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IDEAL GENERATED BY A TRANSLATIONAL INVARIANT FUZZY SUBSET AND AN ELEMENT OF A $\Gamma-$SEMIGROUP

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ABSTRACT. In this section, we introduce the notions of a left and a right translational invariant fuzzy subsets of a $\Gamma-$semigroup M, as well as the concept of a unit with respect to a fuzzy subset, and study their properties. We also prove that if μ is a translational invariant fuzzy subset of a commutative $\Gamma-$semigroup with unity, then the principal ideal generated by an element and μ that contains a unity element is a prime ideal of the $\Gamma-$semigroup.

1. INTRODUCTION

The notion of a ternary algebraic system was introduced by Lehmer [5] in 1932. In 1995, Murali Krishna Rao [9, 10, 11] introduced the notion of a $\Gamma-$semiring as a generalization of a $\Gamma-$ring, a ring, a ternary semigroup and a semiring. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. In 1981, Sen [24] introduced the notion of a $\Gamma-$semigroup as a generalization of a semigroup. Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and charecterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. Quasi ideals are generalization of right ideals and left ideals whereas bi-ideals are generalization of quasi ideals. Steinfeld[26] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [2] introduced the concept of quasi ideal for a semigroup. Murali Krishna Rao [13] introduced the concept of bi-interior ideal as a generalization of quasi ideal, bi-ideal and interior ideal of a semigroup and study the properties of bi-interior ideals. Murali Krishna Rao [14, 15, 16, 17, 18] introduced and studied of a bi-quasi ideal, a soft bi-interior ideal, a fuzzy prime ideal, a fuzzy soft tri-quasi ideal and a fuzzy filter of semigroups and $\Gamma-$semirings. A. K. Ray [21] introduced the notion

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of a translational invariant fuzzy subset and A. K. Ray and Ali [22] generalized the results of ring theory by using the notion of translational invariant fuzzy subsets. In this paper, we introduce the notion of units, associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semigroup generated by a translational invariant fuzzy subset and an element. We study the properties of image and pre-image of a translational invariant fuzzy subset under the $\Gamma-$semigroup homomorphism.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. [24] A semigroup is an algebraic system $(M, .)$ consisting of a non-empty set M together with an associative binary operation $'' \cdot ''$.

Definition 2.2. [24] Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup if it satisfies

(i) $x\alpha y \in M$
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3. [24] Let M be a $\Gamma-$semigroup. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.4. [24] A $\Gamma-$semigroup M is said to be left (right) singular if for each $a \in M$ there exists $\alpha \in \Gamma$ such that $a\alpha b = a(a\alpha b = b)$, for all $b \in M$.

Definition 2.5. [24] A $\Gamma-$semigroup M is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$, for all $\alpha \in \Gamma$.

Definition 2.6. [24] Let M be a $\Gamma-$semigroup. An element $a \in M$ is said to be an idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.

Definition 2.7. [24] Let M be a $\Gamma-$semigroup. An element $a \in M$ is said to be regular element of M if there exist $x \in M,\ \alpha \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.8. [24] Let M be a $\Gamma-$semigroup. Every element of M is a regular element of M then M is said to be regular $\Gamma-$semigroup M.

Definition 2.9. [24] An element a of a $\Gamma-$semigroup M is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

Definition 2.10. [24] Let M be a $\Gamma-$semigroup. An ideal P of M is called a prime ideal of M if for any $a, b \in M$ and $\gamma \in \Gamma,\ a\gamma b \in P \Rightarrow a$ or $b \in P$.

Definition 2.11. [15] A non-empty subset A of $\Gamma-$semigroup M is called

(i) a $\Gamma-$subsemigroup of M if $A\Gamma A \subseteq A$
(ii) a quasi ideal of M if $A\Gamma M \cap M\Gamma A \subseteq A$
(iii) a bi-ideal of M if $A\Gamma M\Gamma A \subseteq A$
(iv) an interior ideal of M if $M\Gamma A\Gamma M \subseteq A$
(v) a left (right) ideal of M if $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$
(vi) an ideal if A is a $\Gamma-$subsemigroup of $M,\ A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition 2.12. [28] Let M be a nonempty set. A mapping $f : S \to [0, 1]$ is called a fuzzy subset of M.

3. Ideal of a $\Gamma-$semigroup M generated by an element and translational invariant fuzzy subset of M

In this section, we introduce the notion of a left and a right translational invariant fuzzy subset of a $\Gamma-$semigroup M, the notion of a unit with respect to a fuzzy subset and study their properties. And also we prove that if μ is a translational invariant fuzzy subset of a commutative $\Gamma-$semigroup with unity then principal ideal generated by an element and μ contains an unity element is not a proper ideal of a $\Gamma-$semigroup.

The set $\{r \in M \mid \mu(r) = \mu(x\alpha a),\ a \in M,\ \text{for some } x \in M,\ \text{for all } \alpha \in \Gamma\}$, is denoted by $L(a, \mu)$ and the set $\{r \in M \mid \mu(r) = \mu(a\alpha x),\ a \in M,\ \text{for some } x \in M,\ \text{for all } \alpha \in \Gamma\}$, is denoted by $R(a, \mu)$.

Definition 3.1. Let M be a $\Gamma-$semigroup and μ be a fuzzy subset of M. Then μ is said to be left translational invariant if

$\mu(x) = \mu(y) \Rightarrow \mu(a\alpha x) = \mu(a\alpha y),\ x, y \in M,\ \text{for all } a \in M,\ \alpha \in \Gamma.$

Definition 3.2. Let M be a $\Gamma-$semigroup and μ be a fuzzy subset of M. Then μ is said to be right translational invariant if

$\mu(x) = \mu(y) \Rightarrow \mu(x\alpha a) = \mu(y\alpha a),\ x, y \in M,\ \text{for all } a \in M,\ \alpha \in \Gamma.$

Example 3.3. Let M be a set of non negative integers and Γ be the additive commutative semigroup of all commutative non negative even integers. Then M is a $\Gamma-$semigroup if $a\gamma b$ is defined as usual multiplication of integers a, γ, b where $a, b \in M$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of M, defined by

$$\mu(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0.5 \text{ if } x \text{ is even} \\ 0.1 \text{ if } x \text{ is odd} \end{cases}$$

$< 2 >= \{0, 2, 4, ...\},\ < 6 >= \{0, 6, 12, ...\}$
$I(6, \mu) = \{x \mid \mu(x) = \mu(y\alpha 6),\ \text{for all } \alpha \in \Gamma,\ \text{for some } y \in M\} =$ All even integers
$< 6 >\subseteq I(6, \mu) \subseteq M.$

Theorem 3.1. *Let μ be a left translational invariant fuzzy subset of a $\Gamma-$semigroup M. Then for any $a \in M$, the set $L(a, \mu)$ is a left ideal of a $\Gamma-$semigroup M.*

Proof. Let μ be a left translational invariant fuzzy subset of the $\Gamma-$semigroup $M,\ s \in L(a, \mu)$. Then

$$\mu(s) = \mu(y\alpha a),\ \text{for some } x, y \in M,\ \text{for all } \alpha \in \Gamma$$

$$\text{Suppose } s \in L(a, \mu),\ r \in M, \beta \in \Gamma.$$
$$\text{Then } \mu(s) = \mu(y\alpha a),\ \text{for some } y \in M,\ \text{for all } \alpha \in \Gamma$$
$$\Rightarrow \mu(r\beta s) = \mu\big(r\beta(y\alpha a)\big),\ \text{for all } \alpha \in \Gamma$$
$$\Rightarrow \mu(r\beta s) = \mu\big((r\beta y)\alpha a\big),\ \text{for all } \alpha \in \Gamma.$$

Therefore $r\beta s \in L(a, \mu)$. Hence $L(a, \mu)$ is a left ideal of M. □

The proof of the following theorem is similar as that of Theorem 3.1

Theorem 3.2. *Let M be a $\Gamma-$semigroup and μ be a right translational invariant fuzzy subset of M and $a \in M$. Then the set $R(a, \mu)$ is a right ideal of a $\Gamma-$semigroup M.*

Corollary 3.3. *Let M be a commutative $\Gamma-$semigroup and μ be a translational invariant fuzzy subset of M. Then for any $a \in M,\ L(a, \mu)$ is an ideal of a $\Gamma-$semigroup M.*

If $L(a,\mu)=R(a,\mu)$ then the ideal $L(a,\mu)$ is denoted by $I(a,\mu)$ and $I(a,\mu)$ is called a $\mu-$principal ideal of M generated by a and μ.

Theorem 3.4. *Let μ be a translational invariant fuzzy subset of a $\Gamma-$semigroup M.*

(i) *If $a\in L(b,\mu)$ then $L(a,\mu)\subseteq L(b,\mu)$*
(ii) *If $a\in R(b,\mu)$ then $R(a,\mu)\subseteq R(b,\mu)$.*

Proof.

$$\begin{aligned}
\text{Suppose } a\in L(b,\mu) &\Rightarrow \mu(a)=\mu(x\alpha b) \text{ and for some } x\in M,\ \text{for all } \alpha\in\Gamma\\
\text{Let } t\in L(a,\mu) &\Rightarrow \mu(t)=\mu(y\alpha a) \text{ and for some } y\in M,\ \text{for all } \alpha\in\Gamma\\
\text{Now } \mu(a)=\mu(x\alpha b) &\Rightarrow \mu(y\beta a)=\mu(y\beta x\alpha b),\ \text{for all } \alpha,\beta\in\Gamma\\
&\Rightarrow \mu(t)=\mu(y\beta x\beta b),\ \text{for all } \alpha,\beta\in\Gamma\\
&\Rightarrow t\in L(b,\mu).
\end{aligned}$$

Hence $L(a,\mu)\subseteq L(b,\mu)$. Similarly we can prove (ii). □

Theorem 3.5. *Let μ be a translational invariant fuzzy subset of a $\Gamma-$semigroup M and $a,b\in M$. If $\mu(a)=\mu(b)$ then $L(a,\mu)=L(b,\mu)$ and $R(a,\mu)=R(b,\mu)$.*

Proof. Let μ be a left translational invariant fuzzy subset of a $\Gamma-$semigroup M, $\mu(a)=\mu(b)$ and $x\in L(a,\mu)$. Then $\mu(x)=\mu(r\alpha a)$, for some $r\in M$.

$$\begin{aligned}
\text{Since } &\mu(a)=\mu(b),\ \mu(r\alpha a)=\mu(r\alpha b)\\
&\Rightarrow \mu(x)=\mu(r\alpha b),\ \text{for all } \alpha\in\Gamma\\
&\Rightarrow x\in L(b,\mu).
\end{aligned}$$

Hence $L(a,\mu)\subseteq L(b,\mu)$. Suppose $y\in L(b,\mu)$.

$$\begin{aligned}
\text{Then } &\mu(y)=\mu(s\alpha b),\ \text{ for some } s\in M,\ \text{for all } \alpha\in\Gamma\\
&\Rightarrow \mu(y)=\mu(s\alpha b)=\mu(s\alpha a).\\
&\Rightarrow y\in L(a,\mu).\\
\text{Therefore } &L(b,\mu)\subseteq L(a,\mu).
\end{aligned}$$

Hence $L(a,\mu)=L(b,\mu)$. Similarly we can prove $R(a,\mu)=R(b,\mu)$. □

The proof of the following theorems are straight forward verification.

Theorem 3.6. *Let M be a $\Gamma-$semigroup and μ be a translational invariant fuzzy subset of M. For any $a\in M$, the left ideal $M\Gamma a=\{r\alpha a\mid r\in M,\ \alpha\in\Gamma\}$ of M is contained in left ideal $L(a,\mu)$ and the right ideal $a\Gamma M=\{a\alpha r\mid r\in M,\ \alpha\in\Gamma\}$ is contained in right ideal $R(a,\mu)$.*

Theorem 3.7. *If M is a commutative $\Gamma-$semigroup with unity, μ is a translational invariant fuzzy subset of M and $a\in M$ then the principal ideal $\langle a\rangle=I(a,\mu)$.*

Definition 3.4. Let M be a $\Gamma-$semigroup with unity element e, μ be a translational invariant fuzzy subset of a $\Gamma-$semigroup M and $\mu(0)\neq\mu(e)$. An element $a\in M$ with $\mu(a)\neq\mu(0)$ is called a $\mu-$unit of M if there exists an element $u\in M$ such that $\mu(u)\neq\mu(0)$ and $\mu(a\alpha u)=\mu(u\alpha a)=\mu(e)$, for all $\alpha\in\Gamma$.

Theorem 3.8. *Let M be a $\Gamma-$semigroup with unity e and a be $\mu-$unit of M. Then $L(a,\mu)=R(a,\mu)=M$.*

Proof. Suppose a is a $\mu-$unit of the $\Gamma-$semigroup M. Then there exists $u \in M$ such that $\mu(u) \neq \mu(0),\ \mu(a\alpha u) = \mu(u\alpha a) = \mu(e),$ for all $\alpha \in \Gamma$. Let $x \in M$. Then there exists $\gamma \in \Gamma$ such that $x\gamma e = e\gamma x = x$. Now

$$\begin{aligned}
&\mu(e) = \mu(a\alpha u), \text{ for all } \alpha \in \Gamma\\
\Rightarrow&\mu(e\beta x) = \mu(a\alpha u\beta x), \text{ for all } \alpha, \beta \in \Gamma\\
\Rightarrow&\mu(x) = \mu(a\alpha u\gamma x), \text{ for all } \alpha, \gamma \in \Gamma\\
\Rightarrow&x \in R(a, \mu)
\end{aligned}$$

Therefore $M \subseteq R(a, \mu)$.

Similarly we can prove $M \subseteq L(a, \mu)$. Hence $L(a, \mu) = R(a, \mu) = M$. □

Theorem 3.9. *Let M be a $\Gamma-$semigroup with unity $e, a \in M$ and μ be a right translational invariant fuzzy subset of M. If $e \in R(a, \mu)$ then $R(a, \mu) = M$.*

Proof. Suppose e is the unity element of the $\Gamma-$semigroup $M, x \in M$ and $e \in R(a, \mu)$. Since $x \in M$, by definition of unity, there exists $\gamma \in \Gamma$ such that $e\gamma x = x$. Then $e \in R(a, \mu)$

$$\begin{aligned}
\Rightarrow&\mu(e) = \mu(a\alpha y) \text{ for some } y \in M, \text{ for all } \alpha \in \Gamma\\
\Rightarrow&\mu(e\gamma x) = \mu(a\alpha y\gamma x), \text{ for all }, \alpha \in \Gamma\\
\Rightarrow&\mu(x) = \mu(a\alpha y\gamma x), \text{ for all } \alpha \in \Gamma.
\end{aligned}$$

Therefore $x \in R(a, \mu)$. Hence $R(a, \mu) = M$. □

Corollary 3.10. *Let M be a commutative $\Gamma-$semigroup with unity $e, a \in M$ and μ be a translational invariant fuzzy subset of M. If $e \in I(a, \mu)$ then $I(a, \mu) = M$.*

Theorem 3.11. *Let M be a $\Gamma-$semigroup with unity $e, a \in M$ and μ be a right translational invariant fuzzy subset of M. If $x \in R(a, \mu)$ is an invertible then $R(a, \mu) = M$.*

Proof. Let M be a $\Gamma-$semigroup with unity $e, a \in M$ and μ be a right translational invariant fuzzy subset of M. Suppose $x \in R(a, \mu)$ is invertible. Then there exist $y \in M,\ \alpha \in \Gamma$ such that $x\alpha y = e$. Since by Theorem 3.2, $R(a, \mu)$ is a right ideal of a semigroup M. Therefore $e = x\alpha y \in R(a, \mu)$. By Theorem 3.9, $R(a, \mu) = M$. □

Corollary 3.12. *Let M be a commutative semigroup with unity, $a \in M$ and μ be a translational invariant fuzzy subset of M. If $x \in I(a, \mu)$ is invertible then $I(a, \mu) = M$.*

4. Prime ideal I(a, μ)

In this section, we introduce the notion of associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semigroup generated by translational fuzzy subset and an element. We study the properties of image and pre-image of translational invariant fuzzy subset under the $\Gamma-$semigroup homomorphism. We prove that every homomorphic image of an ideal of a $\Gamma-$semigroup generated by$\mu-$prime element and fuzzy translational invariant, homomorphism-invarinat fuzzy subset μ is a prime ideal of a $\Gamma-$semigroup. Throughout in this section M is a commutative $\Gamma-$semigroup.

Definition 4.1. Let M be a $\Gamma-$semigroup, $a, b \in M$ and $\mu(a) \neq \mu(0)$. a is said to be $\mu-$divisor of b if there exists $c \in M$ such that $\mu(b) = \mu(a\alpha c),$ for all $\alpha \in \Gamma$. It is denoted by $(a/_b)_\mu$.

Definition 4.2. Let M be a $\Gamma-$semigroup, $a, b \in M$ and $\mu(a) \neq \mu(0)$. Then a and b are said to be $\mu-$associates if $(a/_b)_\mu$ and $(b/_a)_\mu$.

Theorem 4.1. *Let M be a $\Gamma-$semigroup, $a, b \in M$ and $\mu(a) \neq \mu(0)$. If $(a/_b)_\mu$ then $I(b,\mu) \subseteq I(a,\mu)$.*

Proof. Suppose $(a/_b)_\mu$.

$$\Rightarrow \mu(b) = \mu(a\alpha c), \text{ for some } c \in M, \text{ for all } \alpha \in \Gamma$$
$$\Rightarrow b \in I(a,\mu).$$

By Theorem 3.4, $I(b,\mu) \subseteq I(a,\mu)$.

□

Definition 4.3. Let $f : M \to S$ be a homomorphism of $\Gamma-$semigroups M, S and μ be a fuzzy subset of M. We define a fuzzy subset $f(\mu)$ of S by

$$f(\mu)(x) = \begin{cases} \sup\limits_{y \in f^{-1}(x)} \mu(y), & \text{if } f^{-1}(x) \neq \phi \\ 0\,, & \text{otherwise .} \end{cases}$$

Definition 4.4. A fuzzy subset μ is called a homomorphism invariant ($f-$invariant) if f is a $\Gamma-$semigroup homomorphism.

Theorem 4.2. *Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a $f-$invariant fuzzy subset of M. If $x = f(a)$ then $f(\mu)(x) = \mu(a),\ a \in M$.*

Proof. Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a $f-$invariant fuzzy subset of M. Suppose $x = f(a)$. Then $f^{-1}(x) = a$. Let $t \in f^{-1}(x)$. Then $x = f(t) \Rightarrow f(a) = x = f(t)$. Since μ is a $f-$invariant fuzzy subset of $M \Rightarrow \mu(a) = \mu(t)$. Therefore $f(\mu)(x) = \sup\limits_{t \in f^{-1}(x)} \{\mu(t)\} = \mu(a)$. □

Theorem 4.3. *Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S. If μ is a translational invariant and $f-$invariant fuzzy subset of M then $f(\mu)$ is a translational invariant fuzzy subset of S.*

Proof. Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a translational invariant and $f-$invariant fuzzy subset of M. Let $x, y \in S,\ \alpha \in \Gamma,\ f(\mu)(x) = f(\mu)(y)$. Since f is onto, there exist $a, b \in M$ such that $f(a) = x, f(b) = y$. By Theorem 4.2, we have $f(\mu)(x) = \mu(a)$ and $f(\mu)(y) = \mu(b) \Rightarrow \mu(a) = \mu(b)$. Let $z \in S$. Then there exists $c \in M$ such that $f(c) = z$.

$$\begin{aligned} x\alpha z &= f(a)\alpha f(c) \\ &= f(a\alpha c) \\ y\alpha z &= f(b)\alpha f(c) \\ &= f(b\alpha c) \\ \Rightarrow f(\mu)(x\alpha z) &= \mu(a\alpha c) \\ \text{and } f(\mu)(y\alpha z) &= \mu(b\alpha c). \end{aligned}$$

$$\text{Since } \mu \text{ is translational fuzzy invariant, } \mu(a) = \mu(b)$$
$$\Rightarrow \mu(a\alpha c) = \mu(b\alpha c), \text{ for all } \alpha \in \Gamma$$
$$\Rightarrow f(\mu)(x\alpha z) = f(\mu)(y\alpha z), \text{ for all } \alpha \in \Gamma.$$

Hence $f(\mu)$ is a translational invariant fuzzy subset of S. □

Definition 4.5. Let $\phi : M \to M'$ be a homomorphism of $\Gamma-$semigroups and μ be a fuzzy subset of M'. We define a fuzzy subset $\phi^{-1}(\mu)$ of M by $\phi^{-1}(\mu)(x) = \mu(\phi(x)),$ for all $x \in M$. We call $\phi^{-1}(\mu)$ is a pre image of μ.

Theorem 4.4. *Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a translational invariant fuzzy subset of S. Then $f^{-1}(\mu)$ is a translational invariant fuzzy subset of a $\Gamma-$semigroup M.*

Proof. Let f be a homomorphism from a $\Gamma-$semigroup M into a $\Gamma-$semigroup S and μ be a translational invariant fuzzy subset of S. Let $a, b \in M$ and $f^{-1}(\mu)(a) = f^{-1}(\mu)(b)$. Then $\mu(f(a)) = \mu(f(b))$. Suppose $x \in M$ then $f(x) = y \in S, \alpha \in \Gamma$. Since μ is translational invariant fuzzy subset of S, we have

$$\begin{aligned} &\mu(f(a)\alpha y) = \mu(f(b)\alpha y) \\ \Rightarrow &\mu(f(a)\alpha f(x)) = \mu(f(b)\alpha f(x)) \\ \Rightarrow &\mu(f(a\alpha x)) = \mu(f(b\alpha x)) \\ \Rightarrow &f^{-1}(\mu)(a\alpha x) = f^{-1}(\mu)(b\alpha x). \end{aligned}$$

Hence $f^{-1}(\mu)$ is a translational invariant fuzzy subset of M. □

Theorem 4.5. *Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S. If μ is a $f-$invariant, translational invariant fuzzy subset of M then*

$$f(I(a,\mu)) = I(f(a), f(\mu)), \textit{ for all } a \in M.$$

Proof. Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a $f-$invariant, translational invariant fuzzy subset of M.

$$\begin{aligned} &\text{Let } y \in I(f(a), f(\mu)) \\ &\Leftrightarrow f(\mu)(y) = f(\mu)(s\alpha f(a)), \text{ for some } s \in S, \text{ for all } \alpha \in \Gamma \\ &\Leftrightarrow y, s \in S, f \text{ is onto, there exist } x, r \in M \text{ such that } f(x) = y, f(r) = s. \\ &\Leftrightarrow f(\mu)f(x) = f(\mu)\big(f(r)\alpha f(a)\big), \text{ for all } \alpha \in \Gamma \\ &\Leftrightarrow f(\mu)f(x) = f(\mu)\big(f(r\alpha a)\big) \\ &\Leftrightarrow \mu(x) = \mu(r\alpha a), \text{ for all } \alpha \in \Gamma \\ &\Leftrightarrow x \in I(a,\mu) \\ &\Leftrightarrow y = f(x) \in f\big(I(a,\mu)\big). \end{aligned}$$

Hence $I\big(f(a), f(\mu)\big) = f\big(I(a,\mu)\big)$. □

Theorem 4.6. *Let M be a $\Gamma-$semigroup and $a, b \in M, \mu(a), \ \mu(b) \neq \mu(0)$. If a and b are $\mu-$assoicates then $I(a,\mu) = I(b,\mu)$.*

Proof. Let M be a $\Gamma-$semigroup and $a, b \in M, \mu(a), \ \mu(b) \neq \mu(0)$. Suppose a and b are $\mu-$assoicates. Then by Definition 4.2, $(a/b)_\mu$ and $(b/a)_\mu$. By Theorem 4.1,

$$\begin{aligned} I(b,\mu) &\subseteq I(a,\mu) \\ \text{and } I(a,\mu) &\subseteq I(b,\mu) \\ \text{Hence } \ I(b,\mu) &= I(a,\mu). \end{aligned}$$

□

Theorem 4.7. *Let a, b be in a $\Gamma-$semigroup M and $\mu(a), \mu(b) \neq \mu(0)$. If $\mu(a) = \mu(b\alpha u)$, for all $\alpha \in \Gamma$, for some $\mu-$unit $u \in M$ then a and b are $\mu-$associates.*

Proof. Let a, b be in a semigroup M and $\mu(a), \mu(b) \neq \mu(0)$. Suppose $\mu(a) = \mu(b\alpha u)$, for all $\alpha \in \Gamma$, for some $\mu-$unit $u \in M$. Then $(b/_a)_\mu$ Since u is a $\mu-$unit, there exists $v \in M$ and $\mu(v) \neq \mu(0)$ such that $\mu(u\beta v) = \mu(e)$, for all $\beta \in \Gamma$.

$$\begin{aligned} \mu(a) &= \mu(b\alpha u), \text{ for all } \alpha \in \Gamma. \\ \Rightarrow \mu(a\beta v) &= \mu(b\alpha u\beta v), \text{ for all } \alpha, \beta \in \Gamma \\ &= \mu(b\alpha e), \text{ for all } \alpha \in \Gamma \\ &= \mu(b), \\ &\Rightarrow (a/_b)_\mu. \end{aligned}$$

Hence a and b are $\mu-$associates. □

Definition 4.6. Let M be a $\Gamma-$semigroup and μ be a translational invariant fuzzy subset of M. Suppose element a is not a unit and $\mu(a) \neq \mu(0)$. Then a is said to be a $\mu-$prime element if $\left(a/_{b\alpha c}\right)_\mu \Rightarrow \left(a/_b\right)_\mu$ or $\left(a/_c\right)_\mu$ for all $b, c \in M, \ \alpha \in \Gamma$.

Theorem 4.8. *Let M be a $\Gamma-$semigroup,μ be a translational invariant fuzzy subset of M, $a \in M, \mu(a) \neq \mu(0)$ and $I(a, \mu) \neq M$. Then a is a $\mu-$prime element if and only if the ideal $I(a, \mu)$ is a prime ideal of a $\Gamma-$semigroup$M$.*

Proof. Let M be a $\Gamma-$semigroup, μ be a translational invariant fuzzy subset of M, $a \in M, \mu(a) \neq \mu(0)$ and $I(a, \mu) \neq M$. Suppose a is a $\mu-$prime element. By Corollary 3.3, $I(a, \mu)$ is an ideal of $\Gamma-$semigroup M.
Let $x, y \in M, \ \alpha \in \Gamma$ and $x\alpha y \in I(a, \mu)$.

$$\begin{aligned} &\Rightarrow \mu(x\alpha y) = \mu(a\beta r), \text{ for some } r \in M, \ \text{ for all } \beta \in \Gamma. \\ &\Rightarrow \left(a/_{x\alpha y}\right)_\mu \\ &\Rightarrow \left(a/_x\right)_\mu \text{ or } \left(a/_y\right)_\mu, \ \text{ since } a \text{ is a } \mu-\text{prime}. \end{aligned}$$

$$\begin{aligned} \text{If } \left(a/_x\right)_\mu & \text{ then } \mu(x) = \mu(a\alpha c), \text{ for all } \alpha \in \Gamma \text{ for some } c \in M \\ &\Rightarrow x \in I(a, \mu) \\ \text{If } \left(a/_y\right)_\mu & \text{ then } y \in I(a, \mu). \end{aligned}$$

Hence $I(a, \mu)$ is a prime ideal of M.

Conversely suppose that $I(a, \mu)$ is a prime ideal of a $\Gamma-$semigroup M.
Let $x, y \in M, \ \alpha \in \Gamma$ and $\left(a/_{(x\alpha y)}\right)_\mu$

$$\begin{aligned} &\Rightarrow \mu(x\alpha y) = \mu(a\beta c), \text{ for all } \beta \in \Gamma \text{ for some } c \in M \\ &\Rightarrow x\alpha y \in I(a, \mu). \\ &\Rightarrow x \in I(a, \mu) \text{ or } y \in I(a, \mu), \text{ since } I(a, \mu) \text{ is a prime ideal.} \end{aligned}$$

$$\begin{aligned} \text{If } x \in I(a, \mu) \text{ then } \mu(x) = \mu(y\beta a), \text{ for all } \beta \in \Gamma \text{ for some } y \in M \\ \Rightarrow \left(a/_x\right)_\mu. \end{aligned}$$

Similarly we can show that if $y \in I(a, \mu)$ then $(a/_y)_\mu$.
Hence a is a $\mu-$prime element of M. □

Definition 4.7. A fuzzy subset μ is called homomorphism-invariant ($f-$invariant) if f is a $\Gamma-$semigroup homomorphism.

Theorem 4.9. *Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a translational invariant and $f-$invariant fuzzy subset of M. If a is a $\mu-$ prime element of $\Gamma-$semigroup M then $f(a)$ is a $f(\mu)-$prime element of a $\Gamma-$semigroup S.*

Proof. Let μ be a translational invariant and $f-$invariant fuzzy subset of a semigroup M and f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S. By Theorem 4.3, $f(\mu)$ is a translational invariant fuzzy subset of S. Suppose a is a $\mu-$prime element of a $\Gamma-$semigroup M and $\left(f(a)/_{y\alpha z}\right)_{f(\mu)}, y, z \in S, \alpha \in \Gamma$.
Since f is an onto homomorphism there exist $b, c \in M$ such that
$f(b) = y, f(c) = z$. So $f(b\alpha c) = f(b)\alpha f(c) = y\alpha z$.

$$\begin{aligned}\left(f(a)/_{y\alpha z}\right)_{f(\mu)} &\Rightarrow \text{ there exists } d \in M \text{ such that } f(\mu)\big(f(a)\beta f(d)\big) = f(\mu)f(b\alpha c), \text{ for all } \beta \in \Gamma\\ &\Rightarrow \mu(a\beta d) = \mu(b\alpha c), \text{ for all } \beta \in \Gamma\\ &\Rightarrow \left(a/_{b\alpha c}\right)_{\mu}.\end{aligned}$$

Since a is $\mu-$prime element, we have $\left(a/_b\right)_{\mu}$ or $\left(a/_c\right)_{\mu}$.

$$\begin{aligned}&\Rightarrow \mu(a\alpha s) = \mu(b) \text{ or } \mu(a\alpha r) = \mu(c), \text{ for all } \alpha \in \Gamma, \text{ for some } s, r \in M\\ &\Rightarrow f(\mu)\Big(f(a\alpha s)\Big) = f(\mu)\Big(f(b)\Big) \text{ or } f(\mu)\Big(f(a\alpha r)\Big) = f(\mu)\Big(f(c)\Big), \text{ for all } \alpha \in \Gamma\\ &\Rightarrow f(\mu)\Big(f(a)\alpha f(s)\Big) = f(\mu)\Big(f(b)\Big) \text{ or } f(\mu)\Big(f(a)\alpha f(r)\Big) = f(\mu)\Big(f(c)\Big), \text{ for all } \alpha \in \Gamma\\ &\Rightarrow \left(f(a)/_{f(b)}\right)_{f(\mu)} \text{ or } \left(f(a)/_{f(c)}\right)_{f(\mu)}\\ &\Rightarrow \left(f(a)/_{y}\right)_{f(\mu)} \text{ or } \left(f(a)/_{z}\right)_{f(\mu)}.\end{aligned}$$

Hence $f(a)$ is a $f(\mu)-$prime element of S. □

Theorem 4.10. *Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a translational invariant and $f-$invarianat fuzzy subset of a $\Gamma-$semigroup M. If a is a $\mu-$prime element of a $\Gamma-$semigroup M then the homomorphic image of $I(a, \mu)$ is a prime ideal of a $\Gamma-$semigroup S.*

Proof. Let f be a homomorphism from a $\Gamma-$semigroup M onto a $\Gamma-$semigroup S and μ be a translational invariant and $f-$invarianat fuzzy subset of a $\Gamma-$semigroup M. By Theorem 4.3, $f(\mu)$ is a fuzzy translational invariant fuzzy subset of a $\Gamma-$semigroup S. By Theorem 4.8, the $\mu-$principal ideal $I(a, \mu)$ is a prime ideal of a $\Gamma-$semigroup M where a is a $\mu-$prime element and μ is a $f-$invariant, translational invariant fuzzy subset of M. By Theorem 4.5, $f(I(a, \mu)) = I(f(a), f(\mu))$. By Theorem 4.9, $f(a)$ is $f(\mu)-$prime element of a $\Gamma-$semigroup S. Therefore, by Theorem 4.8, $I(f(a), f(\mu))$ is a prime ideal of S. □

5. Conclusions

In this paper, we introduced the notion of a left and a right translational invariant fuzzy subset of a $\Gamma-$semigroup M, the notion of a unit with respect to fuzzy subset and studied their properties. We proved that if μ is a translational invariant fuzzy subset of a commutative $\Gamma-$semigroup with unity then principal ideal generated by an element and μ, contains an unity element is not a proper ideal of a $\Gamma-$semigroup. we introduced the notion of associates, prime elements with respect to a fuzzy subset, an ideal of a $\Gamma-$semigroup generated by translational fuzzy subset and an element. We studied the properties of image and pre-image of translational invariant fuzzy subset under the $\Gamma-$semigroup homomorphism. We proved that every homomorphic image of an ideal of a $\Gamma-$semigroup generated by$\mu-$prime element and translational invariant fuzzy subset μ is a prime ideal of a $\Gamma-$semigroup. Our future work on this topic, we will extend these results to other algebraic structures and ordered $\Gamma-$semigroups.

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