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ON THE RIEMANN PROBLEM

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ABSTRACT. The Riemann problem is stated as follows: find an analytic in a domain $D_+ \cup D_-$ function $\Phi(z)$ such that $(*)$ $\Phi^+(t) = G(t)\Phi^-(t) + g(t),\ t \in S$. Here S is the boundary of D_+, D_- complements the complex plane to $D_+ \cup S$, the functions $G = G(t)$ and $g = g(t)$ belong to $H^\mu(S)$, the space of Hölder-continuous functions. The theory of problem (*) is developed also for continuous G. If $G = 1$, $S \in C^\infty$ and g is a tempered distribution, then problem $(*)$ has a solution in tempered distributions. It is proved that problem $(*)$ for $G \in L_p(S)$ and g a tempered distribution does not make sense. It is proved that if $G \in C^\infty(S)$, $G \neq 0$ on S, and g is a distribution of the class $\mathcal{D}'$, then the Riemann problem makes sense and a method for solving this problem is given. It is proved that if $S = \mathbb{R} = (-\infty, \infty)$ and $G \in L_p(\mathbb{R})$, where $p \geq 1$ is a fixed number, then $|\ln G|$ does not belong to $L_q(\mathbb{R})$ for any $q \geq 1$.

1. INTRODUCTION

There is a large literature on the boundary value problems for analytic functions, see [1], [5]. The problem discussed in our paper is the Riemann problem. It consists of finding a piece-wise analytic function $\Phi(z)$ from the boundary condition:

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in S. \tag{1.1}$$

Here $\Phi(z)$ is analytic in the domain $D_+ \cup D_-$, S is the boundary of D_+, D_- is the complement to $D_+ \cup S$ in the complex plane. For simplicity we assume that D_+ is a bounded connected domain and $S \in C^\infty$ is its closed smooth boundary. The $\Phi^+(t)$ is the limiting value of $\Phi(z)$ as $z \to t$, $t \in S$, z tends to t along the interior normal N_t to S at the point t. Similarly $\Phi^-(t)$ is defined.

Our aim is to investigate under what non-smooth assumptions on G and g problem (1.1) can or cannot make sense.

Let us recall the known results about Riemann problem (1.1), see [1], [5]. If $G = 1$, then

$$\Phi(z) = \frac{1}{2\pi i}\int_S \frac{g(s)ds}{s - z}. \tag{1.2}$$

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This formula solves problem (1.1) even if g is a tempered distribution, see [7]. It is proved in [6], [7], that

$$\Phi^+(t) = \Phi(t) + \frac{g(t)}{2}, \quad \Phi^-(t) = \Phi(t) - \frac{g(t)}{2}, \tag{1.3}$$

and

$$\Phi(t) = Bg := \frac{1}{i\pi}\int_S \frac{g(s)ds}{s-t}. \tag{1.4}$$

The operator B on the space of tempered distributions $\mathcal{S}'$ is defined, see [7], as

$$(Bg, \psi) = -(g, B\psi), \quad \psi \in C^\infty(S), \tag{1.5}$$

where ψ is the test function and the (g, ψ) is the value of the linear functional on $C^\infty(S)$ on the test function ψ. The sign minus in formula (1.5) should be clear because for g smooth the necessity of this sign is obvious. If we assume that the boundary S is infinitely smooth, then $B\psi \in C^\infty(S)$ provided that $\psi \in C^\infty(S)$.

If $G \in H^\mu$ and $g \in H^\mu$, where H^μ is the space of Hölder-continuous functions ([1]), then to solve problem (1.1) one proceeds as in [1]. Define the index of G as follows:

$$ind\, G = \nu := \frac{1}{2\pi}\Delta_S argG := \frac{1}{2\pi}\int_S d\, argG, \tag{1.6}$$

where $arg\, G$ is the argument of G.

Assume for simplicity that $\nu = 0$ and G does not vanish on S. The case of ν equal to some integer is reduced to the case $\nu = 0$ as in [1]. If $\nu = 0$, one can solve problem (1.1) with $g = 0$ by taking logarithm and getting problem (1.1) with $G = 1$:

$$\ln \Phi^+ = \ln \Phi^- + \ln G. \tag{1.7}$$

Since $ind\, G = \nu = 0$, the function $\Phi(z)$ does not have zeros in D^+ and in D^-, so the function $\ln \Phi(z)$ is analytic in D^+ and in D^-. Therefore,

$$\ln \Phi(z) = \frac{1}{2\pi i}\int_S \frac{\ln G(s)ds}{s-z} := \Gamma(z). \tag{1.8}$$

Let us denote

$$X(z) := e^{\Gamma(z)}, \quad X^\pm(t) := e^{\Gamma_\pm(t)}. \tag{1.9}$$

Then

$$G(t) = \frac{X^+(t)}{X^-(t)}. \tag{1.10}$$

Since $\Gamma(\infty) = 0$, one has $X(\infty) = 1$. Problem (1.1) can be written as

$$\frac{\Phi^+}{X^+} = \frac{\Phi^-}{X^-} + \frac{g}{X^+}. \tag{1.11}$$

If $X^\pm$ are bounded and $\min_{s\in S}|X^\pm(s)| \geq \epsilon > 0$, then $\frac{g(t)}{X^+(t)} \in L_p(S)$ if $g \in L_p(S)$. But if g is a tempered distribution on S, then $\frac{g(t)}{X^+(t)}$ is not a tempered distribution on S because $\frac{\psi}{X^+(t)}$ is not, in general, a $C^\infty(S)$ function, so it is not a test function for a tempered distribution, see [3] for the properties of tempered distributions.

The conclusion is:

If g is a tempered distribution, then $\frac{g(t)}{X^+(t)}$ is not, in general, a tempered distribution on S if X^+ is not a $C^\infty(S)$ function bounded away from zero.

Under the same assumptions, namely, $S \in C^\infty(S)$, $G \in C^\infty(S)$ and $X^+ \neq 0$ on S, the above argument shows that the Riemann problem makes sense for $g \in \mathcal{D}'$.

The distributions of classes S' and $\mathcal{D}'$ are studied in [2]

2. A special result

Consider the following question:

Suppose $S = (-\infty, \infty) := \mathbb{R}$ and D_+ is the upper half-plane of the complex plane.

Problem. *If $G \in L_p := L_p(\mathbb{R})$, $p \geq 1$ is a fixed number, can $\ln G$ belong to $L_q = L_q(\mathbb{R})$ for some $q \geq 1$?*

Theorem 1. The answer is no.

Proof. If $|G| \in L_p$, then for an arbitrary fixed $\epsilon \in (0,1)$ one has

$$meas E_\epsilon = \infty, \tag{2.1}$$

where

$$E_\epsilon = meas\{x, x \in \mathbb{R}, |G(x)| \leq \epsilon\}. \tag{2.2}$$

Therefore,

$$\ln |G(x)| \leq \ln \epsilon = -\ln \frac{1}{\epsilon}. \tag{2.3}$$

So,

$$|\ln |G|| = -\ln |G| \geq \ln \frac{1}{\epsilon}. \tag{2.4}$$

Consequently,

$$\int_{\mathbb{R}} |\ln |G(s)||^q ds \geq \int_{E_\epsilon} |\ln |G(s)||^q ds \geq (\ln \frac{1}{\epsilon})^q meas\, E_\epsilon = \infty. \tag{2.5}$$

This means that $|\ln |G||$ does not belong to L_q for any $q \geq 1$.

Theorem 1 is proved. □

3. Conclusion and Discussion

The Riemann problem is stated as follows: find an analytic in a domain $D_+ \cup D_-$ function $\Phi(z)$ such that $(*)$ $\Phi^+(t) = G(t)\Phi^-(t) + g(t)$ $t \in S$. Here S is the boundary of D^+, D^- complements the complex plane to $D^+ \cup S$, the functions $G = G(t)$ and $g = g(t)$ belong to $H^\mu(S)$, the space of Hölder-continuous functions. The theory of problem (*) is developed also for continuous G.

If $G = 1$ and g is a tempered distribution, then problem $(*)$ has a solution in tempered distributions. The generalization to $G \in L_p(S)$ and g a tempered distribution is not possible.

A similar result is valid for $G \in C^\infty(S)$, $G(s) \neq 0$ on S, and g is a tempered distribution or $g \in \mathcal{D}'$.

It is proved that if $S = \mathbb{R} = (-\infty, \infty)$ and $G \in L_p(\mathbb{R})$, where $p \geq 1$ is a fixed number, then $|\ln\, G|$ does not belong to $L_q(\mathbb{R})$ for any $q \geq 1$.

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