ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 7, Number 3 (2024), 267-280
DOI: 10.62072/acm.2024.070306
ISSN: 2582-0818

INVESTIGATE SEVERAL NEW NEUTROSOPHIC NORMED SPACES OF I-CONVERGENCE OF THE TRIPLE SEQUENCES

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ABSTRACT. This article's main goal is to introduce and investigate several new neutrosophic normed spaces of I-convergence of the triple sequences. By using the compact operator, these spaces are defined. These spaces have several basic characteristics, such as fuzzy topology and verifiable inclusion relations.

1. INTRODUCTION

Fast [6] and Schoenberg [21] were the first to introduce statistical convergence independently introduce the idea of statistical convergence. Salat et al.[11] established the concept of I-convergence, a statistical convergence generality. Later, Edely, Mursaleen[14], and Tripathy independently developed the idea of statistical convergence for double sequences, while Mursaleen and Saves [20] independently developed it for fuzzy numbers. Regarding this, for double sequences [3], there are fact I and I^*- convergence, which are two quite different types of convergence.

Converged triple sequences were introduced by Gurdal, Sahiner, and Duden[16] in 2007. Numerous authors have further explored this idea; see ([4, 5, 17, 2]). Tripathy and Goswami are familiar with the idea of the me-convergence of triple sequences in probabilistic normed spaces.

An extension of the intuitionistic fuzzy set theory that Atanassov K.T. [1] first suggested in 1986. Because it allocates the degree of membership to the components so that unique persons may be distinguished in a set, fuzzy set theory is a potent tool for characterizing ambiguity and vagueness. According to a vast body of research that has recently emerged in the scientific discipline, the idea of fuzzy sets has surprisingly grown into the current norm for young scientists or researchers. Many authors now use the concept of fuzzy topology as a crucial tool in their work. The most recent advancements in fuzzy topology are the concepts of the intuitionistic fuzzy normed space[13] and the intuitionistic fuzzy-2 normed space [15].

2020 *Mathematics Subject Classification.* 26E50; 28A80.
Key words and phrases. I-convergence; Neutrosophic normed space; Statistically convergent.
Received: August 20, 2024. Accepted: September 20, 2024. Published: September 30, 2024.

After some time, Smarandache [18, 19] [17, 18] created the concept of Neutrosophic Sets [$\mathcal{NS}$], which is an alternative form of notation for classical set theory by including an intermediate membership function. This set is a formal setting designed to quantify the truth, ambiguity, and falsity. The idea of convergent triple sequences was introduced by Gurdal, Sahiner, and Duden [16] in 2007. Numerous authors have further explored this idea; see, ([4, 5, 17, 2]). Tripathy and Goswami are accustomed to the concept of triple sequences I-convergent in probabilistic normed spaces. Tripathy and Shiner [17] examined the I-convergence qualities in triple sequence spaces and presented some insightful findings.

In this work, we introduce the paper to investigate some new neutrosophic normed spaces of I-convergence of triple sequences defined by compact operator. Fuzzy topology and verifiable inclusion relations are some essential characteristics of these spaces.

Despite the potential of neutrosophic normed spaces, their study for triple sequences with I-convergence remains limited. By defining new spaces using compact operators, this research explores their fundamental properties and implications offering new perspectives in sequence analysis and fuzzy logic.

2. Preliminaries

In this section, some basic definitions are imparted that are helpful to understand the main section.

Definition 2.1. [10] A collection of sets $\mathcal{I}_d \subseteq 2^{\Im}$ allegedly is an ideal for a not an empty set $\Im$, if it fulfils:

(i) $\phi \in \mathcal{I}_d$;
(ii) if $\mathfrak{P}, \mathfrak{Q} \in \mathcal{I}_d \Rightarrow \mathfrak{P} \cup \mathfrak{Q} \in \mathcal{I}_d$;
(iii) if $\mathfrak{P} \in \mathcal{I}_d$ and $\mathfrak{Q} \subseteq \mathfrak{P} \Rightarrow \mathfrak{Q} \in \mathcal{I}_d$.

An ideal $\mathcal{I}_d$ is called a non-trivial ideal if $\Im \notin \mathcal{I}_d$.

Definition 2.2. [10] A collection of sets $\mathcal{F} \subseteq 2^{\Im}$ is allegedly a filter for a not an empty set $\Im$, if it fulfils:

(i) $\phi \notin \mathcal{F}$;
(ii) if $\mathfrak{P}, \mathfrak{Q} \in \mathcal{F} \Rightarrow \mathfrak{P} \cap \mathfrak{Q} \in \mathcal{F}$;
(iii) if $\mathfrak{P} \in \mathcal{F}$ and $\mathfrak{P} \subseteq \mathfrak{Q} \Rightarrow \mathfrak{Q} \in \mathcal{F}$.

For each ideal $\mathcal{I}_d$ there is a filter $\mathcal{F}(\mathcal{I}_d)$ corresponding to $\mathcal{I}_d$.
$\mathcal{F}(\mathcal{I}_d) = \{\mathfrak{P} \subseteq \mathbb{N} : \mathfrak{P}^c \in \mathcal{I}_d\}$, where $\mathfrak{P}^c = \mathbb{N} - \mathfrak{P}$.

Definition 2.3. [2] A triple sequence $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})$ is allegedly $\mathcal{I}_d$ convergent to a number ξ if for each $\varepsilon_0 > 0$, like that $\{(\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} - \xi| \geq \varepsilon_0\} \in \mathcal{I}_d$ and to be symbolized as, $\mathcal{I}_d \lim \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} = \xi$.

Definition 2.4. [12] A binary operation $\circledast : [0,1] \times [0,1] \to [0,1]$is known as a triangular norm if $\circledast$ fulfils:

- $\circledast$ is commutative and associative,
- $\circledast$ is continuous,
- $\ell \circledast 1 = \ell$ for all $\ell \in [0,1]$,
- $\ell_1 \circledast g_1 \leq \ell_2 \circledast g_2$ whenever $\ell_1 \leq \ell_2$ and $g_1 \leq g_2$ for all ℓ_1, g_1, ℓ_2, g_2 in $[0,1]$.

Examples: (i) $\ell \circledast g = \ell g$ (ii) $\ell \circledast g = \min\{\ell, g\}$.

Definition 2.5. [12] A binary operation $\diamond : [0,1]\times[0,1] \to [0,1]$ is known as a triangular norm if $\diamond$ fulfils:

- $\diamond$ is commutative and associative,
- $\diamond$ is continuous,
- $f \diamond 0 = f$ for all $f \in [0,1]$,
- $f_1 \diamond g_1 \leq f_2 \diamond g_2$ whenever $f_1 \leq f_2$ and $g_1 \leq g_2$ for all f_1, g_1, f_2, g_2 in $[0,1]$.

Examples: (i) $f \diamond g = \min\{f + g, 1\}$ (ii) $f \diamond g = \max\{f, g\}$.

Definition 2.6. The 7-tuple $(\Im, \mathcal{J}, \mathcal{G}, \mathcal{H}, \circledast, \odot, \diamond)$ is allegedly $\mathcal{NNS}$ if $\Im$ is a linear space, $\circledast$ is a continuous triangular norm, $\odot$ and $\diamond$ are continuous triangular conorm, $\mathcal{J}, \mathcal{G}$ and $\mathcal{H}$ are fuzzy sets on $\Im \times (0,\infty)$ fulfils the subsequent criteria:
For every $\zeta_1, \Upsilon_1 \in \Im$ and $s, \varpi_1 > 0$;

(a) $0 \leq \mathcal{J}(\zeta_1, \varpi_1) \leq 1; 0 \leq \mathcal{G}(\zeta_1, \varpi_1) \leq 1; 0 \leq \mathcal{H}(\zeta_1, \varpi_1) \leq 1$,
(b) $\mathcal{J}(\zeta_1, \varpi_1) + \mathcal{G}(\zeta_1, \varpi_1) + \mathcal{H}(\zeta_1, \varpi_1) \leq 3$,
(c) $\mathcal{J}(\zeta_1, \varpi_1) > 0$,
(d) $\mathcal{J}(\zeta_1, \varpi_1) = 1$ if and only if $\zeta_1 = 0$,
(e) $\mathcal{J}(f\zeta_1, \varpi_1) = \mathcal{J}\left(\zeta_1, \frac{\varpi_1}{|f|}\right)$ for each $f \neq 0$,
(f) $\mathcal{J}(\zeta_1, \varpi_1) \circledast \mathcal{J}(\Upsilon, s) \leq \mathcal{J}(\zeta_1 + \Upsilon, \varpi_1 + s)$,
(g) $\mathcal{J}(\zeta_1, \varpi_1) : (0,\infty) \to [0,1]$ is continuous,
(h) $\lim_{\varpi_1 \to \infty} \mathcal{J}(\zeta_1, \varpi_1) = 1$ and $\lim_{\varpi_1 \to 0} \mathcal{J}(\zeta_1, \varpi_1) = 0$,
(i) $\mathcal{G}(\zeta_1, \varpi_1) < 1$,
(j) $\mathcal{G}(\zeta_1, \varpi_1) = 0$ if and only if $\zeta_1 = 0$,
(k) $\mathcal{G}(f\zeta_1, \varpi_1) = \mathcal{G}\left(\zeta_1, \frac{\varpi_1}{|f|}\right)$ for each $f \neq 0$,
(l) $\mathcal{G}(\zeta_1, \varpi_1) \odot \mathcal{G}(\Upsilon, s) \geq \mathcal{G}(\zeta_1 + \Upsilon, \varpi_1 + s)$,
(m) $\mathcal{G}(\zeta_1, \varpi_1) : (0,\infty) \to [0,1]$ is continuous,
(n) $\lim_{\varpi_1 \to \infty} \mathcal{G}(\zeta_1, \varpi_1) = 0$ and $\lim_{\varpi_1 \to 0} \mathcal{G}(\zeta_1, \varpi_1) = 1$,
(o) $\mathcal{H}(\zeta_1, \varpi_1) < 1$,
(p) $\mathcal{H}(\zeta_1, \varpi_1) = 0$ if and only if $\zeta_1 = 0$,
(q) $\mathcal{H}(f\zeta_1, \varpi_1) = \mathcal{H}\left(\zeta_1, \frac{\varpi_1}{|f|}\right)$ for each $f \neq 0$,
(r) $\mathcal{H}(\zeta_1, \varpi_1) \diamond \mathcal{G}(\Upsilon, s) \geq \mathcal{H}(\zeta_1 + \Upsilon, \varpi_1 + s)$,
(s) $\mathcal{H}(\zeta_1, \varpi_1) : (0,\infty) \to [0,1]$ is continuous,
(t) $\lim_{\varpi_1 \to \infty} \mathcal{H}(\zeta_1, \varpi_1) = 0$ and $\lim_{\varpi_1 \to 0} \mathcal{H}(\zeta_1, \varpi_1) = 1$.

Definition 2.7. Let $(\Im, \mathcal{J}, \mathcal{G}, \mathcal{H}, \circledast, \odot, \diamond)$ be an $\mathcal{NNS}$. Then a sequence $\zeta = (\zeta_{\check{k}})$ is allegedly converging towards a number ξ with regard to the $\mathcal{N}n$ $(\mathcal{J}, \mathcal{G}, \mathcal{H})$ if for all $\varepsilon_0 > 0$ and $\varpi_1 > 0$, there exists $\check{k}_0 \in \mathbb{N}$ in such a way
$\mathcal{J}(\zeta_{\check{k}} - \xi, \varpi_1) > 1 - \varepsilon_0, \mathcal{G}(\zeta_{\check{k}} - \xi, \varpi_1) < \varepsilon_0$ and $\mathcal{H}(\zeta_{\check{k}} - \xi, \varpi_1) < \varepsilon_0$, for all $\check{k} \geq \check{k}_0$ and to be represented as, $(\mathcal{J}, \mathcal{G}, \mathcal{H}) \lim \zeta = \xi$.

Definition 2.8. Let $(\Im, \mathcal{J}, \mathcal{G}, \mathcal{H}, \circledast, \odot, \diamond)$ be an $\mathcal{NNS}$. Then a sequence $\zeta = (\zeta_{\check{k}})$ is allegedly statistically convergent($\mathcal{SC}o$) to a number ξ with regard to the $\mathcal{N}n$ $(\mathcal{J}, \mathcal{G}, \mathcal{H})$ if for everyone $\varepsilon_0 > 0$ and $\varpi_1 > 0$, we have

$$\delta\left(\left\{\begin{array}{c}\check{k} \in \mathbb{N} : \mathcal{J}(\zeta_{\check{k}} - \xi, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathcal{G}(\zeta_{\check{k}} - \xi, \varpi_1) \geq \varepsilon_0 \text{ and } \mathcal{H}(\zeta_{\check{k}} - \xi, \varpi_1) \geq \varepsilon_0\end{array}\right\}\right) = 0$$

and to be represented as, $st_{(\mathcal{J},\mathcal{G},\mathcal{H})} - \lim \zeta_{\check{k}} = \xi$.

Definition 2.9. Let $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, \circledast, \odot, \diamond)$ be an $\mathcal{NNS}$. Then a sequence $\zeta = (\zeta_{\breve{k}})$ is allegedly I-convergent to a number ξ with regard to the $\mathcal{N}n$ $(\mathscr{J}, \mathscr{G}, \mathscr{H})$ if for everyone $\varepsilon_0 > 0$ and $\varpi_1 > 0$, we have

$$\left\{ \begin{array}{c} \breve{k} \in \mathbb{N} : \mathscr{J}(\zeta_{\breve{k}} - \xi, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\zeta_{\breve{k}} - \xi, \varpi_1) \geq \varepsilon_0 \text{ and } \mathscr{H}(\zeta_{\breve{k}} - \xi, \varpi_1) \geq \varepsilon_0 \end{array} \right\} \in I.$$

and to be represented as, $I_{(\mathscr{J},\mathscr{G},\mathscr{H})} \lim \zeta_{\breve{k}} = \xi$.

3. Triple sequences in $\mathcal{NNS}$ by compact linear operator

Definition 3.1. Let $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, \circledast, \odot, \diamond)$ be an $\mathcal{NNS}$. Then a triple sequence $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})$ is allegedly $\mathcal{SC}o$ to a number ξ with regard to the $\mathcal{N}n$ $(\mathscr{J}, \mathscr{G}, \mathscr{H})$ if for everyone $\varepsilon_0 > 0$ and $\varpi_1 > 0$, we have

$$\delta\left(\left\{ \begin{array}{c} (\breve{\imath}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathscr{J}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \geq \varepsilon_0 \text{ and } \mathscr{H}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \geq \varepsilon_0 \end{array} \right\}\right) = 0$$

or equivalently,

$$\lim_{pqr} \frac{1}{pqr} \left| \left\{ \begin{array}{c} n \leq p, k \leq q, l \leq r : \mathscr{J}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \geq \varepsilon_0 \text{ and } \mathscr{H}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \geq \varepsilon_0 \end{array} \right\} \right| = 0$$

and to be symbolized as, $st^3_{(\mathscr{J},\mathscr{G},\mathscr{H})} \lim \zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} = \xi$.

Definition 3.2. Let $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, \circledast, \odot, \diamond)$ be an $\mathcal{NNS}$. Then a triple sequence $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})$ is allegedly statistically Cauchy(SCa) with regard to the $\mathcal{N}n$ $(\mathscr{J}, \mathscr{G}, \mathscr{H})$ if for everyone $\varepsilon_0 > 0$ and $\varpi_1 > 0$ there exists $p = p(\varepsilon_0), q = q(\varepsilon_0), r = r(\varepsilon_0)$ like that, we have

$$\delta\left(\left\{ \begin{array}{c} (\breve{\imath}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathscr{J}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \zeta_{pqr}, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \zeta_{pqr}, \varpi_1) \geq \varepsilon_0 \text{ and } \mathscr{H}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \zeta_{pqr}, \varpi_1) \geq \varepsilon_0 \end{array} \right\}\right) = 0.$$

Definition 3.3. Let $(\Im, \mathscr{J}, \mathscr{G}, \mathscr{H}, \circledast, \odot, \diamond)$ be an $\mathcal{NNS}$. Then a triple sequence $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})$ is allegedly I_3-convergent to a number ξ with regard to the $\mathcal{N}n$ $(\mathscr{J}, \mathscr{G}, \mathscr{H})$ if for everyone $\varepsilon_0 > 0$ and $\varpi_1 > 0$, we have

$$\left\{ \begin{array}{c} (\breve{\imath}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathscr{J}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \geq \varepsilon_0 \text{ and } \mathscr{H}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} - \xi, \varpi_1) \geq \varepsilon_0 \end{array} \right\} \in I_3.$$

and to be symbolized as, $I_3^{(\mathscr{J},\mathscr{G},\mathscr{H})} \lim \zeta_k = \xi$.
Where, I_3 is a non trivial ideal of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$.

A **Compact Linear Operator** is a relation $\mathscr{R} : \mathscr{M} \to \mathscr{N}$ which meets two criteria:

(i) $\mathscr{R}$ is linear;
(ii) For each limited sequence $(\zeta_{\breve{k}}) \in \mathscr{M}$, $\mathscr{R}(\zeta_{\breve{k}})$ has a convergent subsequence in $\mathscr{N}$.

where $\mathscr{M}$ and $\mathscr{N}$ are normed linear spaces. The collection of all bounded linear operators $\mathscr{B}(\mathscr{M}, \mathscr{N})$ is normed linear space normed by $\|\mathscr{R}\| = \sup_{\zeta \in V, \|\zeta\|=1} \|\mathscr{R}\zeta\|$

Remark: $\mathscr{B}(\mathscr{M}, \mathscr{N})$ is a closed subspace of the set of all compact linear operators $\mathfrak{C}(\mathscr{M}, \mathscr{N})$. Additionally, $\mathfrak{C}(\mathscr{M}, \mathscr{N})$ is a Banach space if $\mathscr{N}$ is a Banach space.

The I-convergence of triple sequences described by compact operators in $\mathcal{NNS}$ is the topic of this article. In addition, we define an open ball with a non-zero radius that is centered at a triple sequence and investigate the topology of the defined spaces.

4. Main results

In this section, we provide the subsequent classes of sequence spaces.

$$ {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T}) = \Big\{(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\vartheta : \big\{(\breve{\imath},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \leq 1-\varepsilon_0 \text{ or } \mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \text{ or } \mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0\big\} \in I_3\Big\} $$

$$ {}_3S^I_{0(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T}) = \Big\{(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\vartheta : \big\{(\breve{\imath},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) \leq 1-\varepsilon_0 \text{ or } \mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) \geq \varepsilon_0 \text{ or } \mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) \geq \varepsilon_0\big\} \in I_3\Big\} $$

We also define;

$$ {}_3\mathcal{B}_\zeta(\delta,\varpi_1)(\mathcal{T}) = \Big\{(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\vartheta : \big\{(\breve{\imath},\breve{\varrho},\breve{\sigma}) : \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}(\mathcal{T}(\zeta)-\mathcal{T}(\Upsilon),\varpi_1) > 1-\delta, \mathcal{G}(\mathcal{T}(\zeta)-\mathcal{T}(\Upsilon),\varpi_1) < \delta \text{ and } \mathcal{H}(\mathcal{T}(\zeta)-\mathcal{T}(\Upsilon),\varpi_1) < \delta\big\} \in I_3\Big\} $$

which is an open ball with centre at $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})$ and radius δ with respect to ϖ_1.

Theorem 4.1. *If a triple sequence $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ is I_3-convergent to a number ξ with regard to the $\mathcal{N}n$ $(\mathcal{J},\mathcal{G},\mathcal{H})$, then the limit ξ is distinctive.*

Proof. Let $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ like that $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}\lim \zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} = \xi_1$ and $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}\lim \zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} = \xi_2$, for a given ε_0, we have $\delta > 0$ like that $(1-\delta)\circledast(1-\delta) > 1-\varepsilon_0, \delta\odot\delta < \varepsilon_0$ and $\delta\Diamond\delta < \varepsilon_0$, then we define for $\varpi_1 > 0$,

$$ \mathcal{D}_1 = \Big\{(\breve{\imath},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}\left(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi_1,\frac{\varpi_1}{2}\right) \leq 1-\delta \text{ or } \mathcal{G}\left(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi_1,\frac{\varpi_1}{2}\right) > \delta \text{ or } \mathcal{H}\left(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi_1,\frac{\varpi_1}{2}\right) > \delta\Big\} $$

$$ \mathcal{D}_2 = \Big\{(\breve{\imath},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}\left(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi_2,\frac{\varpi_1}{2}\right) \leq 1-\delta \text{ or } \mathcal{G}\left(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi_2,\frac{\varpi_1}{2}\right) > \delta \text{ or } \mathcal{H}\left(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}})-\xi_2,\frac{\varpi_1}{2}\right) > \delta\Big\} $$

Since $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}\lim \zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} = \xi_1$ and $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}\lim \zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} = \xi_2$
$\Rightarrow \mathcal{D}_1 \in I_3$ and $\mathcal{D}_2 \in I_3$ for all $\varpi_1 > 0$.
Consider $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2, \Rightarrow \mathcal{D} \in I_3$ and $\mathcal{D}^c \in \mathcal{F}(I_3)$.
If, $(p,q,r) \in \mathcal{D}^c$, then

$$ \begin{aligned} \mathcal{J}(\xi_1-\xi_2,\varpi_1) &\geq \mathcal{J}\left(\mathcal{T}\zeta_{pqr}-\xi_1,\frac{\varpi_1}{2}\right) \circledast \mathcal{J}\left(\mathcal{T}\zeta_{pqr}-\xi_2,\frac{\varpi_1}{2}\right) \\ &> (1-\delta)\circledast(1-\delta) \\ &> 1-\varepsilon_0. \end{aligned} $$

$\varepsilon_0 > 0$ being arbitrary

$$ \Rightarrow \mathcal{J}(\xi_1-\xi_2,\varpi_1) = 1 \text{ for all } \varpi_1 > 0 \Rightarrow \xi_1 = \xi_2. $$

Also,

$$ \begin{aligned} \mathcal{G}(\xi_1-\xi_2,\varpi_1) &\leq \mathcal{G}\left(\mathcal{T}\zeta_{pqr}-\xi_1,\frac{\varpi_1}{2}\right) \odot \mathcal{G}\left(\mathcal{T}\zeta_{pqr}-\xi_2,\frac{\varpi_1}{2}\right) \\ &< (\delta)\odot(\delta) \end{aligned} $$

$$< \varepsilon_0.$$

$\varepsilon_0 > 0$ being arbitrary

$$\Rightarrow \mathscr{G}(\xi_1 - \xi_2, \varpi_1) = 0, \quad \text{for all } \varpi_1 > 0 \Rightarrow \xi_1 = \xi_2.$$

Similarly it can be proved that,
$\mathscr{H}(\xi_1 - \xi_2, \varpi_1) < \varepsilon_0$ for all $\varpi_1 > 0 \Rightarrow \xi_1 = \xi_2$.
Thus, it may be said that, $I_3^{(\mathscr{J},\mathscr{G},\mathscr{H})} \lim \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} = \xi$ is distinctive. □

Theorem 4.2. *Let $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})$ be a triple sequence in ${}_3S^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathscr{T})$. Consequently, the following claims are equivalent:*

(i) *If $I_3^{(\mathscr{J},\mathscr{G},\mathscr{H})} \lim \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} = \xi$,*

(ii) $\left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathscr{J}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) \geq \varepsilon_0 \text{ and } \mathscr{H}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) \geq \varepsilon_0 \end{array} \right\} \in I_3$

(iii) $\left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathscr{J}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) > 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) < \varepsilon_0 \text{ and } \mathscr{H}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) < \varepsilon_0 \end{array} \right\} \in \mathscr{F}(I_3)$

(iv) $I_3 \lim \mathscr{J}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) = 1, I_3 \lim \mathscr{G}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) = 0$ *and* $I_3 \lim \mathscr{H}(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \varpi_1) = 0.$

Theorem 4.3. *The spaces ${}_3S^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathscr{T})$ and ${}_3S^I_{0(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathscr{T})$ are linear spaces.*

Proof. We demonstrate the linearity of the space ${}_3S^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathscr{T})$.For other space, it comes right after.
Let $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}), \Upsilon = (\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathscr{T})$ and $\mathscr{f}, \mathscr{g}$ be non-zero scalars and $\varepsilon_0 > 0$ be given, then

$$\mathscr{A}_1 = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathscr{J}\left(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}\left(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) \geq \varepsilon_0 \text{ and } \mathscr{H}\left(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) \geq \varepsilon_0 \end{array} \right\} \in I_3$$

$$\mathscr{A}_2 = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathscr{J}\left(\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_2, \frac{\varpi_1}{2|\mathscr{g}|}\right) \leq 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}\left(\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_2, \frac{\varpi_1}{2|\mathscr{g}|}\right) \geq \varepsilon_0 \text{ and } \mathscr{H}\left(\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_2, \frac{\varpi_1}{2|\mathscr{g}|}\right) \geq \varepsilon_0 \end{array} \right\} \in I_3$$

$$\mathscr{A}_1^c = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathscr{J}\left(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) > 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}\left(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) < \varepsilon_0 \text{ and } \mathscr{H}\left(\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) < \varepsilon_0 \end{array} \right\} \in \mathscr{F}(I_3)$$

$$\mathscr{A}_2^c = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathscr{J}\left(\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_2, \frac{\varpi_1}{2|\mathscr{g}|}\right) > 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}\left(\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_2, \frac{\varpi_1}{2|\mathscr{g}|}\right) < \varepsilon_0 \text{ and } \mathscr{H}\left(\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi_2, \frac{\varpi_1}{2|\mathscr{g}|}\right) < \varepsilon_0 \end{array} \right\} \in \mathscr{F}(I_3).$$

Define $\mathscr{A}_3 = \mathscr{A}_1 \cup \mathscr{A}_2$, so that $\mathscr{A}_3 \in I_3 \Rightarrow \mathscr{A}_3^c \in \mathscr{F}(I_3)$ is non-empty. We now demonstrate that for everyone $(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}), (\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) \in_3 S^I_{(\mathscr{J},\mathscr{G},\mathscr{H})}(\mathscr{T})$.

$$\mathscr{A}_3^c \subset \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \\ \mathscr{J}(\mathscr{f}\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) + \mathscr{g}\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})) - (\mathscr{f}\xi_1 + \mathscr{g}\xi_2), \varpi_1) > 1 - \varepsilon_0 \text{ or} \\ \mathscr{G}(\mathscr{f}\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) + \mathscr{g}\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})) - (\mathscr{f}\xi_1 + \mathscr{g}\xi_2), \varpi_1) < \varepsilon_0 \text{ and} \\ \mathscr{H}(\mathscr{f}\mathscr{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) + \mathscr{g}\mathscr{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})) - (\mathscr{f}\xi_1 + \mathscr{g}\xi_2), \varpi_1) < \varepsilon_0 \end{array} \right\}.$$

Let $(p, q, r) \in \mathscr{A}_3^c$. In this case

$$\mathscr{J}\left(\mathscr{T}(\zeta_{pqr}) - \xi_1, \frac{\varpi_1}{2|\mathscr{f}|}\right) > 1 - \varepsilon_0 \text{ or}$$

$$\mathscr{G}\left(\mathcal{T}(\zeta_{pqr})-\xi_1,\frac{\varpi_1}{2|\ell|}\right)<\varepsilon_0 \text{ and}$$
$$\mathscr{H}\left(\mathcal{T}(\zeta_{pqr})-\xi_1,\frac{\varpi_1}{2|\ell|}\right)<\varepsilon_0.$$

and

$$\mathscr{F}\left(\mathcal{T}(\Upsilon_{pqr})-\xi_2,\frac{\varpi_1}{2|g|}\right)>1-\varepsilon_0 \text{ or}$$
$$\mathscr{G}\left(\mathcal{T}(\Upsilon_{pqr})-\xi_2,\frac{\varpi_1}{2|g|}\right)<\varepsilon_0 \text{ and}$$
$$\mathscr{H}\left(\mathcal{T}(\Upsilon_{pqr})-\xi_2,\frac{\varpi_1}{2|g|}\right)<\varepsilon_0.$$

We have

$$\begin{aligned}&\mathscr{F}(\ell\mathcal{T}(\zeta_{pqr})+g\mathcal{T}(\Upsilon_{pqr}))-(\ell\xi_1+g\xi_2),\varpi_1)\\&\quad\geq\mathscr{F}\left(\ell\mathcal{T}(\zeta_{pqr})-\ell\xi_1,\frac{\varpi_1}{2}\right)\circledast\mathscr{F}\left(g\mathcal{T}(\zeta_{pqr})-g\xi_2,\frac{\varpi_1}{2}\right)\\&\quad=\mathscr{F}\left(\mathcal{T}(\zeta_{pqr})-\xi_1,\frac{\varpi_1}{2|\ell|}\right)\circledast\mathscr{F}\left(\mathcal{T}(\zeta_{pqr})-\xi_2,\frac{\varpi_1}{2|g|}\right)\\&\quad>(1-\varepsilon_0)\circledast(1-\varepsilon_0)=1-\varepsilon_0.\end{aligned}$$

Also,

$$\begin{aligned}&\mathscr{G}(\ell\mathcal{T}(\zeta_{pqr})+g\mathcal{T}(\Upsilon_{pqr}))-(\ell\xi_1+g\xi_2),\varpi_1)\\&\quad\leq\mathscr{G}\left(\ell\mathcal{T}(\zeta_{pqr})-\ell\xi_1,\frac{\varpi_1}{2}\right)\odot\mathscr{G}\left(g\mathcal{T}(\zeta_{pqr})-g\xi_2,\frac{\varpi_1}{2}\right)\\&\quad=\mathscr{G}\left(\mathcal{T}(\zeta_{pqr})-\xi_1,\frac{\varpi_1}{2|\ell|}\right)\odot\mathscr{G}\left(\mathcal{T}(\zeta_{pqr})-\xi_2,\frac{\varpi_1}{2|g|}\right)\\&\quad<\varepsilon_0\odot\varepsilon_0=\varepsilon_0.\end{aligned}$$

and

$$\begin{aligned}&\mathscr{H}(\ell\mathcal{T}(\zeta_{pqr})+g\mathcal{T}(\Upsilon_{pqr}))-(\ell\xi_1+g\xi_2),\varpi_1)\\&\quad\leq\mathscr{H}\left(\ell\mathcal{T}(\zeta_{pqr})-\ell\xi_1,\frac{\varpi_1}{2}\right)\Diamond\mathscr{H}\left(g\mathcal{T}(\zeta_{pqr})-g\xi_2,\frac{\varpi_1}{2}\right)\\&\quad=\mathscr{H}\left(\mathcal{T}(\zeta_{pqr})-\xi_1,\frac{\varpi_1}{2|\ell|}\right)\Diamond\mathscr{H}\left(\mathcal{T}(\zeta_{pqr})-\xi_2,\frac{\varpi_1}{2|g|}\right)\\&\quad<\varepsilon_0\Diamond\varepsilon_0=\varepsilon_0.\end{aligned}$$

Hence,

$$\mathscr{A}_3^c\subset\left\{\begin{array}{c}(\breve{\iota},\breve{\varrho},\breve{\sigma})\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}:\\ \mathscr{F}(\ell\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})+g\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}))-(\ell\xi_1+g\xi_2),\varpi_1)>1-\varepsilon_0\text{ or}\\ \mathscr{G}(\ell\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})+g\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}))-(\ell\xi_1+g\xi_2),\varpi_1)<\varepsilon_0\text{ and}\\ \mathscr{H}(\ell\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})+g\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}))-(\ell\xi_1+g\xi_2),\varpi_1)<\varepsilon_0.\end{array}\right\}$$

Hence ${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathcal{T})$ is a linear space. □

Theorem 4.4. *Let $\zeta=(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})$ be a triple sequence in ${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathcal{T})$ like that $(\mathscr{F},\mathscr{G},\mathscr{H})_3\lim\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}=\xi$, then $I_3^{(\mathscr{F},\mathscr{G},\mathscr{H})}\lim\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}=\xi$.*

Proof. Let $(\mathcal{J},\mathcal{G},\mathcal{H})_3 \lim \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} = \xi$, and $\varepsilon_0 > 0$ be given then for everyone $\varpi_1 > 0$, there exists $(p,q,r) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ like that
$\mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi) > 1-\varepsilon_0, \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi) < \varepsilon_0$ and $\mathcal{H}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi) < \varepsilon_0$, for all $(\breve{\iota},\breve{\varrho},\breve{\sigma}) \geq (p,q,r)$.
As a result, we have

$$\mathcal{B} = \left\{ \begin{array}{c} (\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \\ \mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \leq 1-\varepsilon_0 \text{ or} \\ \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \text{ and} \\ \mathcal{H}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \end{array} \right\} \subseteq \left\{ \begin{array}{c} (p',q',r') \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \\ p' < p-1, q' < q-1, r' < r-1 \end{array} \right\}$$

But, I_3 being admissible $\Rightarrow \mathcal{B} \in I_3$. Hence, $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})} \lim \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} = \xi$. □

Theorem 4.5. *Let $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})$ be a triple sequence in ${}_3\vartheta$. If $\Upsilon = (\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})$ in ${}_3\vartheta$ is a $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}$ convergent sequence like that $\{(\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} \neq \Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}\} \in I_3$, then ζ is also $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}$ convergent.*

Proof. Think about the set, $\{(\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} \neq \Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}\} \in I_3$ and $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})} \lim \Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}} = \xi$ then $0 < \varepsilon_0 < 1$ for all $\varpi_1 > 0$, we get

$$\left\{ \begin{array}{c} (\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \mathcal{J}(\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \leq 1-\varepsilon_0 \text{ or} \\ \mathcal{G}(\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \text{ and} \mathcal{H}(\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \end{array} \right\} \in I_3.$$

$0 < \varepsilon_0 < 1$ for all $\varpi_1 > 0$,

$$\left\{ \begin{array}{c} (\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \\ \mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \leq 1-\varepsilon_0 \text{ or} \\ \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \text{ and} \\ \mathcal{H}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \end{array} \right\}$$

$$\subseteq \{(\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}} \neq \Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}}\} \cup \left\{ \begin{array}{c} (\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \\ \mathcal{J}(\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \leq 1-\varepsilon_0 \text{ or} \\ \mathcal{G}(\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \text{ and} \\ \mathcal{H}(\mathcal{T}(\Upsilon_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \end{array} \right\}. \tag{4.1}$$

Since the right-hand side of (4.1) contains an element of I_3, we obtain

$$\left\{ \begin{array}{c} (\breve{\iota},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}: \mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \leq 1-\varepsilon_0 \text{ or} \\ \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \text{ and} \mathcal{H}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})-\xi,\varpi_1) \geq \varepsilon_0 \end{array} \right\} \in I_3.$$

□

Theorem 4.6. *If $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ is a Cauchy sequence with regard to $\mathcal{N}n$ $(\mathcal{J},\mathcal{G},\mathcal{H})$ then it is $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})}$-Cauchy with regard to same norm.*

Proof. The proof was deleted because it seemed obvious. □

Theorem 4.7. *If $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ is a Cauchy sequence with regard to $\mathcal{N}n$ $(\mathcal{J},\mathcal{G},\mathcal{H})$ then the sequence $\zeta = (\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}})$ has a subsequence which is an ordinary Cauchy sequence with regard to the similar norm.*

Proof. The proof was deleted because it seemed obvious. □

Theorem 4.8. *Each open ball ${}_3\mathcal{B}_\zeta(\delta,\varpi_1)(\mathcal{T})$ is an open set in ${}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$.*

Proof. Let ${}_3\mathscr{B}_\zeta(\delta,\varpi_1)(\mathscr{T})$ be an open ball have radius δ, centered at ζ with regard to ϖ_1. i.e

$${}_3\mathscr{B}_\zeta(\delta,\varpi_1)(\mathscr{T}) = \left\{ \Upsilon = (\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in 3\vartheta : \left\{ \begin{array}{c} (\breve{\imath},\breve{\varrho},\breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \\ \mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) > 1-\delta \text{ or} \\ \mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) < \delta \text{ and} \\ \mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) < \delta \end{array} \right\} \in I_3 \right\}.$$

Let $\Upsilon \in {}_3\mathscr{B}^c_\zeta(\delta,\varpi_1)(\mathscr{T})$.
Then,

$$\begin{aligned} &\mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) > 1-\delta, \\ &\mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) < \delta \text{ and} \\ &\mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) < \delta. \end{aligned}$$

Since, $\mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) > 1-\delta,$ there exists $0 < \varpi_0 < \varpi_1$ like that,

$$\begin{aligned} &\mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0) > 1-\delta, \\ &\mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0) < \delta \text{ and} \\ &\mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0) < \delta. \end{aligned}$$

Now, think about $\delta_0 = \mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0)$, we have $\delta_0 > 1-\delta$ a thing exists, $\varpi_2 \in (0,1)$ like that $\delta_0 > 1-\varpi_2 > 1-\delta$.
For $\delta_0 > 1-\varpi_2$, we have $\delta_1,\delta_2,\delta_3 \in (0,1)$ like that
$\delta_0 \circledast \delta_1 > 1-\varpi_2, (1-\delta_0)\odot(1-\delta_2) \leq \varpi_2$ and $(1-\delta_0)\diamond(1-\delta_3) \leq \varpi_2$.
Let $\delta_4 = \max\{\delta_1,\delta_2,\delta_3\}$.
In this case, we look at the open ball ${}_3\mathscr{B}^c_\Upsilon(1-\delta_4,\varpi_1-\varpi_0)(\mathscr{T})$ and we show that ${}_3\mathscr{B}^c_\Upsilon(1-\delta_4,\varpi_1-\varpi_0)(\mathscr{T}) \subset {}_3\mathscr{B}^c_\zeta(\delta,\varpi_1)(\mathscr{T})$.
Let $\chi = (\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\mathscr{B}^c_\Upsilon(1-\delta_4,\varpi_1-\varpi_0)(\mathscr{T})$ then

$$\begin{aligned} &\mathscr{J}(\mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1-\varpi_0) > \delta_4 \text{ or} \\ &\mathscr{G}(\mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1-\varpi_0) < 1-\delta_4 \text{ or} \\ &\mathscr{H}(\mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1-\varpi_0) < 1-\delta_4 \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) \\ &\geq \mathscr{J}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0) \circledast \mathscr{J}(\mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1-\varpi_0) \\ &\geq (\delta_0 \circledast \delta_3) \geq (\delta_0 \circledast \delta_1) \geq (1-\varpi_2) \geq (1-\delta) \end{aligned}$$

also,

$$\begin{aligned} &\mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) \\ &\leq \mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0) \odot \mathscr{G}(\mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1-\varpi_0) \\ &\leq (1-\delta_0)\odot(1-\delta_4) \leq (1-\delta_0)\odot(1-\delta_2) \leq \varpi_2 \geq \delta. \end{aligned}$$

and

$$\begin{aligned} &\mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1) \\ &\leq \mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_0) \diamond \mathscr{H}(\mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}),\varpi_1-\varpi_0) \\ &\leq (1-\delta_0)\diamond(1-\delta_4) \leq (1-\delta_0)\diamond(1-\delta_2) \leq \varpi_2 \geq \delta. \end{aligned}$$

Thus $z \in {}_3\mathscr{B}^c_\zeta(\delta, \varpi_1)(\mathscr{T})$. Hence ${}_3\mathscr{B}^c_\Upsilon(1-\delta_4, \varpi_1-\varpi_0)(\mathscr{T}) \subset {}_3\mathscr{B}^c_\zeta(\delta, \varpi_1)(\mathscr{T})$. □

Remark: ${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ is an $\mathcal{NNS}$. Think about the set

$${}_3\tau^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T}) = \big\{\mathscr{A} \subset {}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T}) : \text{ for everyone } \zeta \in \mathscr{A} \text{ there exists } \varpi_1 > 0 \text{ and } 0 < \delta < 1 \text{ such that } {}_3\mathscr{B}_\zeta(\delta, \varpi_1)(\mathscr{T}) \subset \mathscr{A}\big\}.$$

$\Rightarrow$ ${}_3\tau^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ is clearly a topology on ${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$.

Theorem 4.9. *The topology ${}_3\tau^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ on ${}_3S^I_{0(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ is first countable.*

Proof. $\left\{{}_3\mathscr{B}_\zeta\left(\frac{1}{k}, \frac{1}{k}\right)(\mathscr{T}) : k = 1, 2, 3, \dots\right\}$ forms a local base at ζ therefore the topology ${}_3\tau^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ on ${}_3S^I_{0(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ is first countable. □

Theorem 4.10. *${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ and ${}_3S^I_{0(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ are Hausdorff spaces.*

Proof. We determine the outcome for the space ${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$.
The outcome is obvious for ${}_3S^I_{0(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$.
Let, $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \Upsilon = (\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ like that $(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \neq (\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}})$.
Then $0 < \mathscr{F}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1) < 1$ or $0 < \mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1) < 1$ and $0 < \mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1) < 1$.
Consider, $\delta_1 = \mathscr{F}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1), \delta_2 = \mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1)$ and $\delta_3 = \mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1)$, $\delta = \max\{\delta_1, 1-\delta_2, 1-\delta_3\}$. If we take, $\delta_0 \in (\delta, 1)$ there exists δ_4, δ_5 and δ_6 like that $\delta_4 \circledast \delta_4 \geq \delta_0, (1-\delta_5) \odot (1-\delta_5) \leq (1-\delta_0)$ and $(1-\delta_6) \diamond (1-\delta_6) \leq (1-\delta_0)$.
Again for, $\delta_7 = \max\{\delta_5, \delta_5, \delta_6\}$ and we consider ${}_3\mathscr{B}_\zeta\left(1-\delta_7, \frac{\varpi_1}{2}\right)$ and ${}_3\mathscr{B}_\Upsilon\left(1-\delta_7, \frac{\varpi_1}{2}\right)$.
Clearly, ${}_3\mathscr{B}^c_\zeta\left(1-\delta_7, \frac{\varpi_1}{2}\right) \cap {}_3\mathscr{B}^c_\Upsilon\left(1-\delta_7, \frac{\varpi_1}{2}\right) = \emptyset$.
For, if there exists $\chi = (\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\mathscr{B}^c_\zeta\left(1-\delta_7, \frac{\varpi_1}{2}\right) \cap {}_3\mathscr{B}^c_\Upsilon\left(1-\delta_7, \frac{\varpi_1}{2}\right)$
then

$$\begin{aligned}
\delta_1 &= \mathscr{F}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1)\\
&\geq \mathscr{F}\left(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \frac{\varpi_1}{2}\right) \circledast \mathscr{F}\left(\mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \frac{\varpi_1}{2}\right)\\
&\geq \delta_7 \circledast \delta_7 \geq \delta_4 \circledast \delta_4 \geq \delta_0 > \delta_1,\\
\delta_2 &= \mathscr{G}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1)\\
&\leq \mathscr{G}\left(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \frac{\varpi_1}{2}\right) \odot \mathscr{G}\left(\mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \frac{\varpi_1}{2}\right)\\
&\leq (1-\delta_7) \odot (1-\delta_7)\\
&\leq (1-\delta_5) \odot (1-\delta_5)\\
&\leq (1-\delta_0) < \delta_2 \text{ and}\\
\delta_3 &= \mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \varpi_1)\\
&\leq \mathscr{H}\left(\mathscr{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \frac{\varpi_1}{2}\right) \diamond \mathscr{H}\left(\mathscr{T}(\chi_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathscr{T}(\Upsilon_{\breve{\imath}\breve{\varrho}\breve{\sigma}}), \frac{\varpi_1}{2}\right)\\
&\leq (1-\delta_7) \diamond (1-\delta_7)\\
&\leq (1-\delta_6) \diamond (1-\delta_6)\\
&\leq (1-\delta_0) < \delta_3
\end{aligned}$$

which cannot be done.
Therefore, ${}_3S^I_{(\mathscr{F},\mathscr{G},\mathscr{H})}(\mathscr{T})$ is Hausdorff. □

Theorem 4.11. *${}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ is an $\mathcal{NNS}$ and ${}_3\tau^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ is a topology on ${}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$. Then there exists a sequence $(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$, such that $\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} \to \zeta$ iff $\mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 1, \mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 0$ and $\mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 0$ as $\breve{\imath}, \breve{\varrho}, \breve{\sigma} \to \infty$.*

Proof. Let $\varpi_1 > 0$, let $\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} \to \zeta$ and $0 < \delta < 1$, there exists $(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ like that $(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\mathcal{B}_\zeta(\delta, \varpi_1)(\mathcal{T})$, for everyone $\breve{\imath} \geq p, \breve{\varrho} \geq q, \breve{\sigma} \geq r$,

$$
{}_3\mathcal{B}_\zeta(\delta, \varpi_1)(\mathcal{T}) = \left\{ \begin{array}{c} (\breve{\imath}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \\ \mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) > 1 - \delta \text{ or} \\ \mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) < \delta \text{ and} \\ \mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) < \delta \end{array} \right\} \in I_3,
$$

like that, ${}_3\mathcal{B}^c_\zeta(\delta, \varpi_1)(\mathcal{T}) \in \mathcal{F}(I_3)$.
Then,

$$
\begin{aligned}
1 - \mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) &< \delta, \\
\mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) &< \delta \text{ and} \\
\mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) &< \delta.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 1, \\
&\mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 0 \text{ and} \\
&\mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 0, \text{ as } \breve{\imath}, \breve{\varrho}, \breve{\sigma} \to \infty.
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
&\mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 1, \\
&\mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 0 \text{ and} \\
&\mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) \to 0, \text{ as } \breve{\imath}, \breve{\varrho}, \breve{\sigma} \to \infty.
\end{aligned}
$$

then for, $0 < \delta < 1$, there exists $(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ like that

$$
\begin{aligned}
1 - \mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) &< \delta, \\
\mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) &< \delta \text{ and} \\
\mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) &< \delta, \text{ for all } \breve{\imath} \geq p, \breve{\varrho} \geq q, \breve{\sigma} \geq r.
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
&\mathcal{J}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) > 1 - \delta, \\
&\mathcal{G}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) < \delta \text{ and} \\
&\mathcal{H}(\mathcal{T}(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta), \varpi_1) < \delta, \text{ for all } \breve{\imath} \geq p, \breve{\varrho} \geq q, \breve{\sigma} \geq r
\end{aligned}
$$

Thus, $(\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3\mathcal{B}^c_\zeta(\delta, \varpi_1)(\mathcal{T})$, for everyone $\breve{\imath} \geq p, \breve{\varrho} \geq q, \breve{\sigma} \geq r$.
Hence, $\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}} \to \zeta$. □

Theorem 4.12. *A triple sequence $\zeta = (\zeta_{\breve{\imath}\breve{\varrho}\breve{\sigma}}) \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ is I-convergent iff for everyone $\varepsilon_0 > 0, \varpi_1 > 0$ there exists numbers $p = p(\zeta, \varepsilon_0, \varpi_1), q = q(\zeta, \varepsilon_0, \varpi_1)$ and $r = r(\zeta, \varepsilon_0, \varpi_1)$ like that,*

$$
\left\{ \begin{array}{c} (p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mathcal{J}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) > 1 - \varepsilon_0 \text{ or} \\ \mathcal{G}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0 \text{ and } \mathcal{H}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0 \end{array} \right\} \in \mathcal{F}(I_3).
$$

Proof. Let $I_3^{(\mathcal{J},\mathcal{G},\mathcal{H})} \lim \zeta = \xi$ and let $\varepsilon_0 > 0$ and $\varpi_1 > 0$. For $\varepsilon_0 > 0$ to be given, choose $\varpi_2 > 0$ like that, $(1-\varepsilon_0) \circledast (1-\varepsilon_0) > 1-\varpi_2$, $\varepsilon_0 \odot \varepsilon_0 < \varpi_2$ and $\varepsilon_0 \diamond \varepsilon_0 < \varpi_2$.
We get, $\zeta \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$,

$$P = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}\left(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \frac{\varpi_1}{2}\right) \leq 1-\varepsilon_0 \text{ or} \\ \mathcal{G}\left(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \frac{\varpi_1}{2}\right) \geq \varepsilon_0 \text{ and } \mathcal{H}\left(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \frac{\varpi_1}{2}\right) \geq \varepsilon_0 \end{array} \right\} \in I_3.$$

This implies

$$P^c = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \mathcal{J}\left(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \frac{\varpi_1}{2}\right) > 1-\varepsilon_0 \text{ or} \\ \mathcal{G}\left(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0 \text{ and } \mathcal{H}\left(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0 \end{array} \right\} \in \mathcal{F}(I_3).$$

Conversely, let $p, q, r \in P$. Then

$$\mathcal{J}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) > 1-\varepsilon_0 \text{ or}$$
$$\mathcal{G}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0 \text{ and}$$
$$\mathcal{H}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0.$$

Now, we show that there exists $p = p(\zeta, \varepsilon_0, \varpi_1), q = q(\zeta, \varepsilon_0, \varpi_1)$ and $r = r(\zeta, \varepsilon_0, \varpi_1)$ like that,

$$\left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) : \mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \leq 1-\varpi_2 \text{ or} \\ \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \geq \varpi_2 \text{ and } \mathcal{H}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \geq \varpi_2 \end{array} \right\} \in I_3.$$

Thus, for everyone $\zeta \in {}_3S^I_{(\mathcal{J},\mathcal{G},\mathcal{H})}(\mathcal{T})$ we consider

$$Q = \left\{ \begin{array}{c} (\breve{\iota}, \breve{\varrho}, \breve{\sigma}) : \mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \leq 1-\varpi_2 \text{ or} \\ \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \geq \varpi_2 \text{ and } \mathcal{H}(\mathcal{T}(\zeta_{\breve{\iota}\breve{\varrho}\breve{\sigma}}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \geq \varpi_2 \end{array} \right\} \in I_3.$$

Now, we show that $Q \subset P$.
Let $Q \not\subset P$ then there exists $(\breve{\iota}', \breve{\varrho}', \breve{\sigma}') \in Q$ and $(\breve{\iota}', \breve{\varrho}', \breve{\sigma}') \notin P$.
Consequently, we get,

$$\mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \leq 1-\varpi_2 \text{ or } \mathcal{J}\left(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \xi, \frac{\varpi_1}{2}\right) > 1-\varepsilon_0.$$

In specific,

$$\mathcal{J}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) > 1-\varepsilon_0.$$

Moreover, we get

$$\begin{aligned} 1-\varpi_2 &\geq \mathcal{J}(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \\ &\geq \mathcal{J}\left(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \xi, \frac{\varpi_1}{2}\right) \circledast \mathcal{J}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) \\ &\geq (1-\varepsilon_0) \circledast (1-\varepsilon_0) > 1-\varpi_2, \end{aligned}$$

which cannot be achieved. While on the other hand,

$$\mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \geq \varpi_2 \text{ or } \mathcal{G}\left(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0.$$

Particularly,

$$\mathcal{G}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0.$$

Therefore, we get

$$\begin{aligned} \varpi_2 &\leq \mathcal{G}(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \mathcal{T}(\zeta_{pqr}), \varpi_1) \\ &\leq \mathcal{G}\left(\mathcal{T}(\zeta_{\breve{\iota}',\breve{\varrho}',\breve{\sigma}'}) - \xi, \frac{\varpi_1}{2}\right) \odot \mathcal{G}\left(\mathcal{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) \end{aligned}$$

$$\leq \varepsilon_0 \odot \varepsilon_0 < \varpi_2,$$

which is not possible.
Also, $\mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}',\breve{\varrho}',\breve{\sigma}'}) - \mathscr{T}(\zeta_{pqr}), \varpi_1) \geq \varpi_2$ or $\mathscr{H}\left(\mathscr{T}(\zeta_{\breve{\imath}',\breve{\varrho}',\breve{\sigma}'}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0$.
Particularly, $\mathscr{H}\left(\mathscr{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) < \varepsilon_0$.
Therefore we get,

$$\begin{aligned}\varpi_2 &\leq \mathscr{H}(\mathscr{T}(\zeta_{\breve{\imath}',\breve{\varrho}',\breve{\sigma}'}) - \mathscr{T}(\zeta_{pqr}), \varpi_1) \\ &\leq \mathscr{H}\left(\mathscr{T}(\zeta_{\breve{\imath}',\breve{\varrho}',\breve{\sigma}'}) - \xi, \frac{\varpi_1}{2}\right) \diamond \mathscr{H}\left(\mathscr{T}(\zeta_{pqr}) - \xi, \frac{\varpi_1}{2}\right) \\ &\leq \varepsilon_0 \diamond \varepsilon_0 < \varpi_2,\end{aligned}$$

which cannot be done. Thus, $Q \subset P$ and we have, $P \in I \Rightarrow Q \in I$. □

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