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RIGHT DISTRIBUTIVE BI-ALGEBRAS

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ABSTRACT. BI-algebra was introduced in 2017 by A. Borumand Saeid, H. S. Kim and A. Rezaei. Then this class of logical algebras was the focus of many researchers. In this paper, we register an additional property of ideals in right distributive BI-algebras. Then, in this paper we discuss the following two things: the definition of the concept of atoms in right distributive BI-algebras and the registration of many properties of such a designed concept of atoms. In addition to the previous one, the paper designs an extension of the right distributive BI-algebra $\mathfrak{A} = (A, \cdot, 0)$ by adding one element $w \notin A$ such that w is an atom in $\mathfrak{A}$.

1. INTRODUCTION

In 2017, A. Borumand Saeid, H. S. Kim and A. Rezaei introduced in [2] a new algebra, called a BI-algebra, which is a generalization of both a (dual) implication algebra (in terms of paper [3]) and an implicative BCK-algebra (in sense of [5]), and they discussed the basic properties of BI-algebras, and investigated some ideals and congruence relations. Then, this class of logical algebras was the subject of study in papers [1, 6, 7, 8, 9, 10, 11]. In the report [1], published 2019 by S. S. Ahn, J. M. Ko and A. Borumand Saeid, the concepts of (normal) sub-algebras and (normal) ideals in BI-algebras were introduced and analyzed. In addition to the previous one, in that paper, the authors considered both the design of congruences on BI-algebras and the construction of quotient BI-algebras as well as the properties of homomorphisms between BI-algebras. Articles [6, 7] are dedicated to BI-algebra extensions such as hyper BI-algebras and pseudo BI-algebras.

In this paper, we register an additional property of ideals in right distributive BI-algebras (Theorem 3.1). Relying on this result, the proof of Theorem 5.2 in [2] is demonstrated in a slightly different way than it was done in [2] (Theorem 3.3). Additionally, for a given ideal J in the right distributive BI-algebra $\mathfrak{A}$, another right congruence on $\mathfrak{A}$ is designed (Theorem 3.5). Then, this paper discusses both the creation of the concept of atoms in right distributive BI-algebras (Definition 3.2) and it the registration of many properties of

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such a designed concept (Propositions 3.7 - 3.9 and Theorem 3.10 and Theorem 3.11). In addition to the previous them, the paper designs an extension of the right distributive BI-algebra $\mathfrak{A} = (A, \cdot, 0)$ by adding one element $w \notin A$ such that w is an atom in $\mathfrak{A}$ (Theorem 3.13).

2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, and so on, are labels for the logical functions of conjunction, disjunction, implication, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the mark A is the abbreviation for the formula B.

In this text, to mark recognizable formulas, we will use, as far as possible, their standard abbreviations that appear in a very well-known paper [4].

Definition 2.1. ([2], Definition 3.1) An algebra $\mathfrak{A} =: (A, \cdot, 0)$ of type $(2, 0)$ is called a BI-algebra if the following holds:

(Re) $(\forall x \in A)(x \cdot x = 0)$,

(Im) $(\forall x, y \in A)(x \cdot (y \cdot x) = x)$.

A BI-algebra $\mathfrak{A}$ is said to be right distributive if the following

(Dr) $(\forall x, y, z \in A)((x \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z))$

is valid.

Left distributivity, according to [2], Proposition 3.9, in this class of logical algebras is possible only in the trivial case when $\mathfrak{A} =: (\{0\}, \cdot, 0)$.

Remark. *The concept of BI-algebra should not be confused with the concept of 'basic implication algebra' in the sense of Definition 12 in the paper* [12]*, which is often abbreviated as BI-algebra, as well. Also, in this direction, see the article* [13].

Let $\mathfrak{A} =: (A, \cdot, 0)$ be a BI-algebra. We introduce a relation $\preccurlyeq$ on the set A by

$$(\forall x, y \in A)(x \preccurlyeq y \iff x \cdot y = 0).$$

We note that $\preccurlyeq$ is not a partially order set, but it is only reflexive. It is shown ([2], Proposition 3.14) that if $\mathfrak{A}$ is a right distributive BI-algebra, then the induced relation $\preccurlyeq$ is a transitive relation. So, if $\mathfrak{A}$ is a right distributive BI-algebra, then $\preccurlyeq$ is a quasi-order on A right compatible with the operation in $\mathfrak{A}$ ([2], Proposition 3.12(iv)).

Some of the important properties of this class of logical algebras are given by the following two propositions:

Proposition 2.1 ([2], Proposition 3.7)**.** *Let* $\mathfrak{A} =: (A, \cdot, 0)$ *be a BI-algebra. Then:*

(M) $(\forall x \in A)(x \cdot 0 = x)$,

(L) $(\forall x \in A)(0 \cdot x = 0)$,

(iii) $(\forall x, y \in A)(x \cdot y = (x \cdot y) \cdot y)$,

(vi) $(\forall x, y, z \in A)(x \cdot y = z \Longrightarrow (z \cdot y = z \wedge y \cdot z = y))$.

It is obvious that, according to (L), it holds

(1) $(\forall x \in A)(0 \preccurlyeq x)$.

The properties of this relation $\preccurlyeq$ in the right distributive BI-algebra $\mathfrak{A} = (A, \cdot, 0)$ are summarized in the following proposition:

Proposition 2.2 ([2], Proposition 3.12)**.** *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a right distributive BI-algebra. Then the following holds:*

(2) $(\forall x, y \in A)(y \cdot x \preccurlyeq y)$,

(3) $(\forall x, y \in A)((y \cdot x) \cdot x \preccurlyeq y)$,

(4) $(\forall x, y, z \in A)((x \cdot z) \cdot (y \cdot z) \preccurlyeq x \cdot y)$,

(5) $(\forall x, y \in A)(x \preccurlyeq y \implies x \cdot z \preccurlyeq y \cdot z)$,

(6) $(\forall x, y, z \in A)((x \cdot y) \cdot z \preccurlyeq x \cdot (y \cdot z))$,

(7) $(\forall x, y, z \in A)(x \cdot y = z \cdot y \implies (x \cdot z) \cdot y = 0)$.

It should be noted here that, in the general case, this relation $\preccurlyeq$ is not left compatible with the internal operation in any right distributive BI-algebra.

The concept of ideal in BI-algebras is determined by the following definition:

Definition 2.2. ([2], Definition 4.1) A subset J of a BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ is called an ideal of $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$,

(J1) $(\forall x, y \in A)((x \cdot y \in J \,\wedge\, y \in J) \implies x \in J)$.

For an ideal J in a BI-algebra $\mathfrak{A}$ holds ([2], Proposition 4.5)

(J2) $(\forall x, y \in A)((x \preccurlyeq y \,\wedge\, y \in J) \implies x \in J)$.

The concept of sub-algebras in BI-algebras $\mathfrak{A} =: (A, \cdot, 0)$ is introduced by a standard way. A nonempty subset S of A is a sub-algebra in $\mathfrak{A}$ if it satisfies the condition

(S1) $(\forall x, y \in A)((x \in S \,\wedge\, y \in S) \implies x \cdot y \in S)$.

It can immediately be concluded that the sub-algebra S in a BI-algebra $\mathfrak{A}$ satisfies the condition

(S0) $0 \in S$.

Indeed, let $x \in S$ since S is not empty. Then, according to (S1) and (Re), we have $x \in S \implies 0 = x \cdot x \in S$.

Example 2.3. Let $A = \{1, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	a	b
b	b	b	0	b
c	c	b	c	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a BI-algebra ([2], Example 3.3).

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_4 = \{0, a, b\}$, and $S_6 = \{0, b, c\}$ are sub-algebras in $\mathfrak{A}$. Sub-set $K = \{0, a, c\}$ is not a sub-algebra in $\mathfrak{A}$, because, for example, we have $a \in K \,\wedge\, c \in K$ but $a \cdot c = b \notin K$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$, $J_3 = \{0, c\}$, $J_5 = \{0, a, c\}$ are ideals in $\mathfrak{A}$. Subset $S_4 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a \in S_4$ and

$c \cdot a = b \in S_4$ but $c \notin S_4$. Also, subset $S_6 = \{0, b, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c \in S_6$ and $a \cdot c = b \in J_6$ but $a \notin J_6$.

The previous example enables the conclusion that the concept of sub-algebras and the concept of ideals in BI-algebras are mutually independent. The subset $K = \{0, a, c\}$ is not a sub-algebra in $\mathfrak{A}$, but it is an ideal in $\mathfrak{A}$. On the other hand, the subset $S_6 = \{0, b, c\}$ is a sub-algebra in $\mathfrak{A}$ but it is not an ideal in $\mathfrak{A}$.

Example 2.4. Let $A = \{1, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a right distributive BI-algebra ([2], Example 3.10(i)). The relation $\preccurlyeq$ is given by

$$\preccurlyeq = \{(0,0), (0,a), (0,b), (0,c), (a,a), (a,c), (b,b), (b,c), (c,c)\}.$$

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_4 = \{0, a, b\}$ are sub-algebras in $\mathfrak{A}$. However, subsets $S_5 = \{0, a, c\}$ and $S_6 = \{0, b, c\}$ are not sub-algebras on $\mathfrak{A}$. For S_5, for example, we have $a \in S_5$ and $c \in S_5$ but $c \cdot a = b \notin S_5$. In the second case, for S_6, for example, we have $b \in S_6$ and $c \in S_6$ but $c \cdot b = a \notin S_6$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $S_2 = \{0, b\}$ are ideals in $\mathfrak{A}$. Subset $J_3 = \{0, c\}$ is not an ideal in $\mathfrak{A}$, because, for example, we have $a \cdot c = 0 \in J_3$ and $c \in J_3$ but $a \notin J_3$. Subset $S_4 = \{0, a, b\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a \in S_4$ and $c \cdot a = b \in S_4$ but $c \notin S_4$. Subset $S_5 = \{0, a, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $b \cdot c = 0 \in S_5$ and $c \in S_5$ but $b \notin S_5$. Also, subset $S_6 = \{0, b, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $c \in S_6$ and $a \cdot c = b \in J_6$ but $a \notin J_6$.

Example 2.5. Let $A = \{1, a, b, c\}$ be a set with the operation given with the table

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Then $\mathfrak{A} = (A, \cdot, 0)$ is a right distributive BI-algebra ([1], Example 3.3(2)). The relation $\preccurlyeq$ is given by $\preccurlyeq = \{(0,0), (0,a), (0,b), (0,c), (a,a), (b,b), (c,c)\}$.

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_4 = \{0, a, b\}$, $S_5 = \{0, a, c\}$ and $S_6 = \{0, b, c\}$ are sub-algebras in $\mathfrak{A}$. Each of the aforementioned subsets is an ideal in $\mathfrak{A}$. Sub-algebra S_1 is a normal sub-algebra in $\mathfrak{A}$ but the sub-algebra S_4 is not a normal because, for example, we have $c \cdot c = 0 \in S_4$ and $b \cdot c = b \in S_4$ but $(c \cdot b) \cdot (c \cdot c) = c \cdot 0 = c \notin S_4$.

3. The main results

This section is the central part of this report. It consists of three subsections. In the first of them, one additional property of ideals in right-distributive BI-algebras is shown, which does not appear in previously published texts. As a consequence of this property, it is proved that the family of ideals in right distributive algebras is a subfamily of sub-algebras in that class of algebras. In the general case, these two concepts in BI-algebras are mutually independent.

3.1. **One additional feature of the ideals.** Ideals in right distributive BI-algebras have one additional property as shown in the following theorem:

Theorem 3.1. *Let J be an ideal in a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then:*

(J3) $(\forall x, y \in A)(x \in J \implies x \cdot y \in J)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x \in J$. Then $x \cdot (y \cdot x) = x \in J$ in accordance with (Im). According to (6), we have $(x \cdot y) \cdot x \preccurlyeq x \cdot (y \cdot x)$. Therefore, according to (J2), we have $(x \cdot y) \cdot x \in J$. On the other hand, from this and from $x \in J$ follows $x \cdot y \in J$ according to (J1). □

As a consequence of the previous theorem, we have:

Corollary 3.2. *Any ideal in a right distributive BI-algebra $\mathfrak{A}$ is a sub-algebra in $\mathfrak{A}$.*

Proof. Let J be an ideal in a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ and $x, y \in J$ be arbitrary elements. Then $x \cdot y \in J$ and $y \cdot x \in J$ by (J3). □

Remark. *The reversal of Corollary 3.2 need not be valid as shown in Example 2.4. The subset S_4 in that example is a sub-algebra but it is not an ideal.*

In the light of the previous theorem, now the proof of Theorem 5.2 in [2] can be demonstrated in a slightly different way (without referring to claim (iii) of Proposition 3.12 in [2]).

Theorem 3.3. *Let J be an ideal on a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Then the relation $\rho_J \subseteq A \times A$, determined by*

$$(\forall x, y \in A)((x, y) \in \rho_J \iff (x \cdot y \in J \wedge y \cdot x \in J)),$$

is a right congruence in $\mathfrak{A}$.

Proof. Since the presence of reflexivity and symmetry of this relation is obvious, let's prove transitivity. Let $x, y, z \in A$ be such that $(x, y) \in \rho_J$ and $(y, z) \in \rho_J$. This means $x \cdot y \in J$, $y \cdot x \in J$, $y \cdot z \in J$ and $z \cdot y \in J$. We have $x \cdot y \in J \implies (x \cdot y) \cdot z \in J$ by (J3). Thus $(x \cdot z) \cdot (y \cdot z) \in J$ in accordance with (Dr). Hence, from $(x \cdot z) \cdot (y \cdot z) \in J$ and $y \cdot z \in J$ we get $x \cdot z \in J$, according to (J1). Analogous to the previous one, it can be obtained that $z \cdot x \in J$. So, $(x, z) \in \rho_J$.

Let $x, y, z \in A$ be such that $(x, y) \in \rho_J$. This means $x \cdot y \in J$ and $y \cdot x \in J$. Thus $(x \cdot z) \cdot (y \cdot z) = (x \cdot y) \cdot z \in J$ and $(y \cdot z) \cdot (x \cdot z) = (y \cdot x) \cdot z \in J$ with respect to (Dr) and (J3). Hence, $(x \cdot z, y \cdot z) \in \rho_J$. □

However, we have:

Theorem 3.4. *For every ideal J in a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$, we have:*

$$J \times J \subseteq \rho_J,$$

where ρ_J is the right congruence on $\mathfrak{A}$ determined according to Theorem 3.3.

Proof. Let $x, y \in A$ be arbitrary elements such that $(x, y) \in J \times J$. This means $x \in J$ and $y \in J$. Thus $x \cdot y \in J$ and $y \cdot x \in J$ by (J3). Hence, $(x, y) \in \rho_J$. □

Example 3.1. Let $A = \{1, a, b, c\}$ be as in Example 2.5. The relation ρ_J on the right distributive BI-algebra $\mathfrak{A}$, generated by the ideal $J =: S_4 = \{0, a, b\}$, is given by

$$\begin{aligned}\rho_J &= \{(0,0),(0,a),(0,b),(a,0),(b,0),(a,a),(b,b),(a,b),(b,a),(c,c)\}\\ &= (J \times J) \cup \{(c,c)\}.\end{aligned}$$

In addition to the right congruence ρ_J, described in Theorem 3.3, in right distributive BI-algebras one more right congruence, generated by the ideal J, can be create.

Theorem 3.5. *Let J be an ideal in a right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$. Let us show that the relation $q_J = (J \times J) \cup \{(x, x) : x \in A \setminus J\}$ is a right congruence on $\mathfrak{A}$. In addition to the previous one, $q_J \subseteq \rho_J$ holds.*

Proof. Indeed, since it is obvious that q_J is a reflexive and symmetric relation on A, we have to prove transitivity and compatibility from the right. Let $x, y, z \in A$ be such that $(x, y) \in q_J$ and $(y, z) \in q_J$. This means $(x, y) \in J \times J \vee x = y \in A \setminus J$ and $(y, z) \in J \times J \vee y = z \in A \setminus J$. We have the following options:

- If $(x, y) \in J \times J$ and $(y, z) \in J \times J$, then $(x, z) \in J \times J$.
- If it were $(x, y) \in J \times J$ and $y = z \in A \setminus J$, or, if it were $x = y \in A \setminus J$ and $(y, z) \in J \times J$, they would have a contradiction in both cases.
- If $x = y \in A \setminus J$ and $y = z \in A \setminus J$, we would have $x = z \in A \setminus J$.

With this, the transitivity of the relation q_J is proved.

Let $x, y, z \in A$ be arbitrary elements such that $(x, y) \in q_J$. Then $(x, y) \in J \times J$ or $x = y \in A \setminus J$. This means $x \in J$ and $y \in J$ or $x = y \in A \setminus J$. Thus $x \cdot y \in J$ and $y \cdot x \in J$ in accordance with (J3) or $x \cdot z = y \cdot z$. Therefore, we have $(x \cdot z, y \cdot z) \in q_J$. □

It is quite justified to ask the question:

Does every right congruence ρ_J on the right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$, generated by the ideal J in $\mathfrak{A}$, have the form $(J \times J) \cup \{(x, x) : x \in A \setminus J\}$?

3.2. **Concept of atoms in right distributive BI-algebra.** It is well known that every quasi-order relation $\preccurlyeq$ on the set A generates an equivalence relation $\equiv$ on the set A by the following way $\equiv =: \preccurlyeq \cap \preccurlyeq^{-1}$. It is easy to see that this relation on A is a right congruence on $\mathfrak{A}$. Indeed, this statement follows immediately from (5). So, for example, we have $a \equiv 0 \iff (a \preccurlyeq 0 \wedge 0 \preccurlyeq a)$, that is, we have $a \equiv 0 \iff (a \cdot 0 = 0 \wedge 0 \cdot a = 0)$. Since the option $a \cdot 0 = 0$ is possible only if $a = 0$, we conclude that $a \equiv 0$ cannot be valid for $a \neq 0$. So, for $a \neq 0$, we have $\neg(a \equiv 0) \iff \neg(a \preccurlyeq 0) \vee \neg(0 \preccurlyeq a)$. Thus, $\neg(a \equiv 0) \iff \neg(a \preccurlyeq 0)$.

Definition 3.2. Let $\mathfrak{A} =: (A, \cdot, 0)$ ne a right distributive BI-algebra. An element $a \in A$ such that $\neg(0 \equiv a)$ is an atom in $\mathfrak{A}$ if holds

(A) $(\forall x \in A)(x \preccurlyeq a \implies (x \equiv a \vee x \equiv 0))$.

We denote the set of all atoms of the BI-algebra $\mathfrak{A}$ by $L(A)$.

It is obvious that:

Proposition 3.6. *The set $L(A)$ is an anti-chain.*

Proof. Let $a, b \in A$ be atoms in a BI-algebra $\mathfrak{A}$ such that $\neg(a \equiv b)$. Assume that elements a and b are comparable. For definiteness, suppose that $a \preccurlyeq b$. Then, according to (A), either $0 \equiv a$ or $a \equiv b$, since b is an atom in $\mathfrak{A}$. As both options are impossible, we conclude that elements a and b are not comparable. □

We have:

Proposition 3.7. *If $\{0, a\}$ is an ideal in a BI-distributive algebra $\mathfrak{A}$, then a is an atom in $\mathfrak{A}$.*

Proof. Let $a \in A$ be such that the subset $\{0, a\}$ is an ideal in $\mathfrak{A}$. Let $x \in A$ be such that $x \preccurlyeq a$. This means $x \cdot a = 0 \in \{0, a\}$. Then $x \in \{0, a\}$ directly follows from $x \cdot a = 0 \in \{0, a\}$ and $a \in \{0, a\}$ by (J1). Hence, $x = 0$ or $x = a$. So, a is an atom in $\mathfrak{A}$. □

Additionally, the following proposition gives some properties of the set $L(A)$.

Proposition 3.8. *Let $\mathfrak{A} = (A, \cdot, 0)$ be a right distributive BI-algebra and $a, b \in L(A)$. Then:*

(8) $b \cdot a = b$.

(9) $(\forall x \in A)(b \cdot x = 0 \vee b \cdot x = b)$.

(10) $(\forall x \in A)((b \cdot x) \cdot x = 0 \vee (b \cdot x) \cdot x = b)$.

Proof. Assertion (8) follows directly from (2) with respect to (A), since the option $b \cdot a = 0$ is not possible. The claim (9) also follows directly from (2) with respect to (A). Assertion (10) follows directly from (3) with respect to (A). □

In addition to the previously mentioned properties of the elements of the set $L(A)$, we also have the following property of set $S =: L(A) \cup \{0\}$:

Proposition 3.9. *Let $\mathfrak{A} =: (A, \cdot, 0)$ be a right distributive BI-algebra. Then the set $S =: L(A) \cup \{0\}$ is a sub-algebra in $\mathfrak{A}$.*

Proof. Let $a, b \in S$. If $a = 0$ or $b = 0$, then we have $0 \cdot b = 0 \in S$, respectively, we have $a \cdot 0 = a \in S$. For $a, b \in S$ such that $a \neq 0$ and $b \neq 0$, according to (8), we have $a \cdot b = a \in S$ and $b \cdot a = b \in S$. This shows that S is a sub-algebra in $\mathfrak{A}$. □

Example 3.3. Let $A = \{0, a, b, c\}$ be as in Example 2.4. Then $\mathfrak{A} =: (A, \cdot, 0)$ is a right distributive BI-algebra. Elements a and b are atoms in $\mathfrak{A}$. Therefore, $L(A) = \{a, b\}$. In the table that determines the internal operation in $\mathfrak{A}$, it can be seen that $b \cdot a = b$ and $a \cdot b = a$, while $a \cdot c = 0$ and $b \cdot c = 0$, which illustrates the statement (8).

Example 3.4. Let $A = \{0, a, b, c\}$ be as in Example 2.5. Then $\mathfrak{A} =: (A, \cdot, 0)$ is a right distributive BI-algebra. Elements a, b, c are atoms in $\mathfrak{A}$. Therefore, $L(A) = \{a, b, c\}$.

However, the converse of statement (10) is also valid.

Theorem 3.10. *If the element b of the right distributive BI-algebra $\mathfrak{A} =: (A, \cdot, 0)$ satisfies the condition (10), then b is an atom in $\mathfrak{A}$.*

Proof. Let the formula (10) be valid for the element b of a right distributive BI-algebra $\mathfrak{A}(A, \cdot, 0)$ and let $x \in A$ be such that $x \preccurlyeq b$.

Assume that $(b \cdot x) \cdot x = 0$ holds. Then, by (Dr), we have

$$0 = (b \cdot x) \cdot x = (b \cdot x) \cdot (x \cdot x) = (b \cdot x) \cdot 0 = b \cdot x.$$

This means $b \preccurlyeq x$. Hence, $x \equiv b$.

Now suppose that $(b \cdot x) \cdot x = b$ holds. In this case, we have

$$b = (b \cdot x) \cdot x = (b \cdot x) \cdot (x \cdot x) = (b \cdot x) \cdot 0 = b \cdot x.$$

From here we get $x \cdot (b \cdot x) = x \cdot b = 0$. From here, with respect to (Im), we get $x = 0$. □

Also, the converse of statement (9) is also a valid statement in the right distributive BI-algebra.

Theorem 3.11. *If the element* b *of the right distributive BI-algebra* $\mathfrak{A} =: (A, \cdot, 0)$ *satisfies the condition (9), then* b *is an atom in* $\mathfrak{A}$*.*

Proof. We will prove that conditions (9) and (10) are equivalent in right distributive BI-algebras. Let $\mathfrak{A} = (A, \cdot, 0)$ be a right distributive BI-algebra.

(9) $\Longrightarrow$ (10). If $b \cdot x = 0$, then $(b \cdot x) \cdot x = 0 \cdot x = 0$ by (L). If $b \cdot x = b$, then $b = b \cdot x = (b \cdot x) \cdot x$.

(10) $\Longrightarrow$ (9). According to (Dr), we have $(b \cdot x) \cdot x = (b \cdot x) \cdot (x \cdot x) = (b \cdot x) \cdot 0 = b \cdot x$ with respect to (M). From here it follows that $b \cdot x = 0$ or $b \cdot x = b$ in accordance with (10). $\square$

In the Example 2.5, all elements, except 0, are atoms in the right distributive BI-algebra $\mathfrak{A} = (A, ., 0)$. In such situation, BI-algebra has the following property:

Theorem 3.12. *If in the right distributive BI-algebra* $\mathfrak{A} = (A, \cdot, 0)$ *is* $L(A) = \{a \in A : a \neq 0\}$*, then every sub-algebra in* $\mathfrak{A}$ *is an ideal in* $\mathfrak{A}$*.*

Proof. Let S be a sub-algebra in a right distributive BI-algebra $\mathfrak{A}$ and let $u, v \in A$ be such that $u \cdot v \in S$ and $v \in S$. According to (8), we have $u \cdot v = u \in S$ since elements u and v are atoms in $\mathfrak{A}$. Therefore, we conclude that S is an ideal in $\mathfrak{A}$. $\square$

3.3. **An extension of a BI-algebra.** In this subsection, we design an extension of the right distributive BI-algebra $\mathfrak{A} = (A, \cdot, 0)$ to a right distributive BI-algebra $\mathfrak{B} = (A \cup \{w\}, *, 0)$ by adding the element $w \notin A$.

Theorem 3.13. *Let* $\mathfrak{A} = (A, \cdot, 0)$ *be a right distributive BI-algebra and* $w \notin A$*. We can extend the algebra* $\mathfrak{A}$ *to a right distributive BI-algebra* $\mathfrak{B} = (A \cup \{w\}, *, 0)$ *so that the element* w *is an atom in* $\mathfrak{B}$*.*

Proof. The right $\mathfrak{B}$ can be created in the following way:

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ x & \text{for } x \in A \ \wedge \ y = w, \\ w & \text{for } x = w \ \wedge \ y \in A, \\ 0 & \text{for } x = w \ \wedge \ y = w. \end{cases}$$

It should be checked whether the structure created in this way satisfies the axioms (Re), (Im) and (Dr). We do this by checking the validity of these formulas by putting the variable w instead of one, two or all three variables that appear in them.

(i) The formula (Re) is a valid formula in system $\mathfrak{B}$ because $w * w = 0$ is holds.

(ii) The formula (M) is a valid formula in system $\mathfrak{B}$ because $w * 0 = w$ is holds.

(iii) The formula (L) is a valid formula in system $\mathfrak{B}$ because $0 \cdot w = 0$ is holds.

(iv) The following equalities prove the validity of axiom (Im):
$(\forall y \in A)(w * (y * w) = w * y = w)$,
$(\forall x \in A)(x * (w * x) = x * w = x)$.
$w * (w * w) = w * 0 = w$.

(v) By direct checking, the validity of the formula (Dr) in the system $\mathfrak{B}$ can be checked. For illustration, we show lines of that check:

For $x = w$, we have $(w * y) * z = w * z = w$ and $(w * z) * (y * z) = w * (y * z) = w$.
For $y = w$, we have $(x * w) * z = x \cdot z$ and $(x * z) * (w * z) = (x \cdot z) * w = x \cdot z$.

For $z = w$, we have $(x * y) * w = (x \cdot y) * w = x \cdot y$ and $(x * w) * (y * w) = x \cdot y$.
For $x = w$ and $y = w$, we have $(w*w)*z = 0 \cdot z = 0$ and $(w*z)*(w*z) = w*w = 0$.
For $x = w$ and $z = w$, we have $(w*y)*w = w*w = 0$ and $(w*w)*(y*w) = 0 \cdot y = 0$.
For $y = w$ and $z = w$, we have $(x*w)*w = x*w = x$ and $(x*w)*(w*w) = x \cdot 0 = x$.
Finally, we have $(w * w) * w = 0 * w = 0$ and $(w * w) * (w * w) = 0 \cdot 0 = 0$.

Since $(\forall y \in A)(w * y = w)$ holds, we conclude, according to (9), that w is an atom in the right distributive BI-algebra $\mathfrak{B}$. □

Example 3.5. Let $\mathfrak{A} = (A, \cdot, 0)$ as in Example 2.5. Let $B = \{1, a, b, c, w\}$ be a set with the operation given with the table

$*$	0	a	b	c	w
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
w	w	w	w	w	0

Then $\mathfrak{B} = (A \cup \{w\}, *, 0)$ is a right distributive BI-algebra. It is easy to conclude that the subset $\{0, w\}$ is an ideal in $\mathfrak{B}$. Therefore, according to Proposition 3.7, w is an atom in $\mathfrak{B}$.

4. Conclusions

(Right distributive) BI-algebra was introduced in 2017 by A. Borumand Saeid, H. S. Kim and A. Rezaei which is a generalization of both a (dual) implication algebra and an implicative BCK-algebra, and they discussed the basic properties of BI-algebras, and investigated some ideals and congruence relations. Then this algebraic structure was the subject of interest of several authors.

In this paper we report some new results on ideals and right congruences in right distributive BI-algebras. Thus, among other things, we show that the theorem in [2] on designing of right congruence ρ_J, generated by an ideal J, can be proved differently than it was done in [2]. Additionally, we showed that one can create a right congruence q_J on a right distributive BI-algebra such that $q_J \subseteq \rho_J$. Then, the concept of atom in right distributive BI-algebras was introduced and some of its basic properties were registered. As the conclusion of this report, an extension of the right distributive BI-algebra by adding one element was considered.

Determining a more precise relationship between the mentioned right congruences remains as one of the possibilities in further research.

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