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ABSENCE OF POSITIVE EIGENVALUES OF THE LAPLACIAN IN DOMAINS WITH INFINITE BOUNDARIES

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ABSTRACT. For a wide class of infinite boundaries and the zero boundary condition, a simple proof is given for the absence of the positive eigenvalues of the Laplacian. The objective of this work is to prove Theorem 1 in which such conditions are formulated.

1. INTRODUCTION

Let S^2 be a unit sphere in $\mathbb{R}^3$ and Ω be a connected open set on S^2. The boundary of Ω denote L. This set defines a cone K in $\mathbb{R}^3$. Let $R > 0$ be an arbitrary large number, B_R be a ball centered at the origin and of radius R and B'_R be the complement to B_R in $\mathbb{R}^3$. Let D be a domain with a sufficiently smooth (for example, C^2-smooth) boundary S. This boundary coincides with the boundary of the cone K in B'_R.

We prove that the Dirichlet Laplacian in D does not have positive eigenvalues. Our proof is easily generalised to the case of the domains in the space $\mathbb{R}^n$, $n > 3$.

In the monograph [4],p.333, the absence of positive eigenvalues of the Dirichlet Laplacian is proved for a larger class of domains with infinite boundaries than in this paper, but the proof in [4] is much harder and longer than we offer here. Our proof of Theorem 1 in this paper is exceptionally short and easy to follow.

Consider the problem

$$(\Delta + k^2)u = 0 \quad x \in D, u|_{s \in S} = 0 \quad k^2 > 0. \tag{1.1}$$

Let us assume

$$u \in L^2(D). \tag{1.2}$$

Theorem 1. Under our assumptions about D, the only solution to problem (1.1)-(1.2) *is* $u = 0$.

Proof of Theorem 1 is given in Section 2.

In the remarkable papers [1], [2] the general methods are developed for spectral analysis of the Laplace and Schrödinger operators. Originally these methods were suggested in [3], see also [4] .

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2. Proof of Theorem 1

Let ψ_j be the set of all normalized eigenfunctions in Ω of the Dirichlet operator Δ^*, the angular part of the Laplacian in the spherical coordinates:

$$(\Delta^* + \nu_j)\psi_j = 0, \quad \psi_j|_L = 0, \tag{2.1}$$

where $\nu_j > 0$, $\Delta^* := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi_j}{\partial\theta}) + \frac{\partial}{\partial\phi}(\frac{1}{\sin\theta}\frac{\partial\psi_j}{\partial\phi})$.

Multiply equation (1.1) by ψ_j and integrate over Ω to get

$$u_j'' + \frac{2}{r}u_j' + (k^2 - \frac{\nu_j + \frac{1}{4}}{r^2})u_j = 0, \quad u_j := \int_\Omega u\bar{\psi_j}d\Omega, \tag{2.2}$$

where $u' := \frac{du}{dr}$.

The solution to this equation is:

$$u_j := c_1(j)\frac{J_{a(j)}(kr)}{r^{1/2}} + c_2(j)\frac{N_{a(j)}(kr)}{r^{1/2}}, \tag{2.3}$$

where $J_{a(j)}(kr)$ is the Bessel function regular at $r = 0$ and $N_{a(j)}(kr)$ is the Bessel function linearly independent from $J_{a(j)}(kr)$, and

$$a(j) := (\nu_j + \frac{1}{4})^{1/2}.$$

If $u \in L^2(D)$, then $ru_j(r) \in L^2(R,\infty)$. Therefore, $c_1(j) = c_2(j) = 0$, because of the known asymptotic of the Bessel functions as $r \to \infty$.

Consequently, $u = 0$ in $D \cap B_R'$. Since u solves elliptic equation with constant coefficients, it satisfies the unique continuation property. So, if $u = 0$ in $D \cap B_R'$, then $u = 0$ in D. Theorem 1 is proved. □

Remark 1. It follows from our proof of Theorem 1 that an arbitrary sufficiently smooth perturbation of the boundary inside B_R cannot produce a positive eigenvalue of the Dirichlet Laplacian.

3. Conclusion

For a wide class of infinite boundaries and the zero boundary condition, a simple proof is given for the absence of the positive eigenvalues of the Laplacian. This result is of interest practically and theoretically. Possible developments can include similar results with other boundary conditions. To our knowledge, there are no recent results of this scope.

Conflict of interest. There is no conflict of interest.

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