.9. [1] (**Abelian Multigroup**). Let $A \in MG(X)$. Then A is said to be abelian or commutative if for all $x,\ y \in X,\ C_A(xy) = C_A(yx)$.

Definition 2.10. [10] (**Submultigroup**). Let $A \in MG(X)$. A submultiset B of A is called a submultigroup of A denoted by $B \subseteq A$ if B is a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \subset A$, if $B \subseteq A$ and $A \neq B$.

Definition 2.11. [13] (**Normal Submultigroup**). Let $A,\ B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if

$$C_A(xyx^{-1}) \geq C_A(y)\ \forall\ x,\ y \in X.$$

In general, $C_A(xyx^{-1}) = C_A(y)\ \forall\ x,\ y \in X$.

Definition 2.12. [11] (**Soft Multigroup**). Let X be a group, M be an mgroup over X and $A \subseteq E$ be a set of parameters. A soft Mset (F, A) drawn from M is said to be a soft multigroup (shortly soft Mgroup) over M if and only if $F(\alpha)$ is a submultigroup of M, for all $\alpha \in A$.

Definition 2.13. [11] (**Soft Submultigroup**). Let (F_1, A_1) and (F_2, A_2) be two soft multigroups over M. Then (F_1, A_1) is said to be a soft submgroup of (F_2, A_2) denoted by $(F_1, A_1)\tilde{\leq}(F_2, A_2)$ if $A_1 \subseteq A_2$ and $F_1(\alpha)$ is a submultigroup of $F_2(\alpha)$, $\forall\ \alpha \in A_1$.

Definition 2.14. [11] (**Soft Abelian Multigroup**). A soft multigroup (F, A) over a multigroup M of a group X is called a soft abelian multigroup if $F(\alpha)$ is an abelian submgroup of M, $\forall\ \alpha \in A$.

Definition 2.15. [11]. Let (F, A) be a soft multigroup over M. Then

(i) (F, A) is said to be an identity soft multigroup over M if $F(\alpha) = [C_M(e)]_e\ \forall\ \alpha \in A$, where e is the identity element of X.
(ii) (F, A) is said to be an absolute soft multigroup over M if $F(\alpha) = M,\ \forall\ \alpha \in A$.

3. Product of Soft Multigroups and Its Properties

Definition 3.1. Product of Soft Multigroups Let $M \in MG(X)$ and $N \in MG(X)$ and $N \in MG(Y)$. Suppose $(F, A) \in SMG(M)$ and $(G, B) \in SMG(N)$ respectively. Then the product of (F, A) and (G, B) denoted by $(F, A) \times (G, B)$ is a transformation

$$C_{(F,A)\times(G,B)} : X \times Y \Rightarrow \mathbb{N}$$

defined by $C_{F(\alpha)\times G(\beta)}((a, b)) = \wedge[C_{F(\alpha)}(a), C_{G(\beta)}(b)]\ \forall\ a \in X$ and $b \in Y$.

Example 3.2. Let $X = \{e^{'}, a, b, c\}$ be Klein-4 group and $Y = \{e, b\}$ be group such that $a^2 = b^2 = c^2 = e^{'}$ and (F, A) and (G, B) be soft multigroups over M and N respectively. Let $A = [\alpha_1, \alpha_2]$ and $B = [\beta_1, \beta_2]$. Suppose $M = [e^{'}, a, b, c]_{4,3,3,3}$ and $N = [e, b]_{5,4}$ and $(\alpha_1) = [e^{'}, a, b, c]_{3,2,2,2}$, $F(\alpha_2) = [e^{'}, a, b, c]_{4,3,2,2}$, $G(\beta_1) = [e, b]_{4,3}$ and $G(\beta_1) = [e, b]_{3,2}$. Clearly, $(F, A) \in SMG(M)$ and $(G, B) \in SMG(N)$ respectively.

Now, $N_* \times M_* = \{(e^{'}, e), (e^{'}, b), (a, e), (a, b), (b, e), (b, b), (c, e), (c, b)\}$ is a group such that;
$(e^{'}, b)^2 = (a, e)^2 = (a, b)^2, = (b, e)^2 = (b, b)^2 = (c, e)^2 = (c, b)^2 = (e^{'}, e)$ is the identity element of $N_* \times M_*$.

Therefore, $F(\alpha_1)\times G(\beta_1) = \{(e^{'},e)^3,(e^{'},b)^3,(a,e)^3,(a,b)^3,(b,e)^2,(b,b)^2,(c,e)^2,(c,b)^2\}$
And $F(\alpha_1)\times G(\beta_1) = \{(e^{'},e)^3,(e^{'},b)^2,(a,e)^3,(a,b)^2,(b,e)^2,(b,b)^2,(c,e)^2,(c,b)^2\}$.

$$(F,A)\times(G,B) = \{\alpha_1\beta_1\{(e^{'},e)^3,(e^{'},b)^3,(a,e)^3,(a,b)^3,(b,e)^2,(b,b)^2,(c,e)^2,(c,b)^2\},\{\alpha_2\beta_2$$

$$\{(e^{'},e)^3,(e^{'},b)^2,(a,e)^3,(a,b)^2,(b,e)^2,(b,b)^2,(c,e)^2,(c,b)^2\}\}\}.$$

Theorem 3.1. *Let X and Y be groups and suppose (F,A) and (G,B) are soft multigroups over M and N respectively. Then $(F,A)\times(G,B)$ is a soft multigroup of $X\times Y$.*

Proof. Let $(a,b)\in X\times Y$ and $a=(a_1,a_2)$ and $b=(b_1,b_2)$, we now have for every $\alpha\in A$ and $\beta\in B$

$$\begin{aligned}
C_{F(\alpha)\times G(\beta)}((a,b)) &= C_{F(\alpha)\times G(\beta)}((a_1,a_2)(b_1,b_2))\\
&= C_{F(\alpha)\times G(\beta)}((a_1b_1,a_2b_2))\\
&= C_{F(\alpha)}(a_1b_1)\wedge C_{G(\beta)}(a_2b_2)\\
&\geq C_{F(\alpha)}(a_1)\wedge C_{F(\alpha)}(b_1), C_{G(\beta)}(a_2)\wedge C_{G(\beta)}(b_2)\\
&= C_{F(\alpha)}(a_1)\wedge C_{G(\beta)}(a_2), C_{F(\alpha)}(b_1)\wedge C_{G(\beta)}(b_2)\\
&= C_{F(\alpha)\times G(\beta)}\left((a_1,a_2)\right)\wedge C_{F(\alpha)\times G(\beta)}\left((b_1,b_2)\right)\\
&= C_{F(\alpha)\times G(\beta)}(a)\wedge C_{F(\alpha)\times G(\beta)}(b).
\end{aligned}$$

Also,

$$\begin{aligned}
C_{F(\alpha)\times G(\beta)}(a^{-1}) &= C_{F(\alpha)\times G(\beta)}\left((a_1,a_2)^{-1}\right) = C_{F(\alpha)\times G(\beta)}\left((a_1^{-1},a_2^{-1})\right)\\
&= C_{F(\alpha)}(a_1^{-1})\wedge C_{G(\beta)}(a_2^{-1}) = C_{F(\alpha)}(a_1)\wedge C_{G(\beta)}(a_2)\\
&= C_{F(\alpha)\times G(\beta)}\left((a_1,a_2)\right) = C_{F(\alpha)\times G(\beta)}\left((a_1,a_2)\right)\\
&= C_{F(\alpha)\times G(\beta)}\left((a)\right).
\end{aligned}$$

Hence, $(F,A)\times(G,B)\in(X\times Y)$. □

Theorem 3.2. *Let $(F_1,A_1),(G_1,B_1)\in SMG(M)$ and let $(F_2,A_2)\in SMG(N)$ respectively such that $(F_1,A_1)\subseteq(G_1,B_1)$ and $(F_2,A_2)\subseteq(G_2,B_2)$. Suppose (F_1,A_1) and (F_2,A_2) are normal soft submultigroups of (G_1,B_1) and (G_2,B_2), then $(F_1,A_1)\times(F_2,A_2)$ is a normal soft submultigroup of $(G_1,B_1)\times(G_2,B_2)$.*

Proof. Since $(F_1,A_1)\times(F_2,A_2)$ is a soft multigroup over M and N and also $(G_1,B_1)\times(G_2,B_2)$ is a soft multigroup over M and N by definition 3.1 and theorem 3.1, we now show that $(F_1,A_1)\times(F_2,A_2)$ is a normal soft submultigroup of $(G_1,B_1)\times(G_2,B_2)$. Now let $(a,b)\in M\times N$ such that $a=(a_1,a_2)$ and $b=(b_1,b_2)$ then $\forall\ \alpha\in A_1$ and $\alpha\in A_2$, we have

$$\begin{aligned}
C_{F_1(\alpha)\times F_2(\alpha)}(aba^{-1}) &= C_{F_1(\alpha)\times F_2(\alpha)}\left((a_1,a_2)(b_1,b_2)(a_1,a_2)^{-1}\right)\\
&= C_{F_1(\alpha)\times F_2(\alpha)}\left((a_1,a_2)((b_1,b_2)(a_1^{-1},a_2^{-1})\right)\\
&= C_{F_1(\alpha)\times F_2(\alpha)}(a_1b_1a_1^{-1},a_2b_2a_2^{-1})\\
&= C_{F_1(\alpha)}(a_1b_1a_1^{-1})\wedge C_{F_2(\alpha)}(a_2b_2a_2^{-1})\\
&\geq C_{F_1(\alpha)}(b_1)\wedge C_{F_2(\alpha)}(b_2) = C_{F_1(\alpha)\times F_2(\alpha)}(aba^{-1})\geq C_{F_1(\alpha)\times F_2(\alpha)}(b).
\end{aligned}$$

Hence $(F_1,A_1)\times(F_2,A_2)$ is a normal soft submultigroup of $(G_1,B_1)\times(G_2,B_2)$. □

Theorem 3.3. *Let $(F,A) \in SMG(M)$ and $(G,B) \in SMG(N)$ and $(F,A) \times (G,B)$ is a soft multigroup of $M \times N$, then $\forall\ (a,b) \in M \times N$ the following holds $\forall\ \alpha \in A$ and $\forall\ \beta \in B$*

i. $C_{F(\alpha)\times G(\beta)}((a^{-1},b^{-1})) = C_{F(\alpha)\times G(\beta)}((a,b))$,
ii. $C_{F(\alpha)\times G(\beta)}((a,b)^n) \geq C_{F(\alpha)\times G(\beta)}((a,b))$,
iii. $C_{F(\alpha)\times G(\beta)}((e,e^{'})) \geq C_{F(\alpha)\times G(\beta)}((a,b))$.

Where e and $e^{'}$ are the identity of M and N respectively and $n \in \mathbb{N}$.

Proof. Suppose $a \in M, b \in N$ and $(a,b) \in M \times N$, we get
[i.]

$$\begin{aligned} C_{F(\alpha)\times G(\beta)}((a^{-1},b^{-1})) &= C_{F(\alpha)}(a^{-1}) \wedge C_{G(\beta)}(b^{-1}) \\ &= C_{F(\alpha)}(a) \wedge C_{G(\beta)}(b) = C_{F(\alpha)\times G_\beta}((a,b)) \end{aligned}$$

We now see clearly that $C_{F(\alpha)\times G(\beta)}((a^{-1},b^{-1})) = C_{F(\alpha)\times G(\beta)}((a,b))\ \forall\ (a,b) \in M \times N$.
[ii.]

$$\begin{aligned} C_{F(\alpha)\times G(\beta)}((a,b)^n) &= C_{F(\alpha)\times G(\beta)}((a^n,b^n)) = C_{F(\alpha)\times G(\beta)}((a^{n-1},b^{n-1})(a,b)) \\ &\geq C_{F(\alpha)\times G(\beta)}((a^{n-1},b^{n-1})) \wedge C_{F(\alpha)\times G(\beta)}((a,b)) \\ &\geq C_{F(\alpha)\times G(\beta)}((a^{n-2},b^{n-2})) \wedge C_{F(\alpha)\times G(\beta)}((a,b)) \wedge C_{F(\alpha)\times G(\beta)}((a,b)) \\ &\geq C_{F(\alpha)\times G(\beta)}((a,b)) \wedge C_{F(\alpha)\times G(\beta)}((a,b)) \wedge \cdots \wedge C_{F(\alpha)\times G(\beta)}((a,b)) \\ &= C_{F(\alpha)\times G(\beta)}((a,b)) \end{aligned}$$

[iii.]

$$\begin{aligned} C_{F(\alpha)\times G(\beta)}((e,e^1)) &= C_{F(\alpha)\times G(\beta)}((a,b)(a,b)^{-1}) \\ &= C_{F(\alpha)\times G(\beta)}((a,b)(a^{-1},b^{-1})) \\ &\geq C_{F(\alpha)\times G(\beta)}((a,b)) \wedge C_{F(\alpha)\times G(\beta)}((a^{-1},b^{-1})) \\ &= C_{F(\alpha)\times G(\beta)}((a,b))\ \forall\ (a,b) \in\ M \times N. \end{aligned}$$

$Hence,\ C_{F(\alpha)\times G(\beta)}((e,e^1)) \geq C_{F(\alpha)\times G(\beta)}((a,b))$.

□

Theorem 3.4. *Let (F,A) and (G,B) be soft multisets of $M \in MG(X)$ and $N \in MG(Y)$ respectively such that $\forall\ \alpha \in A$ and $\forall\ \beta \in B$, $C_{F(\alpha)}(x) \leq C_{G(\beta)}(e^{-1})\ \forall\ a \in M$, e^1 being the identity element of Y. If $(F,A) \times (G,B)$ is a soft multigroup of $M \times N$ is a soft multigroup of $M \times N$, then $(F,A) \in SMG(M)$.*

Proof. Let $(F,A) \times (G,B)$ be a soft multigroup over $M \times N$ and $a,b \in M$. Then $(a,e^1),(b,e^1) \in M \times N$. Now using the property in the statement of the theorem, we have $C_{F(\alpha)}(x) \leq C_{G(\beta)}(e^1)$.

Now $\forall\ a \in M$ we have

$$\begin{aligned} C_{F(\alpha)}(ab) &= C_{F(\alpha)}(ab) \wedge C_{G(\beta)}(e^1e^1) = C_{F(\alpha)\times G(\beta)}((abe^1e^1)) \\ &= C_{F(\alpha)\times G(\beta)}((a,e^1)(b,e^1)) \geq C_{F(\alpha)\times G(\beta)}((a,e^1)) \wedge C_{F(\alpha)\times G(\beta)}((b,e^1)) \\ &= C_{F(\alpha)}(a) \wedge C_{G(\beta)}(e^1),\ C_{F(\alpha)}(b) \wedge C_{G(\beta)}(e^1) = C_{F(\alpha)}(a) \wedge C_{G(\beta)}(b). \\ Also,\ C_{F(\alpha)}(a^{-1}) \wedge C_{G(\beta)}(e^{1-1}) &= C_{F(\alpha)\times G(\beta)}((a^{-1},e^{1-1})) \\ &= C_{F(\alpha)\times G(\beta)}((a,e^1)^{-1}) = C_{F(\alpha)\times G(\beta)}((a,e^1)) \\ &= C_{F(\alpha)}(a) \wedge C_{G(\beta)}(e^1) = C_{F(\alpha)}(a). \end{aligned}$$

Hence the desired result. □

Theorem 3.5. *Suppose (F,A), (H,C) are conjugate soft multigroup of $M \in MG(X)$ and (G,B), (V,D) are conjugate soft multigroup of $N \in MG(Y)$, then $(F,A) \times (G,B) \in SMG(M \times N)$ is a conjugate soft multigroup of $(H,C) \times (V,D) \in SMG(M \times N)$.*

Proof. Since (F,A) and (H,C) are conjugate, it implies that for every $g_1 \in M$, $\alpha \in A$ and $y \in C$, we have $C_{F(\alpha)}(a) = C_{H(\gamma)}(g_1ag_1^{-1})\ \forall\ a \in M$. Also, since (G,B) and (V,D) are conjugate, for $g_2 \in N$ and for every $\beta \in B$ and $\delta \in D$, we have

$$C_{G(\beta)}(b) = C_{V(\delta)}(g_2bg_2^{-1})\ \forall\ b \in N$$

So now,

$$\begin{aligned} C_{F(\alpha)\times G(\beta)}((a,b)) &= C_{F(\alpha)}(a) \wedge C_{G(\beta)}(b) \\ &= C_{H(\gamma)}(g_1ag_1^{-1}) \wedge C_{V(\delta)}(g_2bg_2^{-1}) \\ &= C_{H(\gamma)\times V(\delta)}((g_1ag_1^{-1}),(g_2bg_2^{-1})) \\ &= C_{H(\gamma)\times V(\delta)}((g_1^{-1}ag_1),((g_2^{-1}ag_2)) \\ &= C_{H(\gamma)\times V(\delta)}((g_1^{-1},g_2^{-1})(a,b)(g_1g_2)) \\ &= C_{H(\gamma)\times V(\delta)}((g_1g_2)^{-1})(a,b)(g_1g_2)). \\ Hence\ C_{F(\alpha)\times G(\beta)}((a,b)) &= C_{H(\gamma)\times V(\delta)}((g_1g_2)^{-1})(a,b)(g_1g_2)). \end{aligned}$$

□

Theorem 3.6. *Let X and Y be two groups $M \in MG(X)$ and $N \in MG(Y)$ respectively such that $(F,A) \in SMG(M)$ and $(G,B) \in SMG(N)$ then (F,A) and (G,B) are commutative if and only of $(F,A) \times (G,B)$ is a commutative soft multigroup over $M \times N$.*

Proof. Assume that (F,A) and (G,B) are commutative, we now show that $(F,A) \times (G,B)$ is commutative soft multigroup of $M \times N$. Let $(a,b) \in M \times N$ such that $a = (a_1,a_2)$ and $b = (b_1,b_2)$ so for every $\alpha \in A$ and $\beta \in B$, we have

$$\begin{aligned} C_{F(\alpha)\times G(\beta)}((a,b)) &= C_{F(\alpha)\times G(\beta)}((a_1,a_2)) = C_{F(\alpha)\times G(\beta)}((a_1b_1,a_2,b_2)) \\ &= C_{F(\alpha)}(a_1b_1) \wedge C_{G(\beta)}(a_2b_2) = C_{F(\alpha)}(b_1a_1) \wedge C_{G(\beta)}(b_2a_2) \\ &= C_{F(\alpha)\times G(\beta)}(b_1a_1,b_2a_2) \\ &= C_{F(\alpha)\times G(\beta)}(b_1,b_2)(a_1,a_2) = C_{F(\alpha)\times G(\beta)}((ba)) \end{aligned}$$

Hence, $(F,A) \times (G,B)$ is a commutative soft multigroup over $M \times N$.

Conversely, assume $(F,A)\times(G,B)$ is a commutative soft multigroup of $M\times N$, then it is quite clear that both (F,A) and (G,B) are commutative soft multigroup of M and N. □

4. CONCEPT OF CUTS OF SOFT MULTIGROUP AND ITS PROPERTIES

In this section the concept of cuts of soft multigroup is established and some of its properties are investigated.

Definition 4.1. (Strong and Weak Lower Cut). Let $(F,A)\in SMG(X)$. Then the sets $(F,A)_{[n]}$ and $(F,A)_{(n)}$ defined by

$$(F,A)_{[n]}=\{\alpha,\{x\}/C_{F(\alpha)}(x)\leq n, n\in\mathbb{N},\ \forall\ \alpha\in A,\ x\in X\}$$

and

$$(F,A)_{(n)}=\{\alpha,\{x\}/C_{F(\alpha)}(x)\leq n, n\in\mathbb{N},\ \forall\ \alpha\in A,\ x\in X\}$$

are called *strong and weak lower cuts* of (F,A).

Example 4.2. Let $X=\{e,a,b,c\}$ be Klein 4 group, $E=\{\alpha_1,\alpha_2,\alpha_3,\cdots\}$ and $A=\{\alpha_1,\alpha_2\}\subseteq E$. Suppose $F(\alpha_1)=\{e,a,c\}_{4,2,3}$ and $F(\alpha_2)=\{e,b\}_{5,3}$.
$(F,A)_{[1]}=\{e,a,c\}$, $(F,A)_{[2]}=\{e,a,c\}$, $(F,A)_{[3]}=\{e,c\}$, $(F,A)_{[4]}=\{e\}$, $(F,A)_{[5]}=\{\ \}$
$(F,A)_{(1)}=\{e,a,c\}$, $(F,A)_{(2)}=\{e,c\}$, $(F,A)_{(3)}=\{e\}$, $(F,A)_{(4)}=\{\ \}$
For $\alpha_2\in A$
$(F,A)_{[1]}=\{e,b\}$, $(F,A)_{[2]}=\{e,b\}$, $(F,A)_{[3]}=\{e,b\}$, $(F,A)_{[4]}=\{e\}$, $(F,A)_{[5]}=\{e\}$
$(F,A)_{[6]}=\{\ \}$, $(F,A)_{(1)}=\{e,b\}$, $(F,A)_{(2)}=\{e,b\}$, $(F,A)_{(3)}=\{e\}$, $(F,A)_{(4)}=\{e\}$
$(F,A)_{(5)}=\{\ \}$

Definition 4.3. (Strong and Weak Upper Cut). Let $(F,A)\in SMG(X)$. Then the sets $(F,A)^{[n]}$ and $(F,A)^{(n)}$ defined by

$$(F,A)^{[n]}=\{\alpha,\{x\}/C_{F(\alpha)}(x)\leq n, n\in\mathbb{N},\ \forall\ \alpha\in A,\ x\in X\}$$

and

$$(F,A)^{(n)}=\{\alpha,\{x\}/C_{F(\alpha)}(x)\leq n, n\in\mathbb{N},\ \forall\ \alpha\in A,\ x\in X\}$$

are called *strong and weak upper cuts* of (F,A).

Remark. *Let $(F,A),\ (G,B)\in SMG(M)$. Then the followings hold*

1. $(F,A)_{[n]}\subseteq(F,A)_{(n)}$ *and* $(F,A)^{[n]}\subseteq(F,A)^{(n)}$
ii. $(F,A)_{[n]}\subseteq(G,B)_{[n]}$, $(F,A)_{(n)}=(G,B)_{(n)}$, $(F,A)^{[n]}=(G,B)^{[n]}$ *and* $(F,A)^{(n)}=(G,B)^{(n)}$

Proposition 4.1. *Let $(F,A),(G,B)\in SMG(M)$ and $m,n\in\mathbb{N}$. Then the following holds*

i. $(F,A)_{[n]}\subseteq(F,A)_{[m]}$ *iff* $n\geq m$
ii. $(F,A)\subseteq(G,B)$ *iff* $(F,A)^{[n]}=(G,B)^{[n]}$.

Proof. i. For $a\in(F,A)_{[n]}$ implies $C_{(F,A)_{[n]}}(a)\geq n\geq m\ \forall\ \alpha\in A$. Hence $(F,A)_{[n]}\subset(F,A)_{[m]}$. Conversely for $(F,A)_{[n]}\subset(F,A)_{[m]}$ it is clear that $n\geq m$.

ii. Since we already know that $(F,A)_{[n]}\subset(F,A)_{[m]}$ implies that $C_{F(\alpha)}(a)\leq C_{G(\beta)}(a)\ \forall x\in X$. Now for $x\in(F,A)_{[n]}$ and for $x\in(G,B)_{[n]}$ it implies that $C_{G(\beta)}(a)\geq C_{F(\alpha)}(a)\geq n$ and so $(F,A)_{[n]}\subseteq(G,B)_{[n]}$.

Conversely, assume that $(G,B)_{[n]} \subseteq (F,A)_{[n]}$ it implies that $\forall\ \beta \in B$ and $\forall\ \alpha \in A\ C_{G(\beta)}(a) \leq C_{F(\alpha)}(a)\ \forall\ x \in X$ and so $x \in (G,B)$. □

Proposition 4.2. *Let $(F,A) \in SMG(M)$ and $a \in X$ be a fixed element then, $(F,A)_{[n]}a = [(F,A)a]_{[n]}$.*

Proof. Let $a \in X$ and $n \in \mathbb{N}$. We define

$$\begin{aligned}(F,A)_{[n]}a &= \{\alpha \in A, b \in X \mid C_{F(\alpha)}(b) \geq n\}a = \{ba \in X \mid C_{F(\alpha)}(a) \geq n\} \\ &= \{\forall\alpha \in A, g \in X | C_{F(\alpha)}(ga^{-1}) \geq n, ab = g\ \forall\ a,b \in X\} \\ &= \{g \in X \mid C_{F(\alpha)a}(g) \geq n,\ \forall\ a \in X\} = [(F,A)a]_{[n]}.\end{aligned}$$

□

Proposition 4.3. *Let $\{(F,A_i)\}_{i\in I}$ be a family of soft multigroups over M. For $n \leq C_{F(\alpha_i)}(e)\ \forall\ \alpha_i \in A$*

$$\cap_{i\in I}[(F,A)_i]_{[n]}\ is\ a\ soft\ subgroup\ of\ X$$

Proof. Assume that $C = \cap_{i\in I}(F,A)_i$, then $C_G(x) = C_G(e) = C_G(xx^{-1}) = C_{F(\alpha_i)}(xx^{-1}) \geq \wedge_{i\in I}C_{F(\alpha_i)}(x) \geq n$. Now, let $x,y \in X$ then we have $\forall\ \alpha_i \in A$

$$C_G(xy) = \bigwedge_{i\in I} C_{F(\alpha_i)}(xy) \geq n \geq \bigwedge_{i\in I}\left(C_{F(\alpha_i)}(x), C_{F(\alpha_i)}(y)\right) \geq n$$

$$\wedge_{i\in I}C_{F(\alpha_i)}(x) \geq n,\ \wedge_{i\in I}C_{F(\alpha_i)}(y) \geq n = (C_G(x) \wedge C_G(x)) \geq n$$

This implies that $C_G(x) \geq n$ and $C_G(x) \geq n$ and so, $\forall\ x,y \in \cap_{i\in I}[(F,A)_i]_{[n]}$ Also,

$$\begin{aligned}C_G(xy^{-1}) &= \wedge C_{F(\alpha_i)}(xy^{-1}) \geq n \geq \wedge\left(C_{F(\alpha_i)}(x) \wedge C_{F(\alpha_i)}(y)\right) \geq n \\ &= (C_G(x) \wedge C_G(x)) \geq n.\end{aligned}$$

Hence, $xy^{-1} \in \cap_{i\in I}[(F,A)_i]_{[n]}$. Therefore, $\cap_{i\in I}[(F,A)_i]_{[n]}$ is a soft subgroup of X. □

Proposition 4.4. *Let (F,A) and (G,B) be soft multigroups over M. Then,*

i. $[(F,A) \cup (G,B)]_n\ (G,B)_n = (F,A)_n\ (G,B)_n$
ii. $[(F,A) \cup (G,B)]_n\ (F,A)_n = (G,B)_n\ (F,A)_n$

Proof. i. Let $a \in [(F,A)\cup(G,B)]_n\ (G,B)_n$, then $a \in [(F,A)\cup(G,B)]_n\ (G,B)_n \wedge a \notin (G,B)_n$

$\Leftrightarrow$ $(a \in (F,A)_n \forall\ a \in (G,B)_n) \wedge\ a \notin (G,B)_n$
$\Leftrightarrow$ $[(\forall\ \alpha \in A, a \in M, (C_{F(\alpha}(a) \geq n) \vee (\forall\beta \in B,\ a \in M, (C_{G(\beta}(a) \geq n)]$
$\wedge$ $(\forall\beta \in B, a \in M, C_{G(\beta)}(a) < n)$
$\Leftrightarrow$ $[(\forall\alpha \in A, a \in M, (C_{F(\alpha)}(a) \geq n) \wedge (\forall\beta \in B, a \in M, C_{G(\beta)}(a) < n)]$
$\vee$ $[(\forall\beta \in B, a \in M, (C_{G(\beta)}(a) \geq n) \wedge (\forall\beta \in B, a \in M, C_{G(\beta)}(a) < n)]$
$\Leftrightarrow$ $[(\forall\alpha \in A, a \in M, (C_{F(\alpha)}(a) \geq n) \wedge (\forall\beta \in B, a \in M, C_{G(\beta)}(a) < n)]$
$\Leftrightarrow$ $[(\forall\alpha \in A, a \in M, (C_{F(\alpha)}(a) \geq n) \wedge (\forall\beta \in B, a \in M, C_{G(\beta)}(a) < n)]$
$\Leftrightarrow$ $a \in (F,A)_n \wedge a \notin (G,B)_n$
$\Leftrightarrow$ $a \in (F,A)_n\ (G,B)_n.Thus,\ [(F,A) \cup (G,B)]_n\ (G,B)_n = (F,A)_n\ (G,B)_n$.

ii. Let $a \in [(F,A) \cup (G,B)]_n\ (F,A)_n$, then $a \in\ [(F,A) \cup (G,B)]_n\ (F,A)_n$
$\Leftrightarrow [(\forall\ \alpha \in A, a \in M, C_{F(\alpha)}(a) \geq n) \vee (\forall\ \beta \in B, a \in M, C_{G(\beta)}(a) \geq n)]$
$\wedge(\forall\alpha \in A, a \in M, C_{F(\alpha)}(a) < n)$
$\Leftrightarrow [(\forall\ \alpha \in A, a \in M, C_{F(\alpha)}(a) \geq n) \vee (\forall\ \alpha \in A, a \in M, C_{F(\alpha)}(a) < n)]$

$\vee[(\forall\, \beta \in B, a \in M, C_{G(\beta)}(a) \geq n) \wedge (\forall\, \alpha \in A, a \in M, C_{F(\alpha)}(a) < n)]$
$\Leftrightarrow [(\forall\, \alpha \in A, a \in M, C_{F(\alpha)}(a) \geq n) \wedge (\forall\, \alpha \in A, a \in M, C_{F(\alpha)}(a) < n)] \vee \emptyset$
$\Leftrightarrow [(\forall\, \alpha \in A, a \in M, C_{F(\alpha)}(a) \geq n) \wedge (\forall\, \alpha \in A, a \in M, C_{F(\alpha)}(a) < n)]$
$\Leftrightarrow a \in (G, B)_n \wedge a \notin (F, A)_n$
$\Leftrightarrow a \in (G, B)_n\ (F, A)_n$. Thus, $[(F, A) \cup (G, B)]_n\ (F, A)_n = (G, B)_n\ (F, A)_n$.

□

Proposition 4.5. *Let* $(F, A), (G, B) \in SMG(M)$ *and* $m, n \in \mathbb{N}$. *Then the followings holds*

i. $(F, A)_{[n]} \subseteq (F, A)_{[m]}$ *iff* $n \geq m$
ii. $(F, A) \subseteq (G, B)$ *iff* $(F, A)^n < (G, B)^n$

Proof. i. For any $x \in (F, A)_{[n]} \Rightarrow C_{(F,A)_{[n]}} \geq n$ since $n \geq m$ $C_{F(\alpha)}(x) \geq n \geq m$ for every $\alpha \in A$. Hence $(F, A)_{[n]} \subseteq (F, A)_{[m]}$. Conversely, for $(F, A)_{[n]} \subseteq (F, A)_{[m]}$, it is clear that $n \geq m$.

ii. Since we know that $(F, A)_{[n]} \subseteq (G, B)_{[n]}$ it implies that $C_{F(\alpha)}(x) \leq C_{G(\beta)}(x)\ \forall\ x \in X$. Now for $x \in (F, A)_{[n]}$ and for $x \in (G, B)_{[n]}$ implies $C_{G(\beta)}(x) \geq C_{F(\alpha)}(x) \geq n$ and so $(F, A)_{[n]} \subseteq (G, B)_{[n]}$.

Conversely, assume $(G, B)_{[n]} \subseteq (F, A)_{[n]}$ it implies that for every $\beta \in B,\ \alpha \in A$, $C_{G(\beta)}(x) \leq C_{F(\alpha)}(x)\ \forall\ x \in X$. So for $x \in (G, B)_{[n]}$ and $x \in (F, A)_{[n]}$, we have $C_{F(\alpha)}(x) \geq C_{G(\beta)}(x) \geq n$ and so, $(G, B)_{[n]} \subseteq (F, A)_{[n]}$. Hence the desired result.

□

Remark. *Let* $(F, A), (G, B) \in SMG(M)$ *and* $m, n \in N$. *Then the following holds:*

i. $(F, A)^{[n]} \subseteq (F, A)^{[m]}$ *if and only if* $n \geq m$.
ii. $(F, A)^{[n]}, (F, A)^{[m]}$

Proposition 4.6. *Let* $(F, A), (G, B) \in SMG(M)$ *and* $n \in N$. *Then the followings hold*

i. $[(F, A) \cap (G, B)]_{[n]} = (F, A)_{[n]} \cap (G, B)_{[n]}$
ii. $[(F, A) \cup (G, B)]_{[n]} = (F, A)_{[n]} \cup (G, B)_{[n]}$

Proof. i. Assume that $(F, A), (G, B) \in SMG(M)$ it implies that $(F, A) \cap (G, B) \subseteq (F, A)$ and $(F, A) \cap (G, B) \subseteq (G, B)$. By proposition 4.5 $[(F, A) \cap (G, B)]_{[n]} \subseteq (F, A)_{[n]}$ and $[(F, A) \cap (G, B)]_{[n]} \subseteq (G, B)_{[n]}$. It implies that $[(F, A) \cap (G, B)]_{[n]} = (F, A)_{[n]} \cap (G, B)_{[n]}$. Now suppose $y \in (F, A)_{[n]} \cap (G, B)_{[n]}$ it implies $y \in (F, A)_{[n]}$ and $y \in (G, B)_{[n]}$. Then,

$$\begin{aligned} (F, A)_{[n]} \cap (G, B)_{[n]} &= \{\alpha \in A, y \in X / C_{F(\alpha)}(y) \geq n\} \cap \{\beta \in B, y \in X / C_{G(\beta)}(y) \geq n\} \\ &= \{\alpha \in A, \beta \in B, y \in X / C_{F(\alpha)}(y) \geq n \wedge C_{G(\beta)}(y) \geq n\} \\ &= \{\alpha \in A, \beta \in B, y \in X / C_{F(\alpha) \cap G(\beta)}(y) \geq n\} \\ &= \{\alpha \in A, \beta \in B, y \in X / C_{F(\alpha) \cap (G,B)}(y) \geq n\} \subseteq [(F, A) \cap (G, B)]_{[n]} \end{aligned}$$

Consequently, $y \in [(F, A) \cap (G, B)]_{[n]}$ implies $y \in (F, A)_{[n]} \cap (G, B)_{[n]}$. Hence, $[(F, A), (G, B)]_{[n]} = (F, A)_{[n]} \cap (G, B)_{[n]}$.

ii. Since $(F, A), (G, B) \in SMG(M)$, it is clear that $(F, A) \subseteq (F, A) \cup (G, B)$ and $(G, B) \subseteq (F, A) \cup (G, B)$. Now considering Proposition 3.5 it implies that $[(F, A)_{[n]} \subseteq (F, A) \cup (G, B)]_{[n]}$ and $(G, B)_{[n]} \subseteq [(F, A) \cup (G, B)]_{[n]}$.

Also, $y \in [(F,A) \cup (G,B)]_{[n]} \Rightarrow C_{(F,A)\cup(G,B)}(y) \geq n$. That is,
$[(F,A) \cup (G,B)]_{[n]} = \{y \in X/C_{(F,A)\cup(G,B)}(y) \geq n\} = \{y \in X/C_{(F,A)}(y) \cup C_{(G,B)}(y) \geq n\}$
$= \{y \in X/C_{(F,A)}(y) \geq n \cup C_{(G,B)}(y) \geq n\} = \{y \in X/C_{(F,A)}(y) \geq n\} \cup \{y \in X\ C_{(G,B)}(y)\}$
$\subseteq (F,A)_{[n]} \cup (G,B)_{[n]}$.
That is, $y \in (F,A)_{[n]}$ and $y \in (G,B)_{[n]}$. Hence, $[(F,A) \cup (G,B)]_{[n]} = (F,A)_{[n]} \cup (G,B)_{[n]}$.

□

Proposition 4.7. *Suppose* $(F,A) \in SMG(M)$ *and* $(G,B) \in SMG(N)$*, then for all* $n \in \mathbb{N}$*, we have*

i. $((F,A) \times (G,B))^{[n]} = (F,A)^{[n]} \times (G,B)^{[n]}$

ii. $((F,A) \times (G,B))_{[n]} = (F,A)^{[n]} \times (G,B)_{[n]}$

Proof. i. Assume that $(a,b) \in ((F,A) \times (G,B))^{[n]}$, then by definition 3.2, we have $C_{(F,A)\times(G,B)}((a,b))^{[n]} = C_{(F,A)}(a) \wedge C_{(G,B)}(b) > n$. It implies that $C_{(F,A)}(a) > n$ and $C_{(G,B)}(b) > n$. Hence, $a \in (F,A)^{[n]}$ and $b \in (G,B)^{[n]}$. Thus, $(a,b) \in (F,A)^{[n]} \times (G,B)^{[n]}$.

Now, assume $(a,b) \in (F,A)^{[n]} \times (G,B)^{[n]}$, then $C_{(F,A)}(a) > n$ and $C_{(G,B)}(b) > n$. That is, $(C_{(F,A)}(a) \wedge C_{(G,B)}(b) > n)$ which implies that $C_{(F,A)\times C_{(G,B)}}(a,b) > n$ and so, $(a,b) \in (F,A)^{[n]} \times (G,B)^{[n]}$. Therefore, $((F,A) \times (G,B))^{[n]} = (F,A)^{[n]} \times (G,B)^{[n]}$.

ii. Let $(a,b) \in ((F,A) \times (G,B))_{[n]}$. By definition 3.1, we have $C_{(F,A)\times(G,B)_{[n]}}(a,b) = (C_{(F,A)}(a) \wedge C_{(G,B)}(b)) \geq n$, it implies that $C_{(F,A)}(a) \geq n$ and $C_{(G,B)}(b) \geq n$, therefore $a \in (F,A)_{[n]}$ and $b \in (G,B)_{[n]}$.

Also, suppose that $(a,b) \in (F,A)_{[n]} \times (G,B)_{[n]}$, then $C_{(F,A)}(a) \geq n$ and $C_{(G,B)}(b) \geq n$. That is, $C_{(F,A)}(a) \wedge C_{(G,B)}(b) \geq n$ which implies that $C_{C_{(F,A)} \times C_{G,B}}(a,b) \geq n$ and so $(a,b) \in ((F,A) \times (G,B))_{[n]}$
Hence $((F,A) \times (G,B))_{[n]} = (F,A)_{[n]} \times (G,B)_{[n]}$.

□

5. Conclusions and/or Discussions

We explore the concept of soft group and soft multisets in the frame work of soft multigroups and propose the idea of direct product and cut of soft multigroup. Some properties of direct product and cuts of soft multigroups which enhanced the formulation of some related results were established. In a nut shell, the main contribution of this paper is the introduction of direct products and cuts of Soft multigroups respectively. We recommends the concept of direct product of fuzzy soft multigroup, cuts of fuzzy soft multigroups and factor fuzzy soft multigroups for future work since it remain challenging in the frame work of fuzzy soft multigroups.

6. Acknowledgements

The authors are thankful to the Editor-in-Chief, Prof. G. Muhiuddin for the technical comments and to the anonymous reviewers for their relevant suggestions.

REFERENCES

[1] M. Dresher, O. Ore. Theory of Multigroups, American Journal of Mathematics, 60 (1934), p 705 -733.
[2] R. R. Yager. On the Theory of Bags, International J. of Gen. Systems, 13 (1), (1989), p 23 - 37.
[3] W. D. Blizzard. Multiset Theory, Notre Dame Journal of Formal Logic, 30 (1989), p 36 - 66.
[4] W. D. Blizzard. The development of Multiset Theory, Modern logic, 4 (1991), p 319 - 352.
[5] D. Molodtsov. Soft set theory - first results, An International Journal of Computer and Mathematics with Application, 37 (1999), 9 - 31.
[6] P. K. Maji, R. Biswas, and A. Roy. Soft Set Theory. Comput. Math. App., 45 (2003), p 555-562.
[7] S. Miyamoto. Generalization of Multisets. In: In M. Iniguichi S. Tsumito eds. Rough Set Theory and Granular Computing, Springer, (1987), p 59-68.
[8] H. Aktas, N. Cagman. Soft Sets and Soft Groups. Information Sciences, an International Journal of Mathematics, 177(2007), p 2726-2735.
[9] P. Majumdar, S. K. Samanta. Soft Sets and Soft theory. Journal of Mathematics and Computer science, 2(6), (2012), p 1700-1711.
[10] S. K. Nazmul, P. Majumdar, S. K. Samanta. On multisets and Multigroups. Annals of fuzzy mathematics and informatics, 6(3), (2003), p 643-656.
[11] S. K. Nazmul, S. K. Samanta. On Soft Multigroups. Annals of Fuzzy Mathematics and Informatics, 10(2), (2015), p 1-14.
[12] J. A. Awolola, A. M. Ibrahim. Some results on Multigroups. Quasigroups and Related Systems, 24(2016), p 8169-177.
[13] P. A. Ejegwa, A. M. Ibrahim. Normal submultigroups and comultisets of a multigroup. Quasigroups and Related Systems, 25(2017), p 231-244.
[14] I. I. Ahmed. Some properties of Soft Multiset. Science World Journal, 13(4), (2018) p 44 - 46.
[15] A. Rosenfeld. Fuzzy groups. Journal of Mathematical Analysis and Application, 35(2018), p 512 - 517.
[16] Z. Pawlat. Rough set theory and its applications. Journals of telecommunications and information technology, 3(2002), p 7 - 10. doi: 10.22636/jtit.2002.140.

J. A. OTUWE*
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABUJA, NIGERIA.
Email address: talk2josephotuwe2016@gmail.com

I. A. ONYEOZILI*
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABUJA, NIGERIA.
Email address: ijeozili@gmail.com

M. M. MOGBONJU*
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ABUJA, NIGERIA.
Email address: morufu.mogbonju@uniabuja.edu.ng

D. A. OLUYORI †
DEPARTMENT OF APPLIED MATHEMATICS, FEDERAL UNIVERSITY OF TECHNOLOGY, BABURA, JIGAWA STATE, NIGERIA
Email address: daoluyori.mth@futb.edu.ng