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# BI-QUASI-INTERIOR IDEALS

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ABSTRACT. In this paper, as a further generalization of ideals, we introduce the notion of a bi-quasi-interior ideal as a generalization of ideals, right ideals, left ideals, quasi ideals, bi ideals, interior ideals and quasi interior ideals of a $\Gamma-$semigroup and study the properties of bi-quasi-interior ideals of a $\Gamma-$semigroup.

## 1. INTRODUCTION

In 1995, M.Murali Krishna Rao [9,17,18] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$ rings, ternary semirings and semirings. As a generalization of ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [24] in 1964. In 1981. Sen [25] introduced the notion of a $\Gamma-$semigroup as a generalization of a semigroup. The notion of a ternary algebraic system was introduced by Lehmer [7] in 1932. Lister [8] introduced a ternary ring. The set of all negative integers $Z$ is not a semiring with respect to usual addition and multiplication but $Z$ forms a $\Gamma-$semiring where $\Gamma = Z$. The important reason for the development of a $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. The notion of a semiring was introduced by Vandiver in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. Many mathematicians proved important results and charecterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. Henriksen [2] and Shabir et al. studied ideals in semirings We know that the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals.In 1952 the concept of bi-ideals was introduced by Good and Hughes [1]for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [5,6].Bi-ideal is a special case of ( m-n) ideal. Steinfeld first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [3,4] introduced the concept of quasi ideal for a semiring. Murali Krishna Rao [10]studied ideals in ordered $\Gamma-$semiring.Murali Krishna Rao [11,12,13,14,15,16] introduced the notion of left (right) bi-quasi ideal and bi-interior idea of semiring, $\Gamma-$semigroup,$\Gamma-$semigroup and study the properties of left bi-quasi ideals. We characterize the left bi-quasi simple $\Gamma-$semigroup and regular $\Gamma-$semigroup using left bi-quasi ideals of $\Gamma-$semigroups. In this paper,as

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a further generalization of ideals, we introduce the notion of bi- quasi-interior ideal as a generalization of bi-ideal,quasi ideal, interior ideal,bi-interior ideal and bi-quasi ideal of $\Gamma-$semigroup and study the properties of bi-quasi-interior ideals of $\Gamma-$semigroup.Some charecterization of bi-quasi-interior ideals of $\Gamma-$semigroup,regular $\Gamma-$semigroup and simple $\Gamma-$semigroup.

## 2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

**Definition 2.1.** A semigroup is an algebraic system $(M, .)$ consisting of a non-empty set $M$ together with an associative binary operation $'' \cdot ''$ .

**Definition 2.2.** A subsemigroup $T$ of a semigroup $M$ is a non-empty subset $T$ of $M$ such that $TT \subseteq T$.

**Definition 2.3.** A non-empty subset $T$ of a semigroup $M$ is called a left (right) ideal of $M$ if $MT \subseteq T(TM \subseteq T)$.

**Definition 2.4.** A non-empty subset $T$ of a semigroup $M$ is called an ideal of $M$ if it is both a left ideal and a right ideal of$M$.

**Definition 2.5.** A non-empty $Q$ of a semigroup $M$ is called a quasi ideal of $M$ if $QM \cap MQ \subseteq Q$.

**Definition 2.6.** A subsemigroup $T$ of a semigroup $M$ is called a bi-ideal of $M$ if $TMT \subseteq T$.

**Definition 2.7.** A subsemigroup $T$ of a semigroup $M$ is called an interior ideal of $M$ if $MTM \subseteq T$.

**Definition 2.8.** An element $a$ of a semigroup $M$ is called a regular element if there exists an element $b$ of $M$ such that $a = aba$.

**Definition 2.9.** A semigroup $M$ is called a regular semigroup if every element of $M$ is a regular element.

**Definition 2.10.** Let $M$ and $\Gamma$ be non-empty sets. Then we call $M$ a $\Gamma-$semigroup, if there exists a mapping $M \times \Gamma \times M \to M$ (images of $(x, \alpha, y)$ will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$. for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

**Definition 2.11.** Let $M$ be a $\Gamma-$semigroup. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

**Definition 2.12.** A$\Gamma-$semigroup $M$ is said to be left (right) singular if for each $a \in M$ there exists $\alpha \in \Gamma$ such that $a\alpha b = a(a\alpha b = b),$ for all $b \in M$.

**Definition 2.13.** A $\Gamma-$semigroup $M$ is said to be commutative if $a\alpha b = b\alpha a,$ for all $a, b \in M,$ for all $\alpha \in \Gamma$.

**Definition 2.14.** Let $M$ be a $\Gamma-$semigroup. An element $a \in M$ is said to be an idempotent of $M$ if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and $a$ is also said to be $\alpha$ idempotent.

**Definition 2.15.** Let $M$ be a $\Gamma-$semigroup. If every element of $M$ is an idempotent of $M$ then $\Gamma-$semigroup $M$ is said to be band.

**Definition 2.16.** Let $M$ be a $\Gamma-$semigroup. An element $a \in M$ is said to be regular element of $M$ if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

**Definition 2.17.** Let $M$ be a $\Gamma-$semigroup. Every element of $M$ is a regular element of $M$ then $M$ is said to be a regular $\Gamma-$semigroup $M$.

**Definition 2.18.** Let $M$ be a semigroup. A non-empty set $L$ of $M$ is said to be left bi-quasi (right bi-quasi) ideal of $M$ if $L$ is a subsemigroup of $M$ and $ML \cap LML \subseteq L(LM \cap LML \subseteq L)$.

**Definition 2.19.** Let $M$ be a semigroup. A non-empty subset $L$ of $M$ is said to be bi-quasi ideal of $M$ if $L$ is a subsemigroup of $M$ ,$ML \cap LML \subseteq L$ and $LM \cap LML \subseteq L$.

**Definition 2.20.** [12] Let $M$ be a semigroup. A non-empty subset $L$ of $M$ is said to be bi-interior ideal of $M$ if $L$ is a subsemigroup of $M$ and $MLM \cap LML \subseteq L$

**Definition 2.21.** An element $a \in \Gamma-$semigroup $M$ is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

**Definition 2.22.** A non-empty subset $A$ of a $\Gamma-$ semigroup$M$ is called

(i) a $\Gamma-$subsemigroup of $M$ if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $M\Gamma A\Gamma M \subseteq A$.
(v) a left (right) ideal of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an ideal if $A$ is a $\Gamma-$subsemigroup of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.
(vii) a $k-$ideal if $A$ is a $\Gamma-$subsemigroup of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.
((viii) a bi-interior ideal of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.
((ix) a left bi-quasi ideal (right bi-quasi ideal) of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M \cap A\Gamma M\Gamma A \subseteq A)$.
((x) a left quasi-interior ideal (right quasi-interior ideal) of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $M\Gamma A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M\Gamma A\Gamma M \subseteq A)$.
((xi) a left tri- ideal (right tri- ideal) of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $A\Gamma M\Gamma A\Gamma A \subseteq A$ $(A\Gamma A\Gamma M\Gamma A \subseteq A)$.
((xii) a tri- ideal of $M$ if $A$ is a $\Gamma-$subsemigroup of $M$ and $A\Gamma M\Gamma A\Gamma A \subseteq A$ and $A\Gamma A\Gamma M\Gamma A \subseteq A$.
((xiii) a left(right) weak-interior ideal of $M$ if $B$ is a $\Gamma-$subsemigroup of $M$ and $M\Gamma B\Gamma B \subseteq B($ $B\Gamma B\Gamma M \subseteq B)$.
((xiv) a weak-interior ideal of $M$ if $B$ is a$\Gamma-$subsemigroup of $M$ and $B$is a left weak-interior ideal and a right weak-interior ideal of $M$.

## 3. Bi-quasi-interior ideals of $\Gamma-$semigroups

In this section, we introduce the notion of bi-quasi-interior ideal as a generalization of bi-ideal,quasi-ideal and interior ideal of $\Gamma-$semigroup and study the properties of bi-quasi-interior ideal of $\Gamma-$semigroup.

**Definition 3.1.** A non-empty subset $B$ of a $\Gamma-$semigroup $M$ is said to be bi-quasi-interior ideal of $M$ if $B$ is a $\Gamma-$subsemigroup of $M$ and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.

Every bi-quasi-interior ideal of $\Gamma-$semigroup $M$ need not be bi-ideal,quasi-ideal, interior ideal. bi-interior ideal and bi-quasi ideals of a $\Gamma-$semigroup $M$.

**Example 3.2.** Let $M = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta\}$. The ternary operation is defined by $M \times \Gamma \times M \longrightarrow M$ by the following table:

| $\alpha$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $b$ | $a$ | $b$ | $a$ | $d$ | $d$ |
| $c$ | $a$ | $d$ | $c$ | $d$ | $e$ |
| $d$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $e$ | $a$ | $d$ | $c$ | $d$ | $e$ |

| $\beta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $b$ | $a$ | $b$ | $a$ | $d$ | $d$ |
| $c$ | $a$ | $d$ | $c$ | $d$ | $e$ |
| $d$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $e$ | $a$ | $d$ | $c$ | $d$ | $e$ |

Then $M$ is a $\Gamma-$semigroup. Let $B = \{a, c\}$. Then $B$ is not a right ideal, $B$ is not an interior ideal, $B$ is not a quasi ideal, $B$ is not a quasi interior ideal and $B$ is a bi quasi ideal.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

**Theorem 3.1.** *Let $M$ be a $\Gamma-$semigroup. Then the following are hold.*

(1) *Every left ideal is a bi-quasi-interior ideal of $M$.*
(2) *Every right ideal is a bi-quasi-interior ideal of $M$.*
(3) *Every quasi ideal is bi-quasi-interior of $M$.*
(4) *Every ideal is a bi-quasi-interior ideal of $M$.*
(5) *Intersection of a right ideal and a left ideal of $M$ is a bi-quasi-interior ideal of $M$.*
(6) *If $L$ is a left ideal and $R$ is a right ideal of $\Gamma-$semigroup $M$ then $B = R\Gamma L$ is a bi-quasi-interior ideal of $M$.*
(7) *If $B$ is a bi-quasi-interior ideal and $T$ is a $\Gamma-$subsemigroup of $M$ then $B \cap T$ is a bi-quasi-interior ideals of $\Gamma-$semigroup $M$.*
(8) *Let $M$ be a $\Gamma-$semigroup and $B$ be a $\Gamma-$subsemigroup of $M$.If $B\Gamma M\Gamma M\Gamma M\Gamma B \subseteq B$ then $B$ is a bi-quasi-interior ideal of $M$.*
(9) *Let $M$ be a $\Gamma-$semigroup and $B$ be a $\Gamma-$subsemigroup of $M$.If $M\Gamma M\Gamma M\Gamma M\Gamma B \cap B\Gamma M\Gamma M\Gamma M\Gamma M \subseteq B$ then $B$ is a bi-quasi-interior ideal of $M$.*

**Theorem 3.2.** *Every bi-ideal of a $\Gamma-$semigroup $M$ is a bi-quasi-interior ideal of a $\Gamma-$semigroup $M$.*

*Proof.* Let $B$ be a bi-ideal of $\Gamma-$semigroup $M$. Then $B\Gamma M\Gamma B \subseteq B$.
Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq B$.
Hence every bi-ideal of the $\Gamma-$semigroup $M$ is a bi-quasi-interior ideal of $M$. □

**Theorem 3.3.** *Every interior ideal of a $\Gamma-$semigroup $M$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $I$ be an interior ideal of $\Gamma-$semigroup $M$.
Then $I\Gamma M\Gamma I\Gamma M\Gamma I \subseteq M\Gamma I\Gamma M \subseteq I$.
Hence $I$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$. □

**Theorem 3.4.** *Let $M$ be a $\Gamma-$semigroup and $B$ be a $\Gamma-$subsemigroup of $M$. $B$ is a bi-quasi-interior ideal of $M$ if and only if there exist a left ideal $L$ and a right ideal$R$ such that $R\Gamma L \subseteq B \subseteq R \cap L$ .*

*Proof.* Suppose $B$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup$M$.
Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Let $R = B\Gamma M\Gamma B\Gamma M$ and $L = M\Gamma B$.
Then $L$ and $R$ are left and right ideals of $M$ respectively.

Therefore $R\Gamma L \subseteq B \subseteq R \cap L$ .
Conversely suppose that there exist $L$ and $R$ are left and right ideals of $M$ respectively such that $R\Gamma L \subseteq B \subseteq R \cap L$ .Then
$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L)$
$\subseteq (R)\Gamma M\Gamma(R)\Gamma M\Gamma(L)$
$\subseteq R\Gamma L \subseteq B$ .
Hence $B$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$. □

**Theorem 3.5.** *Let $A$ and $C$ be bi-quasi-interior ideals of a $\Gamma-$semigroup $M$ and $B = A\Gamma C$. If $C\Gamma C = C$ then $B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $A$ and $C$ be bi-quasi-interior ideals of the $\Gamma-$semigroup $M$ and $B = A\Gamma C$. $B\Gamma B = A\Gamma C\Gamma A\Gamma C = A\Gamma C\Gamma C\Gamma C\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq A\Gamma C = B$. Obviously $B = A\Gamma C$ is a $\Gamma-$subsemigroup of $M$

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &= A\Gamma C\Gamma M\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C \\ &\subseteq A\Gamma M\Gamma A\Gamma M\Gamma A\Gamma C \subseteq A\Gamma C = B. \end{aligned}$$

Hence $B$ is a bi-quasi-interior ideal of $M$. □

**Corollary 3.6.** *Let $A$ and $C$ be bi-quasi-interior ideals of a $\Gamma-$semigroup $M$ and $B = C\Gamma A$. If $C\Gamma C = C$ then $B$ is a bi-quasi-interior ideal of $M$.*

**Theorem 3.7.** *Let $A$ and $C$ be $\Gamma-$subsemigroups of $M$ and $B = A\Gamma C$. If $A$ is the left ideal then $B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $A$ and $C$ be $\Gamma-$subsemigroups of $M$ and $B = A\Gamma C$ Suppose $A$ is the left ideal of $M$. $B\Gamma B = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$.

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &= A\Gamma C\Gamma M\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C \\ &\subseteq A\Gamma C = B. \end{aligned}$$

Hence $B$ is a bi-quasi-interior ideal of $M$. □

**Corollary 3.8.** *Let $A$ and $C$ be $\Gamma-$subsemigroups of $\Gamma-$semigroup $M$ and $B = A\Gamma C$. If $C$ is a right ideal then $B$ is a bi-quasi-interior ideal of $M$.*

**Theorem 3.9.** *Let $M$ be a $\Gamma-$semigroup and $T$ be a non-empty subset of $M$. Then every subsemiring of $T$ containing $T\Gamma M\Gamma T\Gamma M\Gamma T$ is a bi-quasi-interior ideal of a $\Gamma-$semigroup $M$.*

*Proof.* Let $B$ be a subsemiring of $T$ containing $T\Gamma M\Gamma T\Gamma M\Gamma T$. Then

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &\subseteq T\Gamma M\Gamma T\Gamma M\Gamma T \\ &\subseteq B. \end{aligned}$$

Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.
Hence $B$ is a bi-quasi-interior ideal of $M$. □

**Theorem 3.10.** *$B$ is a bi-quasi-interior ideal of a $\Gamma-$semigroup $M$ if and only if $B$ is a left ideal of some right ideal of $\Gamma-$semigroup$M$.*

*Proof.* Suppose $B$ is a left ideal of some right ideal $R$ of the $\Gamma-$semigroup$M$.Then $R\Gamma B \subseteq B, R\Gamma M \subseteq B$.Hence $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq R\Gamma M\Gamma B \subseteq R\Gamma B \subseteq B$. Therefore $B$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$. Conversely suppose that $B$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$ .Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Therefore $B$ is a left ideal of right ideal $B\Gamma M\Gamma B\Gamma M$ of the $\Gamma-$semigroup$M$. □

**Corollary 3.11.** *$B$ is a bi-quasi-interior ideal of $\Gamma-$semigroup $M$ if and only if $B$ is a right ideal of some left ideal of $\Gamma-$semigroup$M$.*

**Theorem 3.12.** *If $B$ is a bi-quasi-interior ideal of a $\Gamma-$semigroup $M,$ $T$ is a $\Gamma-$subsemigroup of $M$ and $T \subseteq B$ then $B\Gamma T$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Obviously, $B\Gamma T$ is a $\Gamma-$subsemigroup of $(M,+)$. $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T.$
Hence $B\Gamma T$ is a $\Gamma-$subsemigroup of $M$.

$$\begin{aligned} \text{We have } & M\Gamma B\Gamma T\Gamma M \subseteq M\Gamma B\Gamma M \\ \text{and } & B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq B\Gamma M\Gamma B \\ \Rightarrow & B\Gamma T\Gamma M\Gamma B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T. \end{aligned}$$

Hence $B\Gamma T$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$. □

**Theorem 3.13.** *Let $B$ be bi-ideal of $\Gamma-$semigroup $M$ and $I$ be interior ideal of $M$. Then $B \cap I$ is bi-quasi-interior idea of $M$.*

*Proof.* Obviously $B \cap I$ is a $\Gamma-$subsemigroup of $M$. Suppose $B$ is a bi-ideal of $M$ and $I$ is an interior ideal of $M$. Then

$$(B \cap I)\Gamma M\Gamma(B \cap I)\Gamma M\Gamma(B\Gamma I) \subseteq B(B \cap I)\Gamma M\Gamma(B \cap I)\Gamma M\Gamma(B\Gamma I)\Gamma M \subseteq I$$

Therefore $(B \cap I)\Gamma M\Gamma(B \cap I)\Gamma M\Gamma(B\Gamma I)\Gamma M \subseteq B \cap I$.
Hence $B \cap I$ is a bi-quasi-interior ideal of $M$. □

**Theorem 3.14.** *Let $M$ be a $\Gamma-$semigroup and $T$ be an additive subsemigroup of $M$. Then every additive subsemigroup of $T$ containing $T\Gamma M\Gamma T\Gamma M\Gamma T$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $C$ be an additive subsemigroup of $T$ containing $T\Gamma M\Gamma T\Gamma M\Gamma T$. Then

$$\begin{aligned} C\Gamma M\Gamma C\Gamma M\Gamma C &\subseteq T\Gamma M \cap T\Gamma M\Gamma T \\ &\subseteq C. \end{aligned}$$

Hence $C$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$. □

**Theorem 3.15.** *Let $M$ be a $\Gamma-$semigroup. If $M = M\Gamma a$, for all $a \in M$. Then every bi-quasi-interior ideal of $M$ is a quasi ideal of $M$.*

*Proof.* Let $B$ be a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$ and $a \in B$. Then

$$\begin{aligned} & B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ \Rightarrow & M\Gamma a \subseteq M\Gamma B,\ (B\Gamma M = M) \\ \Rightarrow & M \subseteq M\Gamma B \subseteq M \\ \Rightarrow & M\Gamma B = M \\ \Rightarrow & B\Gamma M = B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ \Rightarrow & M\Gamma B \cap B\Gamma M \subseteq M\Gamma M \cap B\Gamma M \subseteq B. \end{aligned}$$

Therefore $B$ is a quasi ideal of $M$. Hence the theorem. □

**Theorem 3.16.** *The intersection of $\{B_\lambda \mid \lambda \in A\}$ bi-quasi-interior ideal of a $\Gamma-$semigroup $M$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $B = \bigcap\limits_{\lambda \in A} B_\lambda$. Then $B$ is a $\Gamma-$subsemigroup of $M$.
Since $B_\lambda$ is a bi-quasi-interior ideal of $M$, we have

$$B_\lambda \Gamma M \Gamma B_\lambda \Gamma M \Gamma B_\lambda \subseteq B_\lambda, \text{ for all } \lambda \in A$$
$$\Rightarrow \cap B_\lambda \Gamma M \Gamma \cap B_\lambda \Gamma M \cap B_\lambda) \subseteq \cap B_\lambda$$
$$\Rightarrow B \Gamma M \Gamma B \Gamma M \Gamma B \subseteq B.$$

Hence $B$ is a bi-quasi-interior ideal of $M$. □

**Theorem 3.17.** *Let $B$ be a bi-quasi-interior ideal of a $\Gamma-$semigroup $M$,$e \in B$ and $e$ be $\beta-$idempotent. Then $e\Gamma B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $B$ be a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$. Suppose $x \in B \cap e\Gamma M$. Then $x \in B$ and $x = e\alpha y, \alpha \in \Gamma, y \in M$.

$$\begin{aligned} x &= e\alpha y \\ &= e\beta e\alpha y \\ &= e\beta(e\alpha y) \\ &= e\beta x \in e\Gamma B. \end{aligned}$$
$$\text{Therefore } B \cap e\Gamma M \subseteq e\Gamma B$$
$$e\Gamma B \subseteq B \text{ and } e\Gamma B \subseteq e\Gamma M$$
$$\Rightarrow e\Gamma B \subseteq B \cap e\Gamma M$$
$$\Rightarrow e\Gamma B = B \cap e\Gamma M.$$

Hence $e\Gamma B$ is a bi-quasi-interior ideal of $M$. □

**Corollary 3.18.** *Let $M$ be a $\Gamma-$semigroup $M$ and $e$ be $\alpha-$idempotent. Then $e\Gamma M$ and $M\Gamma e$ are bi-quasi-interior ideals of $M$ respectively.*

**Theorem 3.19.** *If $B$ be a left bi-quasi ideal of a $\Gamma-$semigroup $M$, then $B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Suppose $B$ is a left bi-quasi ideal of the $\Gamma-$semigroup $M$.Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B$ and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B$.
Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B \cap M\Gamma B\Gamma M \subseteq B$
Hence $B$ is a bi-quasi-interior ideal of $M$. □

**Corollary 3.20.** *If $B$ be a right bi-quasi ideal of a $\Gamma-$semigroup $M$, then $B$ is a bi-quasi-interior ideal of $M$.*

**Corollary 3.21.** *If $B$ be a bi-quasi ideal of a $\Gamma-$semigroup $M$, then $B$ is a bi-quasi-interior ideal of $M$.*

**Theorem 3.22.** *If $B$ be a bi–interior ideal of a $\Gamma-$ semiring of $M$, then $B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Suppose $B$ is a bi–interior ideal of the $\Gamma-$semigroup $M$.Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M$ and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M$.
Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \cap M\Gamma B\Gamma M \subseteq B$
Hence $B$ is a bi-quasi-interior ideal of $M$. □

## 4. BI-QUASI-INTERIOR SIMPLE $\Gamma-$SEMIGROUP

In this section,we introduce the notion of bi-quasi-interior simple $\Gamma-$semigroup and characterize the bi-quasi-interior simple $\Gamma-$semigroup using bi-quasi-interior ideals of $\Gamma-$semigroup and study the properties of minimal bi-quasi-interior ideals of $\Gamma-$semigroup..

**Definition 4.1.** A $\Gamma-$semigroup $M$ is said to be bi-quasi-interior simple $\Gamma-$semigroup if $M$ has no bi-quasi-interior ideals other than $M$ itself.

**Theorem 4.1.** *If $M$ is a $\Gamma-$group then $M$ is bi-quasi-interior simple $\Gamma-$semigroup.*

*Proof.* Let $B$ be a proper bi-quasi-interior ideal of $\Gamma-$group and $0 \neq a \in B$. Since $M$ is a $\Gamma-$group, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma, x \in M$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Then $x \in B\Gamma M$. Therefore $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly we can prove $M\Gamma B = M$.

$$\begin{aligned} M &= M\Gamma B \\ &= B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ M &\subseteq B \\ \text{Therefore} M &= B. \end{aligned}$$

Hence $\Gamma-$group $M$ has no properbi-quasi-interior ideals. □

**Theorem 4.2.** *Let $M$ be a simple $\Gamma-$semigroup. Every bi-quasi-interior ideal is bi-ideal of $M$.*

*Proof.* Let $M$ be a simple $\Gamma-$semigroup and $B$ be a bi-quasi-interior ideal of $M$.
Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$ and $M\Gamma B\Gamma M$ is an ideal of $M$.
Since $M$ is a simple $\Gamma-$semigroup, we have $M\Gamma B\Gamma M = M$. Hence

$$\begin{aligned} &B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B \\ \Rightarrow &B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Hence the theorem. □

**Theorem 4.3.** *Let $M$ be a $\Gamma-$semigroup. Then $M$ is a bi-quasi-interior simple $\Gamma-$semigroup if and only if $(a)_{bqi} = M$, for all $a \in M$, where $(a)_{bqi}$ is the bi-quasi-interior ideal generated by $a$.*

*Proof.* Let $M$ be a $\Gamma-$semigroup. Suppose that $(a)_{bqi}$ is a bi-quasi-interior ideal generated by $a$ and $M$ is a bi-quasi -interior simple $\Gamma-$semigroup. Then $(a)_{bqi} = M$, for all $a \in M$.

Conversely suppose that $B$ is a bi-quasi-interior ideal of $\Gamma-$semigroup $M$ and $(a)_{bqi} = M$, for all $a \in M$. Let $b \in B$.
Then $(b)_{bqi} \subseteq B \Rightarrow M = (b)_{bqi} \subseteq B \subseteq M$.
Therefore $M$ is a bi-quasi-interior simple $\Gamma-$semigroup. □

**Theorem 4.4.** *Let $M$ be a $\Gamma-$semigroup. $M$ is a bi-quasi-interior simple $\Gamma-$semigroup if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest bi-quasi-interior ideal generated by $a$.*

*Proof.* Let $M$ be a $\Gamma-$semigroup. Suppose $M$ is a bi-quasi-interior simple $\Gamma-$semigroup, $a \in M$ and $B = M\Gamma a$.
Then $B$ is a left ideal of $M$.
Therefore, by Theorem[3.4 ], $B$ is a bi-quasi-interior ideal of $M$.

Therefore $B = M$. Hence $M\Gamma a = M$, for all $a \in M$.

$$M\Gamma a \subseteq < a > \subseteq M$$
$$\Rightarrow M \subseteq < a > \subseteq M.$$
$$\text{Therefore } M = < a > .$$

Suppose $< a >$ is the smallest bi-quasi-interior ideal of $M$ generated by $a$ and $< a >= M$ and $A$ is the bi-quasi-interior ideal and $a \in A$. Then

$$< a > \subseteq A \subseteq M$$
$$\Rightarrow M \subseteq A \subseteq M.$$

Therefore $A = M$. Hence $M$ is a bi-quasi-interior simple $\Gamma-$semigroup. □

**Theorem 4.5.** *Let $M$ be a $\Gamma-$semigroup. Then $M$ is a bi-quasi-interior simple $\Gamma-$semrring if and only if $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.*

*Proof.* Suppose $M$ is left bi-quasi simple $\Gamma-$semigroup and $a \in M$.
Therefore $a\Gamma M\Gamma a\Gamma M\Gamma a$ is a bi-quasi-interior ideal of $M$.
Hence $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.

Conversely suppose that $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.
Let $B$ be a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$ and $a \in B$.

$$M = a\Gamma M\Gamma a\Gamma M\Gamma a$$
$$\subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$$
$$\text{Therefore} M = B.$$

Hence $M$ is a bi-quasi-interior simple $\Gamma-$semigroup. □

**Theorem 4.6.** *If $B$ is a minimal bi-quasi-interior ideal of a $\Gamma-$semigroup $M$ then any two non-zero elements of $B$ generated the same right ideal of $M$.*

*Proof.* Let $B$ be a minimal bi-quasi-interior ideal of $M$ and $x \in B$. Then $(x)_R \cap B$ is a bi-quasi-interior ideal of $M$. Therefore $(x)_R \cap B \subseteq B$.
Since $B$ is a minimal bi-quasi-interior ideal of $M$, we have $(x)_R \cap B = B \Rightarrow B \subseteq (x)_R$.
Suppose $y \in B$. Then $y \in (x)_R$, $(y)_R \subseteq (x)_R$.
Therefore $(x)_R = (y)_R$. Hence the theorem. □

**Corollary 4.7.** *If $B$ is a minimal bi-quasi-interior ideal of a $\Gamma-$semigroup $M$ then any two non-zero elements of $B$ generates the same left ideal of $M$.*

**Definition 4.2.** A $\Gamma-$semigroup $M$ is a left (right) simple $\Gamma-$semigroup if $M$ has no proper left (right) ideals of $M$

A $\Gamma-$semigroup $M$ is said to be simple $\Gamma-$semigroup if $M$ has no proper ideals.

**Theorem 4.8.** *If $\Gamma-$semigroup $M$ is a left simple $\Gamma-$semigroup then every bi-quasi-interior ideal of $M$ is a right ideal of $M$.*

*Proof.* Let $B$ be a bi-quasi-interior of left simple $\Gamma-$semigroup. Then $M\Gamma B$ is a left ideal of $M$ and $M\Gamma B \subseteq M$. Therefore $M\Gamma B = M$. Then

$$M\Gamma B\Gamma M = M\Gamma M = M$$
$$\Rightarrow B\Gamma M = B\Gamma M\Gamma B$$
$$\Rightarrow B\Gamma M = B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$$
$$\Rightarrow B\Gamma M \subseteq B.$$

Hence every bi-quasi-interior ideal is a right ideal of $M$.

□

**Corollary 4.9.** *If $\Gamma-$semigroup $M$ is a right simple $\Gamma-$semigroup then every bi-quasi-interior ideal of $M$ is a left ideal of $M$.*

**Corollary 4.10.** *Every bi-quasi-interior ideal of left and right simple $\Gamma-$semigroup $M$ is an ideal of $M$.*

**Theorem 4.11.** *Let $M$ be a $\Gamma-$semigroup and $B$ be a bi-quasi-interior ideal of $M$. Then $B$ is minimal bi-quasi-interior ideal of $M$ if and only if $B$ is a bi-quasi-interior simple $\Gamma-$susemiring.*

*Proof.* Let $B$ be a minimal bi-quasi-interior ideal of $\Gamma-$semigroup $M$ and $C$ be a bi-quasi-interior ideal of $B$. Then $C\Gamma B\Gamma C\Gamma B\Gamma C \subseteq C$.
Therefore $C\Gamma B\Gamma C\Gamma B\Gamma C\Gamma B$ is a bi-quasi-interior ideal of $M$.
Since $C$ is a bi-quasi-interior ideal of $B$,

$$C\Gamma B\Gamma C\Gamma B\Gamma C = B$$
$$\Rightarrow B = C\Gamma B\Gamma C\Gamma B\Gamma C \subseteq C$$
$$\Rightarrow B = C.$$

Conversely suppose that $B$ is a bi-quasi-interior simple $\Gamma-$subsemigroup of $M$. Let $C$ be a bi-quasi-interior ideal of $M$ and $C \subseteq B$.

$$C\Gamma B\Gamma C\Gamma B\Gamma C = C$$
$$\Rightarrow C\Gamma B\Gamma C\Gamma B\Gamma C \subseteq C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B,$$
$\Rightarrow B = C$.Since $B$ is a bi-quasi-interior simple $\Gamma-$semigroup.

Hence $B$ is a minimal bi-quasi-interior ideal of $M$. □

**Theorem 4.12.** *Let $M$ be a $\Gamma-$semigroup and $B = R\Gamma L$, where $L$ and $R$ are minimal left and right ideals of $M$ repectively. Then $B$ is a minimal bi-quasi-interior ideal of $M$.*

*Proof.* Obviously $B = R\Gamma L$ is bi-quasi-interior ideal of $M$. Let $A$ be bi-quasi-interior ideal of $M$ such that $A \subseteq B$.
We have $M\Gamma A$ is a right ideal. Then

$$\begin{aligned} M\Gamma A &\subseteq M\Gamma B \\ &= M\Gamma R\Gamma L \\ &\subseteq L, \text{ since } L \text{ is a left ideal of } M. \\ \text{Similarly, we can prove } A\Gamma M &\subseteq R \\ \text{Therefore } M\Gamma A &= L,\ A\Gamma M = R \\ \text{Hence }\ B &= A\Gamma M\Gamma M\Gamma A \\ &\subseteq A\Gamma M\Gamma A\Gamma M\Gamma A. \\ &\subseteq A \end{aligned}$$

Therefore $A = B$. Hence $B$ is a minimal bi-quasi-interior ideal of $M$.

□

## 5. Regular $\Gamma-$semigroup

In this section,we characterize regular $\Gamma-$semigroup using bi-quasi-interior ideals of $\Gamma-$semigroup.

**Theorem 5.1.** *Let $M$ be a regular $\Gamma-$semigroup. Then every bi-quasi-interior ideal of $M$ is an ideal of $M$.*

*Proof.* Let $B$ be a bi-quasi-interior ideal of $M$. Then

$$
\begin{aligned}
&B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B\\
\Rightarrow &B\Gamma M \subseteq B\Gamma M\Gamma B, \text{ since } M \text{ is regular}\\
\Rightarrow &B\Gamma M \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B.
\end{aligned}
$$

Similarly, we can show that $M\Gamma B \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.
Hence the theorem. □

**Theorem 5.2.** *Let $M$ be a regular $\Gamma-$semigroup and $I$ be an interior ideal of $M$. Then $M\Gamma I\Gamma M = I$.*

*Proof.* Let $a \in M$. Then there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.
$\Rightarrow$ there exist $\delta, \gamma \in \Gamma$ and $y \in M$ such that $x\beta a = x\beta a\delta y\gamma x\beta a$.

$$
\begin{aligned}
\text{Therefore } a &= a\alpha x\beta a\\
&= a\alpha x\beta a\delta y\gamma x\beta a \in M\Gamma I\Gamma M\\
\text{Hence } I &\subseteq M\Gamma I\Gamma M.
\end{aligned}
$$

We have $M\Gamma I\Gamma M \subseteq I$. Hence $M\Gamma I\Gamma M = I$. □

**Theorem 5.3.** *Let $M$ be a regular $\Gamma-$semigroup. Then $B$ is a bi-quasi-interior ideal of $M$ if and only if $B\Gamma M\Gamma B\Gamma M\Gamma B = B$, for all bi-quasi-interior ideals $B$ of $M$.*

*Proof.* Suppose $M$ is a regular $\Gamma-$semigroup, $B$ is a bi-interior ideal of $M$ and $x \in B$. Then $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$ and there exist $y \in M$, $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x\alpha y\beta x \in B\Gamma M\Gamma B\Gamma M\Gamma B$. Therefore $x \in B\Gamma M\Gamma B\Gamma M\Gamma B$.
Hence $B\Gamma M\Gamma B\Gamma M\Gamma B = B$.

Conversely suppose that $B\Gamma M\Gamma B\Gamma M\Gamma B = B$, for all bi-quasi-interior ideals $B$ of $M$.
Let $B = R \cap L$, where $R$ is a right ideal and $L$ is a left ideal of $M$.
Then $B$ is a bi-interior ideal of $M$.
Therefore $(R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L) = R \cap L$

$$
\begin{aligned}
R \cap L &= (R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L)\\
&\subseteq R\Gamma M\Gamma L\Gamma M\Gamma L\\
&\subseteq R\Gamma L\\
&\subseteq R \cap L \text{ (since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R).
\end{aligned}
$$

Therefore $R \cap L = R\Gamma L$. Hence $M$ is a regular $\Gamma-$semigroup. □

**Theorem 5.4.** *Let $B$ be the bi-quasi-interior ideal of a regular $\Gamma-$semigroup $M$. If $B$ is a bi-quasi-interior ideal of $M$ and $B$ is regular $\Gamma-$subsemigroup of $M$ then any bi-quasi-interior ideal of $B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Let $A$ be a bi-quasi-interior ideal of the regular $\Gamma-$subsemigroup $B$ of $M$. Then by Theorem[5.4 ], $A\Gamma B\Gamma A\Gamma B\Gamma A = A$. We have $B\Gamma M\Gamma B\Gamma M\Gamma B = B$ and $A \subseteq A\Gamma B$.

$$
\begin{aligned}
&\Rightarrow A\Gamma M\Gamma A\Gamma M\Gamma A \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B = B\\
&\Rightarrow A\Gamma B\Gamma A\Gamma B\Gamma A = A \subseteq B\\
&\Rightarrow A = A\Gamma B\Gamma A\Gamma B\Gamma A \subseteq A\Gamma M\Gamma A\Gamma M\Gamma A\\
&\Rightarrow A\Gamma M\Gamma A\Gamma M\Gamma A = A..
\end{aligned}
$$

Hence $A$ is a bi-quasi-interior ideal of $M$. □

**Theorem 5.5.** *$M$ is regular $\Gamma-$semigroup if and only if $A\Gamma B = A \cap B$ for any right ideal $A$ and left ideal $B$ of $\Gamma-$semigroup $M$.*

*Proof.* Let $A, B$ be a right ideal and a left ideal of the regular ordered $\Gamma-$semigroup $M$ respectively.
Obviously $A\Gamma B \subseteq A \cap B$. Let $x \in A \cap B$.
Since $M$ is a regular, there exist $\alpha, \beta \in \Gamma$ and $y \in M$ such that $x = x\alpha y\beta x$. Since $x\alpha y \in A$ and $x \in B$, $x\alpha y\beta x \in A\Gamma B$. Thus $x \in A\Gamma B$. Hence $A\Gamma B = A \cap B$.
Conversely, suppose that $A\Gamma B = A \cap B$ for any right ideal $A$ and left ideal $B$ of $M$. Let $x \in M$ and $I$ be the right ideal generated by $x$ and $J$ be the left ideal generated by $x$. We have $x \in I \cap J = I\Gamma J$. Therefore $x = x\alpha y = z\beta x, \alpha, \beta \in \Gamma, y, z \in M$ which implies that $x = x\alpha y\gamma z\beta x$, for some $\gamma \in \Gamma$. Hence $M$ is a regular ordered $\Gamma-$semigroup. □

**Theorem 5.6.** *Let $B$ be $\Gamma-$subsemigroup of a regular $\Gamma-$semigroup $M$. If $B$ can be represented as $B = R\Gamma L$, where $R$ is a right ideal and $L$ is a left ideal of $M$ then $B$ is a bi-quasi-interior ideal of $M$.*

*Proof.* Suppose $B = R\Gamma L$, where $R$ is right ideal of $M$ and $L$ is a left ideal of $M$.

$$\begin{aligned} B\Gamma M\Gamma B\Gamma M\Gamma B &= R\Gamma L\Gamma M\Gamma R\Gamma LM\Gamma R\Gamma L \\ &\subseteq R\Gamma L = B. \end{aligned}$$

Hence $B$ is a bi-quasi-interior ideal of the $\Gamma-$semigroup $M$.

Conversely suppose that $B$ is a bi-quasi-interior ideal of the regular $\Gamma-$semigroup $M$. We have $B\Gamma M\Gamma B\Gamma M\Gamma B = B$. Let $R = B\Gamma M$ and $L = M\Gamma B$.
Then $R = B\Gamma M$ is a right ideal of $M$ and $L = M\Gamma B$ is a left ideal of $M$.

$$\begin{aligned} & B\Gamma M \cap M\Gamma B \subseteq B\Gamma M\Gamma B\Gamma M\Gamma B = B \\ & \Rightarrow B\Gamma M \cap M\Gamma B \subseteq B \\ & \Rightarrow R \cap L \subseteq B. \\ \text{We have } & B \subseteq B\Gamma M = R \text{ and } B \subseteq M\Gamma B = L \\ & \Rightarrow B \subseteq R \cap L \\ & \Rightarrow B = R \cap L = R\Gamma L, \text{ since } M \text{ is a regular } \Gamma-\text{semigroup.} \end{aligned}$$

Hence $B$ can be represented as $R\Gamma L$, where $R$ is the right ideal and $L$ is the left ideal of $M$. Hence the theorem. □

The following theorem is a necessary and sufficient condition for a $\Gamma-$semigroup $M$ to be regular using bi-quasi-interior ideal.

**Theorem 5.7.** *$M$ is a regular $\Gamma-$semriring if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-quasi-interior ideal $B$, ideal $I$ and left ideal$L$ of $M$.*

*Proof.* Suppose $M$ be a regular $\Gamma-$semigroup, $B, I$ and $L$ are bi-quasi-interior ideal, ideal and left ideal of $M$ respectively.
Let $a \in B \cap I \cap L$. Then $a \in a\Gamma M\Gamma a$, since $M$ is regular.

$$\begin{aligned} a \in a\Gamma M\Gamma a &\subseteq a\Gamma M\Gamma a\Gamma M\Gamma a \\ &\subseteq B\Gamma I\Gamma B \end{aligned}$$

Hence $B \cap I \cap L \subseteq B\Gamma I\Gamma L$.
Conversely suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-quasi-interior ideal $B$, ideal $I$ and left ideal $L$ of $M$. Let $R$ be a right ideal and $L$ be left ideal of $M$. Then by assumption,

$$R \cap L = R \cap M \cap L \subseteq R\Gamma M\Gamma L$$

$$R\Gamma L.$$

We have $R\Gamma L \subseteq R,\ R\Gamma L \subseteq L.$

Therefore $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$.
Thus $M$ is a regular $\Gamma-$semigroup. □

## 6. Conclusions and/or Discussions

As a further generalization of ideals, we introduced the notion of a bi-quasi-interior ideal of a $\Gamma-$semigroup as a generalization of ideal ,left ideal, right ideal, bi-ideal,quasi ideal and interior ideal of a $\Gamma-$semigroup and studied some of their properties. We introduced the notion of bi-quasi-interior simple a $\Gamma-$semigroup and characterized the bi-quasi-interior a simple $\Gamma-$semigroup, a regular $\Gamma-$semigroup using bi-quasi-interior ideals of a $\Gamma-$semigroup. We proved every bi-quasi ideal of a $\Gamma-$semigroup and bi-interior ideal of a $\Gamma-$semigroup are bi-quasi-interior ideals and studied some of the properties of bi-interior ideals of a $\Gamma-$semigroup. In continuity of this paper, we study prime, maximal and minimal bi-quasi-interior ideals of a $\Gamma-$semigroup.

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