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INVERSE OBSTACLE SCATTERING WITH NON-OVERDETERMINED DATA

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ABSTRACT. A new proof is given for the uniqueness theorem for inverse obstacle scattering with non-overdetermined scattering data. It is proved that the knowledge of the scattring amplitude for a fixed wave number, fixed direction of the incident field and all directions of the scattered field in an arbitrary small cone determine the boundary of the obstacle uniquely for the Dirichle boundary condition on the obstacle.

1. INTRODUCTION

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected boundary $S \in C^{1,a}$, $a > 0$, $D' := \mathbb{R}^3 \setminus D$. Consider the scattering problem:

$$(\nabla^2 + k^2)u = 0 \text{ in } D', \tag{1.1}$$

$$u = 0 \text{ on } S, \tag{1.2}$$

$$u = e^{ik\alpha\cdot x} + v(x, \alpha, k), \tag{1.3}$$

where v satisfies the radiation condition

$$v_r - ikv = o(\frac{1}{r}), \;\; r := |x| \to \infty. \tag{1.4}$$

It follows from (1.1)–(1.4) that

$$v = \frac{e^{ikr}}{r} A(\beta, \alpha, k) + o(\frac{1}{r}) \text{ as } r \to \infty, \;\; \frac{x}{|x|} = \beta, \tag{1.5}$$

where $A(\beta, \alpha, k)$ is called the scattering amplitude, $\alpha, \beta \in S^2$, S^2 is the unit sphere in $\mathbb{R}^3$.

In what follows we assume that α and k *are fixed.* Under this assumption, the scattering amplitude is a function of β only, $A = A(\beta)$. The values of the scattering amplitude $A(\beta)$ are non-overdetermined scattering data. The meaning of this terminology is simple: *the dimension of the unknown object is equal to the number of the variables in the data.* This number is equal to two. The dimension of S, the unknown quantity in inverse scattering, is equal also to two. So the inverse scattering problem of finding S from the data $A(\beta)$ is the problem with non-overdetermined scattering data.

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The uniqueness of the solution to the inverse obstacle scattering problem with non-overdetermined scattering data was discussed in [1], [2]. To our knowledge the author gave first uniqueness results for non-overdetermined scattering data, and currently, to our knowledge, there are no other results available. Our goal in this paper is to give a simplified proof of these results.

We may assume that $A(\beta)$ is known not for all $\beta \in S^2$, but for all $\beta \in S_1^2$, where S_1^2 is an arbitrary small open subset of S^2. Since $A(\beta)$ is an analytic function of β on S^2, the knowledge of $A(\beta)$, on S_1^2 detrmines $A(\beta)$ on all of S^2. So, for the proof of the uniqueness theorem for the inverse obstacle scattering problem, we may assume that $A(\beta)$ is known for all $\beta \in S^2$.

We assume in (1.2) the Dirichlet boundary condition, but our argument is valid for the following boundary conditions on S:

$$u_N = 0 \text{ or } u_N + \eta(s)u = 0,\ Im\,\eta(s) \geq 0. \tag{1.6}$$

Let us state the theorem of A.G.Ramm from [1], p. 104. We give in this paper a new short proof of this theorem.

Theorem 1. The knowledge of $A(\beta)$ for all $\beta \in S_1^2$ determines S and the boundary condition of the types (1.2) *or* (1.6) *uniquely.*

2. Proof

First, let us prove that the data $A(\beta)$ determines uniquely the Green's function $G(x, y_0)$, where $y_0 = -\alpha\tau + \nu$, where $\tau > 0$ is any positive number, α is fixed, and $\nu \in D'$ is an arbitrary vector from a bouned subdomain in D". The Green's function solves the problem:

$$(\nabla_x^2 + k^2)G(x, y) = -\delta(x - y) \text{ in } D', \tag{2.1}$$

G satisfies boundary condition (1.2) and the radiation condition (1.4).

It is known that G exists and is unique, see [1]. If $|y| \to \infty$, $\frac{y}{|y|} = -\alpha$, then

$$G(x, y) = \frac{e^{ik|y|}}{4\pi|y|}u(x, \alpha, k) + O(\frac{1}{|y|^2}), \tag{2.2}$$

uniformly with respect to x, changing in a bounded domain, and $u = u(x, \alpha, k)$ solves problem (1.1)–(1.5), see [1].

Suppose there are two G_j, $j = 1, 2$, corresponding to domains D_j with boundaries S_j, and $A_1(\beta) = A_2(\beta)$. Denote $D_{12} := D_1 \cup D_2$, B'_R is an exterior of the ball B_R containing D_{12}. Then $G := G_1 - G_2$ satisfies equation (1.1) in B'_R, and

$$u := u_1 - u_2 = [A_1(\beta) - A_2(\beta)]\frac{e^{ik|x|}}{|x|} + o(\frac{1}{|x|}) = o(\frac{1}{|x|}), \tag{2.3}$$

because $A_1 = A_2$. Any function u, satisfying (1.1) in B'_R and decaying at infinity as $o(\frac{1}{|x|})$, is equal to zero in B'_R, see, for example, [1]. This and formula (2.2) imply that

$$G_1(x, y_0) = G_2(x, y_0) \text{ in } B'_R. \tag{2.4}$$

If $D_1 \neq D_2$, then there are three possibilities:

a) S_1 and S_2 intersect,

b) $D_1 \subset D_2$,

c) D_1 and D_2 have no common points.

Our argument is virtually the same in these cases. So, let us discuss first case a). Take a point $s \in S_2$, $s \notin D_1$. Choose τ and ν so that $y_0 \to s$, $y_0 \in D'_{12}$. By analytic continuation the relation $G_1 = G_2$ in B'_R remains valid in D'_{12}. Since $G_2(s, y_0) = 0$ on S_2, one has $\lim_{y_0 \to s} G_2(s, y_0) = 0$. On the other hand, $G_1(s, y_0) = O(\frac{1}{|s-y_0|})$ as $y_0 \to s$. Since $G_1(s, y_0) = G_2(s, y_0)$, we have a contradiction as $y_0 \to s$: $0 = \infty$. This contradiction proves that case a) is not possible. Therefore, $S_1 = S_2$, so there is at most one S corresponding to the data $A(\beta)$.

Similar arguments prove that cases b) and c) are not possible either.

If S is uniquely determined by the data $A(\beta)$, then the boundary condition is also uniquely determined. Indeed, u is determined in D' uniquely and is continuous up to the boundary S. Calculate u on S. If $u = 0$ on S, then the Dirichlet boundary condition holds. If $u_N = 0$ on S, then the Neumann boundary condition holds. If $-\frac{u_N}{u} = \eta(s)$, then the third boundary condition holds.

Theorem 1 is proved. □

3. Conclusion

A new proof is given for the uniqueness theorem for inverse obstacle scattering with non-overdetermined scattering data. The uniqueness of the solution to inverse obstacle scattering problem with non-overdetermined scattering data was discussed in [1], [2]. To our knowledge, the author gave first uniqueness results for non-overdetermined scattering data and currently, to our knowledge, there are no other results available. Our goal in this paper is to give a simplified proof of these results.

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