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GENERALIZATIONS OF PSEUDO EBE-ALGEBRAS

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ABSTRACT. After BE-algebra was introduced by Kim et al., its generalization was attempted by several scholars. As a follow-up, Borzooei et al., Rezaei et al. and Sayyad et al. introduced pseudo BE-algebra, eBE-algebra and pseudo eBE-algebra, respectively. The aim of this article is to consider more extended version of pseudo eBE-algebra. The concepts of generalized pseudo eBE-algebra, (generalized) pseudo sub-eBE-algebra, eBE-subalgebra, eBE-filter, right (left) f-section and eBE-upper set are introduced and related properties, interrelationships and characterizations are studied.

1. INTRODUCTION

In mathematics, BCK and BCI-algebras are algebraic structures in universal algebra, which were introduced by Y. Imai, K. Iséki and S. Tanaka in 1966, that describe fragments of the propositional calculus involving implication known as BCK and BCI logics. Every abelian group is a BCI-algebra, with "$*$" defined as group subtraction and "0" defined as the group identity. As a generalization of a BCK-algebra, Kim et al. [2] introduced the notion of BE-algebra. In 2013, Borzooei et al. [1] introduced the concept of pseudo BE-algebra which is a generalization of BE-algebra. Rezaei et al. [3] introduced the notion of eBE-algebra which is another generalization of BE-algebra. Sayyad et al. [4] dealt with pseudo version of eBE-algebra which is an extension of pseudo BE-algebra. We observed that there is a condition that do not play a remarkable role in the paper [4]. Furthermore, it can be observed that this condition play little role in the study of substructures, for example, subalgebra, filter and ideal etc. Algebraic structures with conditions that play no many role will inevitably narrow their objects, so they can weaken the value of their use. Everyone knows that the wider the object for a new algebraic structure, the wider the application. Therefore, it is necessary to increase the value of use by expanding the object of algebraic structures except for the conditions in which the role is insignificant. From this point of view, we would like to introduce a more generalized concept by deleting conditions that do not play an important role. We introduce more general version than pseudo eBE-algebras, so called generalized pseudo BE-algebra, and investigate its properties. This can generalize the several results of paper [4], and allows some of the results of the paper [4] to be

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classified as corollaries. We consider the generalization of pseudo BE-algebra and pseudo eBE-algebra called generalized pseudo BE-algebra and generalized pseudo eBE-algebra, respectively, and introduce a method to establish generalized pseudo eBE-algebra using it. We introduce the concepts of (generalized) pseudo sub-eBE-algebra, eBE-subalgebra and eBE-filter and explore the various properties involved. We investigate the relationship between eBE-subalgebra and eBE-filter. We provide conditions for an eBE-subalgebra to be an eBE-filter, and for a subset to be an eBE-filter. We discuss characterizations of eBE-filter. We also introduce the concept of the right (left) f-section, and explore the necessary and sufficient conditions under which it will be an eBE-filter. We introduce the notion of the eBE-upper set, and we find and present the conditions under which eBE-upper set becomes a eBE-filter. We will use the eBE-upper set to characterize the eBE-filter.

2. PRELIMINARIES-LIST OF KNOWN DEFINITIONS

Definition 2.1 ([2]). An algebra $(X, *, 1)$ of type (2,0) is called a *BE-algebra* if it satisfies:

(BE1) $(\forall \mathring{x} \in X)\ (\mathring{x} * \mathring{x} = 1)$,
(BE2) $(\forall \mathring{x} \in X)\ (\mathring{x} * 1 = 1)$,
(BE3) $(\forall \mathring{x} \in X)\ (1 * \mathring{x} = \mathring{x})$,
(BE4) $(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\ (\mathring{x} * (\mathring{y} * \mathring{z}) = \mathring{y} * (\mathring{x} * \mathring{z}))$.

Definition 2.2 ([3]). Let X be a set with a binary operation "$*$" and a non-empty subset E. Then $(X, *, E)$ is called an *eBE-algebra* if it satisfies:

(eBE1) $(\forall \mathring{x} \in X)\ (\mathring{x} * \mathring{x} \in E)$,
(eBE2) $(\forall \mathring{x} \in X)\ (\mathring{x} * E \subseteq E)$,
(eBE3) $(\forall \mathring{x} \in X)\ (E * \mathring{x} = \{\mathring{x}\})$,
(eBE4) $(\forall \mathring{x}, \mathring{y}, \mathring{z} \in X)\ (\mathring{x} * (\mathring{y} * \mathring{z}) = \mathring{y} * (\mathring{x} * \mathring{z}))$

where $E * \mathring{x} = \{a * \mathring{x} \mid a \in E\}$ and $\mathring{x} * E = \{\mathring{x} * a \mid a \in E\}$.

Definition 2.3 ([1]). Consider a structure $(X, *, \circledast, 1)$ in which X is a non-empty set together with binary operations "$*$" and "$*$" and a special element "1". Then $(X, *, \circledast, 1)$ is called a *pseudo BE-algebra* if it satisfies:

(pBE1) $\mathring{x} * \mathring{x} = 1$ and $\mathring{x} \circledast \mathring{x} = 1$,
(pBE2) $\mathring{x} * 1 = 1$ and $\mathring{x} \circledast 1 = 1$,
(pBE3) $1 * \mathring{x} = \mathring{x}$ and $1 \circledast \mathring{x} = \mathring{x}$,
(pBE4) $\mathring{x} * (\mathring{y} \circledast \mathring{z}) = \mathring{y} \circledast (\mathring{x} * \mathring{z})$,
(pBE5) $\mathring{x} * \mathring{y} = 1 \ \Leftrightarrow\ \mathring{x} \circledast \mathring{y} = 1$

for all $\mathring{x}, \mathring{y}, \mathring{z} \in X$.

Let us simply present the pseudo BE-algebra $(X, *, \circledast, 1)$ as X.

Definition 2.4 ([4]). An algebra $(X, *, \circledast, E)$, where E is a non-empty subset of X and "$*$" and "$\circledast$" are binary operations on X, is called a *pseudo eBE-algebra* if it satisfies:

(peBE1) $\mathring{x} * \mathring{x} \in E$ and $\mathring{x} \circledast \mathring{x} \in E$,
(peBE2) $\mathring{x} * E \subseteq E$ and $\mathring{x} \circledast E \subseteq E$,
(peBE3) $E * \mathring{x} = \{\mathring{x}\}$ and $E \circledast \mathring{x} = \{\mathring{x}\}$,
(peBE4) $\mathring{x} * (\mathring{y} \circledast \mathring{z}) = \mathring{y} \circledast (\mathring{x} * \mathring{z})$,
(peBE5) $\mathring{x} * \mathring{y} \in E \ \Leftrightarrow\ \mathring{x} \circledast \mathring{y} \in E$

for all $\mathring{x}, \mathring{y}, \mathring{z} \in X$ where $\mathring{x} * E := \{\mathring{x} * a \mid a \in E\}$ and $\mathring{x} \circledast E := \{\mathring{x} \circledast a \mid a \in E\}$, $E * \mathring{x} := \{a * \mathring{x} \mid a \in E\}$ and $E \circledast \mathring{x} := \{a \circledast \mathring{x} \mid a \in E\}$.

3. GENERALIZED PSEUDO eBE-ALGEBRAS

In this section, we introduce generalized pseudo eBE-algebra, which generalizes pseudo eBE-algebra studied by Sayyad et al., and investigate its properties.

Definition 3.1. Let X be a set with a binary operation "$*$" and a non-empty subset E. Then $(X, *, E)$ is called a *generalized pseudo BE-algebra* if it satisfies (pBE1), (pBE2), (pBE4) and (pBE5).

Definition 3.2. Consider a structure $(X, *, \circledast, E)$ in which X is a non-empty set together with binary operations "$*$" and "$\circledast$" and a non-empty subset "E". Then $(X, *, \circledast, E)$ is called a *generalized pseudo eBE-algebra* if it satisfies (peBE1), (peBE2), (peBE4) and (peBE5).

Example 3.3. Let $X = \{1, a, b, c, d, e\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e
1	1	a	b	1	e	1
a	1	1	b	1	e	1
b	1	1	b	1	1	1
c	1	a	b	1	e	e
d	1	a	b	1	1	1
e	1	a	b	c	d	1

$\circledast$	1	a	b	c	d	e
1	1	a	b	1	c	1
a	1	1	b	1	c	1
b	1	1	b	1	1	1
c	1	a	b	1	d	e
d	1	a	b	1	1	1
e	1	a	b	c	c	1

Let $E = \{1, b\}$. Then it is routine to verify that $(X, *, \circledast, E)$ is a generalized pseudo eBE-algebra. But it does not satisfy (peBE3), and thus it is not a pseudo eBE-algebra.

In what follows, if there is no concern for confusion, the generalized pseudo eBE-algebra $(X, *, \circledast, E)$ shall be simply marked as (X, E).

We define a binary relation "$\leq_e$" on the generalized pseudo eBE-algebra (X, E) as follows:

$$(\forall x, y \in X)(x \leq_e y \Leftrightarrow x * y \in E \Leftrightarrow x \circledast y \in E). \tag{3.1}$$

It is clear that every pseudo eBE-algebra is a generalized pseudo eBE-algebra, but the converse is not true as seen in Example 3.3. We also know that a generalized pseudo eBE-algebra (X, E) with $E = \{1\}$ is a generalized pseudo BE-algebra.

Proposition 3.1. *Every generalized pseudo eBE-algebra (X, E) satisfies:*

$$(\forall x, y \in X)(x * (y \circledast x) \in E,\ x \circledast (y * x) \in E). \tag{3.2}$$

$$(\forall x, y \in X)(x * ((x * y) \circledast y) \in E,\ x \circledast ((x \circledast y) * y) \in E). \tag{3.3}$$

$$(\forall x, y, z \in X)(x \leq_e y * z \Leftrightarrow y \leq_e x \circledast z). \tag{3.4}$$

$$(\forall x, y, z \in X)(x \leq_e y \Rightarrow x \leq_e z * y,\ x \leq_e z \circledast y). \tag{3.5}$$

$$(\forall x, y, z \in X)(x * y = z \text{ or } x \circledast y = z \Rightarrow y * z \in E,\ y \circledast z \in E). \tag{3.6}$$

$$(\forall x, y \in X)(x * y = x,\ x \notin E \Rightarrow x \circledast y \neq y). \tag{3.7}$$

$$(\forall x, y \in X)(x * y = y,\ x \notin E \Rightarrow x \circledast y \neq x). \tag{3.8}$$

$$(\forall x, y, z, a \in X)(x * y = z,\ x \circledast y = a \Rightarrow x \circledast z = x * a). \tag{3.9}$$

$$(\forall x, y \in X)(x \leq_e (x * y) \circledast x,\ x \leq_e (x \circledast y) * x). \tag{3.10}$$

$$(\forall x, y, z \in X)(x * y \in E \Rightarrow x * (z \circledast y) \in E). \tag{3.11}$$

$$(\forall x, y, z \in X)(x \circledast y \in E \Rightarrow x \circledast (z * y) \in E). \tag{3.12}$$

Proof. Let $x, y \in X$. Then $x * (y \circledast x) = y \circledast (x * x) \in y \circledast E \subseteq E$ and $x \circledast (y * x) = y * (x \circledast x) \in y \circledast E \subseteq E$ by (peBE1) and (peBE2). Hence (3.2) is valid. Also we have $x * ((x * y) \circledast y) = (x * y) \circledast (x * y) \in E$ and $x \circledast ((x \circledast y) * y) = (x \circledast y) * (x \circledast y) \in E$ by (peBE1) and (peBE4), which shows (3.3) is valid. The combination of (peBE4) and (3.1) leads to (3.4). Let $x, y, z \in X$ be such that $x \leq_e y$. Then $x * y \in E$ and $x \circledast y \in E$ by (3.1), which imply from (peBE2) and (peBE4) that $x \circledast (z * y) = z * (x \circledast y) \in z * E \subseteq E$ and $x * (z \circledast y) = z \circledast (x * y) \in z \circledast E \subseteq E$. Hence $x \leq_e z * y$ and $x \leq_e z \circledast y$. Assume that $x * y = z$. Then $y \circledast z = y \circledast (x * y) = x * (y \circledast y) \in x * E \subseteq E$ and hence $y * z \in E$ by (peBE5). In the same way, the case of $x \circledast y = z$ can be calculated. Suppose that $x * y = x$ and $x \notin E$ for all $x, y \in X$. If $x \circledast y = y$, then

$$x \circledast x = x \circledast (x * y) = x * (x \circledast y) = x * y = x \notin E$$

which is a contradiction. Hence (3.7) is valid. Suppose that $x * y = y$ and $x \notin E$ for all $x, y \in X$. If $x \circledast y = x$, then

$$x * x = x * (x \circledast y) = x \circledast (x * y) = x \circledast y = x \notin E$$

which is a contradiction. Hence (3.8) is valid. Let $x, y, z, a \in X$ be such that $x * y = z$ and $x \circledast y = a$. Then

$$x \circledast z = x \circledast (x * y) = x * (x \circledast y) = x * a$$

by (peBE4). For every $x, y \in X$, we have

$$x * ((x * y) \circledast x) = (x * y) \circledast (x * x) \in (x * y) \circledast E \subseteq E$$

and

$$x \circledast ((x \circledast y) * x) = (x \circledast y) * (x \circledast x) \in (x \circledast y) * E \subseteq E.$$

Thus $x \leq_e (x * y) \circledast x$ and $x \leq_e (x \circledast y) * x$. Let $x, y, z \in X$ be such that $x * y \in E$ (resp., $x \circledast y \in E$). Using (peBE2) and (peBE4), we have $x * (z \circledast y) = z \circledast (x * y) \in z \circledast E \subseteq E$ (resp., $x \circledast (z * y) = z * (x \circledast y) \in z * E \subseteq E$). This completes the proof. □

Corollary 3.2 ([4]). *Every pseudo eBE-algebra (X, E) satisfies all the conditions listed in Proposition 3.1.*

Corollary 3.3. *Every generalized pseudo eBE-algebra (X, E) satisfies:*

$$(\forall x, y \in X)(x * (y * x) \in E,\ x \circledast (y \circledast x) \in E). \tag{3.13}$$

$$(\forall x, y \in X)(x \circledast ((x * y) \circledast y) \in E,\ x * ((x \circledast y) * y) \in E). \tag{3.14}$$

$$(\forall x, y, z \in X)(x * y \in E \ \Rightarrow\ x \circledast (z \circledast y) \in E). \tag{3.15}$$

$$(\forall x, y, z \in X)(x \circledast y \in E \ \Rightarrow\ x * (z * y) \in E). \tag{3.16}$$

Proof. It is obvious by (3.2), (3.3), (3.11), (3.12) and (peBE5). □

Corollary 3.4 ([4]). *Every pseudo eBE-algebra (X, E) satisfies all the conditions listed in Corollary 3.3.*

Theorem 3.5. *If (X, E_1) and (X, E_2) are generalized pseudo eBE-algebras, then $(X, E_1 \cap E_2)$ and $(X, E_1 \cup E_2)$ are generalized pseudo eBE-algebras.*

Proof. Assume that (X, E_1) and (X, E_2) are generalized pseudo eBE-algebras. It is clear that $x * x \in E_1 \cap E_2$ and $x * x \in E_1 \cup E_2$ for all $x \in X$. It is also axiomatic that (peBE4) and (peBE5) are established. For any $x \in X$, let $a \in x * (E_1 \cap E_2)$ and $b \in x \circledast (E_1 \cap E_2)$. Then there exist $y \in E_1 \cap E_2$ and $z \in E_1 \cap E_2$ such that $a = x * y$ and $b = x \circledast z$, respectively. It follows that $a = x * y \in x * E_1 \subseteq E_1$, $a = x * y \in x * E_2 \subseteq E_2$,

$b = x \circledast z \in x * E_1 \subseteq E_1$, and $b = x \circledast z \in x * E_2 \subseteq E_2$. Hence $a \in E_1 \cap E_2$ and $b \in E_1 \cap E_2$, and so $x * (E_1 \cap E_2) \subseteq E_1 \cap E_2$ and $x \circledast (E_1 \cap E_2) \subseteq E_1 \cap E_2$. For any $x \in X$, let $a \in x * (E_1 \cup E_2)$ and $b \in x \circledast (E_1 \cup E_2)$. Then there exist $y \in E_1 \cup E_2$ and $z \in E_1 \cup E_2$ such that $a = x * y$ and $b = x \circledast z$, respectively. If $y \in E_1$ or $y \in E_2$, then $a = x * y \in x * E_1 \subseteq E_1$ or $a = x * y \in x * E_2 \subseteq E_1$, and so $a \in E_1 \cup E_2$. Thus $x * (E_1 \cup E_2) \subseteq E_1 \cup E_2$. If $z \in E_1$ or $z \in E_2$, then $b = x \circledast z \in x \circledast E_1 \subseteq E_1$ or $b = x \circledast z \in x \circledast E_2 \subseteq E_1$, and so $b \in E_1 \cup E_2$. Thus $x \circledast (E_1 \cup E_2) \subseteq E_1 \cup E_2$. Therefore $(X, E_1 \cap E_2)$ and $(X, E_1 \cup E_2)$ are generalized pseudo eBE-algebras. □

Let $\{E, B\}$ be a partition of X. If (X, E) is a generalized pseudo eBE-algebra, then (X, B) is not a generalized pseudo eBE-algebra as seen in the following example. Therefore we know that if (X, E) is a generalized pseudo eBE-algebra, then $(X, X \setminus E)$ can't be a generalized pseudo eBE-algebra.

Example 3.4. In Example 3.3, if $E = \{1, b\}$ and $B = \{a, c, d, e\}$ then $\{E, B\}$ is a partition of X. We can observe that (X, E) is a generalized pseudo eBE-algebra. But (X, B) is not a generalized pseudo eBE-algebra since $a * a = 1 \notin B$.

We introduce how to make a generalized pseudo eBE-algebra using a weaker structure, so called generalized pseudo BE-algebra, than a pseudo BE-algebra.

Definition 3.5. Consider a structure $(X, *, \circledast, 1)$ in which X is a non-empty set together with binary operations "$*$" and "$*$" and a special element "1". Then $(X, *, \circledast, 1)$ is called a *generalized pseudo BE-algebra* if it satisfies (pBE1), (pBE2), (pBE4) and (pBE5).

Theorem 3.6. *Let $(X, *, \circledast, 1)$ be a generaliged pseudo BE-algebra and let $Y = X \cup E_0$ where E_0 is a set which is disjoint to X. Define two binary operations "$\tilde{*}$" and "$\tilde{\circledast}$" on Y as follows:*

$$\tilde{*} : Y \times Y \to Y, (x, y) \mapsto \begin{cases} x * y & \text{if } x, y \in X, \\ y & \text{otherwise,} \end{cases}$$

and

$$\tilde{\circledast} : Y \times Y \to Y, (x, y) \mapsto \begin{cases} x \circledast y & \text{if } x, y \in X, \\ y & \text{otherwise.} \end{cases}$$

Then $(Y, \tilde{*}, \tilde{\circledast}, E)$ is a generalized pseudo eBE-algebra where $E = E_0 \cup \{1\}$.

Proof. For any $x \in Y$, we have

$$x \tilde{*} x = \begin{cases} x * x = 1 \in E & \text{if } x \in X, \\ x \in E_0 \subseteq E & \text{if } x \in E_0 \end{cases}$$

and

$$x \tilde{\circledast} x = \begin{cases} x \circledast x = 1 \in E & \text{if } x \in X, \\ x \in E_0 \subseteq E & \text{if } x \in E_0. \end{cases}$$

Hence $(Y, \tilde{*}, \tilde{\circledast}, E)$ satisfies the condition (peBE1). It is clear that $(Y, \tilde{*}, \tilde{\circledast}, E)$ satisfies the condition (peBE2). By the similarly way to the proof of Theorem 3.8 in the paper [4], we can verify that $(Y, \tilde{*}, \tilde{\circledast}, E)$ satisfies the conditions (peBE4) and (peBE5). Therefore $(Y, \tilde{*}, \tilde{\circledast}, E)$ with $E = E_0 \cup \{1\}$ is a generalized pseudo eBE-algebra. □

Corollary 3.7. *Let $(X, *, \circledast, 1)$ be a pseudo BE-algebra and let $Y = X \cup E_0$ where E_0 is a set which is disjoint to X. Then $(Y, \tilde{*}, \tilde{\circledast}, E)$ is a generalized pseudo eBE-algebra where $E = E_0 \cup \{1\}$ where "$\tilde{*}$" and "$\tilde{\circledast}$" are binary operationson Y which are defined in Theorem 3.6.*

We present an example to illustrate the Theorem 3.6 in the following:

Example 3.6. Let $X = \{1, a, b, c, d\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d
1	1	a	1	c	1
a	1	1	d	1	d
b	1	c	1	c	1
c	1	1	d	1	b
d	1	a	1	c	1

$\circledast$	1	a	b	c	d
1	1	c	1	c	1
a	1	1	b	1	d
b	1	c	1	c	1
c	1	1	b	1	d
d	1	c	1	c	1

Then it is routine to verify that $(X, *, \circledast, 1)$ is a generalized pseudo BE-algebra which is not a pseudo BE-algebra. Using Theorem 3.6, we can establish a generalized pseudo eBE-algebra $(Y, \tilde{*}, \tilde{\circledast}, E)$ with $E = \{1, e\}$ in which the operations "$\tilde{*}$" and "$\tilde{\circledast}$" as follows:

$*$	1	a	b	c	d	e
1	1	a	1	c	1	e
a	1	1	d	1	d	e
b	1	c	1	c	1	e
c	1	1	d	1	b	e
d	1	a	1	c	1	e
e	1	a	b	c	d	e

$\circledast$	1	a	b	c	d	e
1	1	c	1	c	1	e
a	1	1	b	1	d	e
b	1	c	1	c	1	e
c	1	1	b	1	d	e
d	1	c	1	c	1	e
e	1	a	b	c	d	e

Definition 3.7. Let $(X, *, \circledast, E)$ be a (generalized) pseudo eBE-algebra. If there are subsets L and B of X and E, respectively, such that $(L, *, \circledast, B)$ is a (generalized) pseudo eBE-algebra, then we say that $(L, *, \circledast, B)$ is a *(generalized) pseudo sub-eBE-algebra* of $(X, *, \circledast, E)$.

It is lear that every (generalized) pseudo eBE-algebra $(X, *, \circledast, E)$ is a (generalized) pseudo sub-eBE-algebra of $(X, *, \circledast, E)$.

Example 3.8. Let $X = \{1, a, b, c, d, e, f\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e	f
1	1	f	e	c	d	e	e
a	1	e	e	c	d	e	e
b	1	e	e	c	d	e	e
c	1	a	b	c	d	e	f
d	1	a	b	c	d	e	b
e	1	a	b	c	d	e	f
f	1	e	e	c	d	e	e

$\circledast$	1	a	b	c	d	e	f
1	1	b	e	c	d	e	e
a	1	e	e	c	d	e	e
b	1	e	e	c	d	e	e
c	1	a	b	c	d	e	f
d	1	a	b	c	d	e	f
e	1	a	b	c	d	e	f
f	1	e	e	c	d	e	e

Then it is routine to verify that $(X, *, \circledast, E)$, where $E = \{1, c, d, e\}$, is a generalized pseudo eBE-algebra, and if we take $L := \{1, a, b, e, f\} \subseteq X$ and $B := \{1, e\} \subseteq E$, then $(L, *, \circledast, B)$ is generalized pseudo sub-eBE-algebra of $(X, *, \circledast, E)$.

Example 3.9. Let $X = \{1, a, b, c, d, e, f\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	a	b	c	d	e	f
b	1	a	b	c	d	e	f
c	1	a	b	c	d	e	f
d	1	a	b	c	a	d	a
e	1	a	b	c	a	a	a
f	1	a	b	c	a	a	a

$\circledast$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	a	b	c	d	e	f
b	1	a	b	c	d	e	f
c	1	a	b	c	d	e	f
d	1	a	b	c	a	f	a
e	1	a	b	c	a	a	a
f	1	a	b	c	a	a	a

Then it is routine to verify that $(X, *, \circledast, E)$ is a pseudo eBE-algebra where $E = \{1, a, b, c\}$. If we take $L := \{1, a, d, e, f\} \subseteq X$ and $B := \{1, a\} \subseteq E$, then $(L, *, \circledast, B)$ is a pseudo sub-eBE-algebra of $(X, *, \circledast, E)$.

Theorem 3.8. *If $(L, *, \circledast, E_1)$ and $(L, *, \circledast, E_2)$ are pseudo sub-eBE-algebras of a pseudo eBE-algebra $(X, *, \circledast, E)$, then $E_1 = E_2$.*

Proof. Assume that $E_1 \setminus E_2 \neq \emptyset$. Then there exists $x \in L$ such that $x \in E_1$ and $x \notin E_2$. By (peBE3), we get $x * x \in E_1 * x = \{x\}$ and $x \circledast x \in E_1 * x = \{x\}$. It follows from (peBE1) that $x = x * x \in E_2$ and $x = x \circledast x \in E_2$. This is a contradiction, and so $E_1 \setminus E_2 = \emptyset$. Similarly, we get $E_2 \setminus E_1 = \emptyset$, and therefore $E_1 = E_2$. □

Corollary 3.9. *If $(X, *, \circledast, E_1)$ is a pseudo sub-eBE-algebra of a pseudo eBE-algebra $(X, *, \circledast, E)$, then $E_1 = E$.*

Theorem 3.8 shows that generalized pseudo sub-eBE-algebra, formed by operations "$*$" and "$\circledast$" for a fixed subset L of X, is unique. But Theorem 3.8 is not established in a generalized pseudo eBE-algebra, that is, there are generalized pseudo sub-eBE-algebras $(L, *, \circledast, E_1)$ and $(L, *, \circledast, E_2)$ with $E_1 \neq E_2$ of a generalized pseudo eBE-algebra $(X, *, \circledast, E)$ as seen in the following example.

Example 3.10. Let $X = \{1, a, b, c, d, e\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e
1	b	b	b	b	c	b
a	a	b	b	b	b	b
b	a	a	b	b	b	b
c	1	a	b	b	b	e
d	1	a	b	b	c	e
e	a	b	b	b	c	b

$\circledast$	1	a	b	c	d	e
1	b	b	b	b	c	b
a	e	b	b	b	b	b
b	1	a	b	b	b	b
c	1	a	b	b	b	e
d	1	a	b	b	c	e
e	e	b	b	b	c	b

Then $(X, *, \circledast, E)$ is a generalized pseudo eBE-algebra where $E = \{b, c, d\}$. It is routine to verify that if we take $L := \{1, a, b, c, e\} \subseteq X$, $E_1 := \{b, c\} \subseteq E$ and $E_2 := \{b\} \subseteq E$, then $(L, *, \circledast, E_1)$ and $(L, *, \circledast, E_2)$ are generalized pseudo sub-eBE-algebras of $(X, *, \circledast, E)$ with $E_1 \neq E_2$.

Let $(X, *, \circledast, E)$ be a (generalized) pseudo eBE-algebra. If we take subsets L and B of X and E, respectively, such that $B = L \cap E$ and L is closed under the operations "$*$" and "$\circledast$", then $(L, *, \circledast, B)$ is a (generalized) pseudo sub-eBE-algebra of $(X, *, \circledast, E)$.

Theorem 3.10. *Let $(X, *, \circledast, E)$ be a pseudo eBE-algebra. If a structure $(L, *, \circledast, B)$ is a pseudo sub-eBE-algebra of $(X, *, \circledast, E)$, then $B = L \cap E$ and L is closed under the operations "$*$" and "$\circledast$".*

Proof. Let $(L, *, \circledast, B)$ be a pseudo sub-eBE-algebra of a pseudo eBE-algebra $(X, *, \circledast, E)$. It is clear that L is closed under the operations "$*$" and "$\circledast$". Let $A = L \cap E$. Since $L \subseteq X$, $(L, *, \circledast, A)$ satisfies the condition (peBE4). For every $x \in L$, we get $x*x \in L\cap E = A$ and $x\circledast x \in L\cap E = A$. Also we have $x*A = x*(L\cap E) = (x*L)\cap(x*E) \subseteq L\cap E = A$ and $x \circledast A = x \circledast (L \cap E) = (x \circledast L) \cap (x \circledast E) \subseteq L \cap E = A$ for all $x \in L$. For every $x \in L$, we obtain $A * x = (L \cap E) * x = (L * x) \cap (E * x) \subseteq L \cap \{x\} = \{x\}$ and $A \circledast x = (L \cap E) \circledast x = (L \circledast x) \cap (E \circledast x) \subseteq L \cap \{x\} = \{x\}$, and thus $A * x = \{x\}$ and $A \circledast x = \{x\}$. Let $x, y \in L$ be such that $x * y \in A$. Then $x * y \in L \cap E$, and so $x * y \in L$ and $x * y \in E$ It follows from (peBE5) that $x * y \in L$ and $x \circledast y \in E$. Hence $x \circledast y \in L \cap E = A$. Similarly if $x \circledast y \in A$, then $x * y \in A$ for all $x, y \in L$. Therefore $(L, *, \circledast, A)$ is a pseudo sub-eBE-algebra of $(X, *, \circledast, E)$. Using Theorem 3.8, we conclude that $B = A = L \cap E$. □

The following example shows that Theorem 3.10 is not established in a generalized pseudo eBE-algebra.

Example 3.11. In Example 3.10, we can observe that $(L, *, \circledast, E_2)$ is a generalized pseudo sub-eBE-algebra of $(X, *, \circledast, E)$. But $E_2 = \{b\} \neq \{b, c\} = L \cap E$.

4. eBE-SUBALGEBRAS AND eBE-FILTERS

Let (X, E) be a generalized pseudo eBE-algebra and consider a subset F of X such that F is a superset of E. Then F is not closed under "$*$" or "$\circledast$" as seen in the following example.

Example 4.1. Consider the generalized pseudo eBE-algebra (X, E) in Example 3.10. A superset $F := \{1, b, c, d, e\}$ of E is not closed uner $*$ since $e * 1 = a \notin F$.

Based on these observations, we can consider a substructure, so called eBE-subalgebra, in generalized pseudo eBE-algebra as follows.

Definition 4.2. Let (X, E) be a generalized pseudo eBE-algebra. Then a subset F of X is called an *eBE-subalgebra* of (X, E) if it is a superset of E which is closed under the operations "$*$" and "$\circledast$".

Note that "eBE-subalgebra" is not the same as "subalgebra" in the paper [4].

Example 4.3. Consider the generalized pseudo eBE-algebra (X, E) in Example 3.8. If we take a superset $F := \{1, b, c, d, e\}$ of E, then it is obvious that F is closed under "$*$" and "$\circledast$". Hence F is an eBE-subalgebra of (X, E).

Theorem 4.1. *If F and G are eBE-subalgebras of a generalized pseudo eBE-algebra (X, E), then so is $F \cap G$.*

Proof. It is straightforward by simple calculations. □

The following example shows that the union of two eBE-subalgebras of a generalized pseudo eBE-algebra (X, E) may not be an eBE-subalgebra of (X, E).

Example 4.4. Let $X = \{1, a, b, c, d, e, f\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e	f
1	f	a	f	a	f	f	f
a	1	e	d	e	1	e	e
b	f	a	f	c	f	f	f
c	1	e	b	e	d	e	e
d	f	a	f	a	f	f	f
e	f	a	f	c	f	f	f
f	1	e	b	e	d	e	e

$\circledast$	1	a	b	c	d	e	f
1	e	a	e	a	e	e	f
a	1	f	d	f	1	e	f
b	e	a	e	c	e	e	f
c	1	f	b	f	d	e	f
d	e	a	e	a	e	e	f
e	e	a	e	c	e	e	f
f	1	f	b	f	d	e	f

Then it is routine to verify that $(X, *, \circledast, E)$, where $E = \{e, f\}$, is a generalized pseudo eBE-algebra, and if we take $F := \{1, a, e, f\}$ and $G := \{b, c, e, f\}$, then F and G are eBE-subalgebras of $(X, *, \circledast, E)$. But $F \cup G := \{1, a, b, c, e, f\}$ is not an eBE-subalgebra of $(X, *, \circledast, E)$ since $a * b = d \notin F \cup G$ and $a \circledast b = d \notin F \cup G$.

In the previous section, we introduced generalized pseudo eBE-algebra by deleting a condition that do not play an important role in pseudo eBE-algebra. It can be observed that this condition also play little role in the study of a substructure, called a filter. So we want to deal with filter theory in a generalized pseudo eBE-algebra.

If we take a superset F of E in a generalized pseudo eBE-algebra (X, E), then the following example shows that F does not satisfy the next assertion:

$$(\forall x, y \in X)(x \in F,\, x \circledast y \in F \;\Rightarrow\; y \in F). \tag{4.1}$$

Example 4.5. Consider the generalized pseudo eBE-algebra (X, E) in Example 3.10. A superset $F := \{b, c, d, e\}$ of E does not satisfy (4.1) since $e \circledast 1 = e \in F$ and $e \in F$ but $1 \notin F$.

The following example shows that in a generalized pseudo eBE-algebra (X, E), the set E does not satisfy the following assertion:

$$(\forall x, y \in X)(x \in E,\, x * y \in E \;\Rightarrow\; y \in E). \tag{4.2}$$

Example 4.6. Consider the generalized pseudo eBE-algebra (X, E) in Example 3.10. Then $E = \{b, c, d\}$ does not satisfy (4.2) since $b * e = b \in E$ and $b \in E$ but $e \notin E$.

Based on these observations, we can consider a substructure, so called eBE-filter, in generalized pseudo eBE-algebra as follows.

Definition 4.7. A subset F of X in a generalized pseudo eBE-algebra (X, E) is called an *eBE-filter* of (X, E) if it is a superset of E satisfying the condition (4.1)

Example 4.8. Let $X = \{1, a, b, c, d, e\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e
1	1	a	b	a	d	d
a	1	a	b	a	d	d
b	1	a	b	a	d	d
c	1	a	b	a	d	e
d	1	a	b	c	b	b
e	1	a	b	a	b	b

$\circledast$	1	a	b	c	d	e
1	1	a	b	a	d	e
a	1	a	b	a	d	d
b	1	a	b	a	d	e
c	1	a	b	a	d	e
d	1	a	b	c	b	b
e	1	a	b	a	b	b

Then it is routine to verify that $(X, *, \circledast, E)$ is a generalized pseudo eBE-algebra where $E = \{1, a, b\}$, and the set $F = \{1, a, b, c\}$ is an eBE-filter of (X, E).

Theorem 4.2. *A subset F of X in a generalized pseudo eBE-algebra (X, E) is an eBE-filter of (X, E) if and only if F is a superset of E and satisfies:*

$$(\forall x, y \in X)(x \in F,\, x * y \in F \;\Rightarrow\; y \in F). \tag{4.3}$$

Proof. Assume that F is an eBE-filter of (X, E). Then F is a superset of E. Let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. By (3.14), we have $x \circledast ((x * y) \circledast y) \in E \subseteq F$. It follows from (4.1) that $(x * y) \circledast y \in F$. Hence $y \in F$ by (4.1).

Conversely, suppose that F is a superset of E that satisfies the condition (4.3). Let $x, y \in X$ be such that $x \in F$ and $x \circledast y \in F$. Then $x * ((x \circledast y) * y) \in E \subseteq F$ by (3.14), which implies from (4.3) that $(x \circledast y) * y \in F$, and hence $y \in F$. Therefore F is an eBE-filter of (X, E). □

Corollary 4.3 ([4]). *Let $(X, *, \circledast, E)$ be a pseudo eBE-algebra and $E \subseteq F \subseteq X$. If F is a filter, then*

$$(\forall x, y \in X)(x \in F,\, x \circledast y \in F \;\Rightarrow\; y \in F).$$

Proposition 4.4. *Every eBE-filter F of a generalized pseudo eBE-algebra (X, E) satisfies:*

$$(\forall x, y \in X)(x \leq_e y,\, x \in F \;\Rightarrow\; y \in F). \tag{4.4}$$

$$(\forall x, y, z \in X)(x, y \in F,\, x \leq_e y * z \;\Rightarrow\; z \in F). \tag{4.5}$$

$$(\forall x, y, z \in X)(x, y \in F,\, x \leq_e y \circledast z \;\Rightarrow\; z \in F). \tag{4.6}$$

Proof. (4.4) is straightforward by Theorem 4.2. Let $x, y, z \in X$ be such that $x, y \in F$ and $x \leq_e y*z$. Then $x \circledast (y*z) \in E \subseteq F$, and so $y*z \in F$ by (4.1). Hence $z \in F$ by Theorem 4.2. Let $x, y, z \in X$ be such that $x, y \in F$ and $x \leq_e y \circledast z$. Then $x * (y \circledast z) \in E \subseteq F$, and so $y \circledast z \in F$ by Theorem 4.2. It follows from (4.1) that $z \in F$. □

Corollary 4.5 ([4]). *If $(X, *, \circledast E)$ is a pseudo eBE-algebra, then every filter F satisfies all the conditions listed in Proposition 4.4.*

The relationship between eBE-filter and eBE-subalgebra is mentioned below.

Theorem 4.6. *In a generalized pseudo eBE-algebra, every eBE-filter is an eBE-subalgebra.*

Proof. Let F be an eBE-filter of a generalized pseudo eBE-algebra (X, E). Then F is a superset of E. Let $x, y \in F$. By (3.2), $y * (x \circledast y) \in E \subseteq F$ and $y \circledast (x * y) \in E \subseteq F$. Since$F$ is an eBE-filter of (X, E), we have $x * y \in F$ and $x \circledast y \in F$. Thus F is an eBE-subalgebra of (X, E). □

In the next example, we can verify that the converse of Theorem 4.6 is not true in general.

Example 4.9. In Example 4.8, we can observe $G = \{1, a, b, d, e\}$ is an eBE-subalgebra of (X, E). Since $b \circledast c = a \in G$ and $b \in G$ but $c \notin G$, we know that G is not an eBE-filter of (X, E).

We provide conditions for an eBE-subalgebra to be an eBE-filter in a generalized pseudo eBE-algebra.

Theorem 4.7. *If an eBE-subalgebra F of a generalized pseudo eBE-algebra (X, E) satisfies:*

$$(\forall x, y \in X)(x \in F,\, y \in F^c \;\Rightarrow\; x \circledast y \in F^c), \tag{4.7}$$

or

$$(\forall x, y \in X)(x \in F,\, y \in F^c \;\Rightarrow\; x * y \in F^c), \tag{4.8}$$

then F is an eBE-filter of (X, E).

Proof. Let F be an eBE-subalgebra of (X, E) and suppose it satisfies (4.7). Let $x, y \in X$ be such that $x \in F$ and $x \circledast y \in F$. If $y \notin F$, i.e., $y \in F^c$, then $x \circledast y \in F^c$ by (4.7). This is a contradiction, and so $y \in F$. Threfore F is an eBE-filter of (X, E). Similarly, we can show that if F satisfies the condition (4.8), then F is an eBE-filter of (X, E). □

Theorem 4.8. *Let F be a subset of X in a generalized pseudo eBE-algebra (X, E). If F is a superset of E and satisfies* (4.5) *or* (4.6)*, then F is an eBE-filter of (X, E).*

Proof. Let F be a superset of E and suppose it satisfies (4.5). Let $x, y \in X$ be such that $x \in F$ and $x \circledast y \in F$. Then, by (3.3), $x * ((x \circledast y) * y) \in E$. Hence $x \leq_e (x \circledast y) * y$. It follows from (4.5) that $y \in F$. Thus F is an eBE-filter of (X, E). Similarly, we can verify that if F satisfies (4.6), then F is an eBE-filter of (X, E). □

Theorem 4.9. *If F_1 and F_2 are eBE-filters of a generalized pseudo eBE-algebra (X, E), then so is their intersection $F_1 \cap F_2$.*

Proof. If F_1 and F_2 are eBE-filters of (X, E), then $F_1 \cap F_2$ is clearly a superset of E. Let $x, y \in F$ be such that $x \in F_1 \cap F_2$ and $x * y \in F_1 \cap F_2$. Then $x \in F_1$, and $x * y \in F_1$, $x \in F_2$, and $x * y \in F_2$. It follows from Theorem 4.2 that $y \in F_1$ and $y \in F_2$, that is, $y \in F_1 \cap F_2$. Hence $F_1 \cap F_2$ is an eBE-filter of (X, E). □

The following example shows that the union of eBE-filters of a generalized pseudo eBE-algebra (X, E) may not be an eBE-filter of (X, E).

Example 4.10. Let $X = \{1, a, b, c, d\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d
1	c	a	a	c	c
a	1	c	1	c	c
b	c	c	c	c	c
c	1	a	b	c	c
d	1	a	b	c	c

$\circledast$	1	a	b	c	d
1	c	a	a	c	c
a	1	c	1	c	c
b	c	c	c	c	c
c	1	a	b	c	c
d	1	a	b	c	d

Then it is routine to verify that $(X, *, \circledast, E)$, where $E = \{c, d\}$, is a generalized pseudo eBE-algebra. Let $F = \{1, c, d\}$ and $G = \{a, c, d\}$. Then F and G are eBE-filters of (X, E). But $F \cup G = \{1, a, c, d\}$ is not an eBE-filter of (X, E) since $a * b = 1 = a \circledast b \in F \cup G$ and $a \in F \cup G$ but $b \notin F \cup G$.

Let (X, E) be a generalized pseudo eBE-algebra. Consider mappings:

$$\overrightarrow{f} : X \to 2^X,\; x \mapsto \overrightarrow{f}(x), \qquad \overleftarrow{f} : X \to 2^X,\; x \mapsto \overleftarrow{f}(x)$$

where 2^X is the power set of X, $\overrightarrow{f}(x) := \{y \in X \mid x \leq_e y\}$ and $\overleftarrow{f}(x) := \{y \in X \mid y \leq_e x\}$ which are called the *right f-section* and *left f-section*, respectively, of x.

It is clear that if (X, E) is a generalized pseudo eBE-algebra, then $x \in \overrightarrow{f}(x) \cap \overleftarrow{f}(x)$ for all $x \in X$ by (peBE1).

The following example shows that there exists $x \in X$ such that the right f-section and the left f-section of X are not eBE-filters.

Example 4.11. Consider the generalized pseudo eBE-algebra (X, E) in Example 3.3. We can observe that $\overrightarrow{f}(e) = \{1, b, e\}$ and $\overleftarrow{f}(e) = \{1, a, b, d, e\}$. But $\overrightarrow{f}(e)$ and $\overleftarrow{f}(e)$ are not eBE-filters of (X, E), since $b \circledast c = 1 \in \overrightarrow{f}(e)$ and $b \in \overrightarrow{f}(e)$ but $c \notin \overrightarrow{f}(e)$, and also $b \circledast c = 1 \in \overleftarrow{f}(e)$ and $b \in \overleftarrow{f}(e)$ but $c \notin \overleftarrow{f}(e)$.

We consider conditions for the right f-section (resp., the left f-section) of an element $x \in X$ to be an eBE-filter.

Theorem 4.10. *Let (X, E) be a generalized pseudo eBE-algebra. Then the right f-section of a fixed element a in (X, E) is an eBE-filter of (X, E) if and only if the next assertion is valid.*

$$(\forall x, y \in X)\,(a \leq_e x * y,\, a \leq_e x \;\Rightarrow\; a \leq_e y)\,. \tag{4.9}$$

Proof. Assume that the right f-section of $a \in X$ is an eBE-filter of (X, E). Let $x, y \in X$ be such that $a \leq_e x * y$ and $a \leq_e x$. Then $x * y \in \overrightarrow{f}(a)$ and $x \in \overrightarrow{f}(a)$. Since $\overrightarrow{f}(a)$ is an eBE-filter of (X, E), it follows from Theorem 4.2 that $y \in \overrightarrow{f}(a)$, that is, $a \leq_e y$.

Conversely, suppose that the condition (4.9) is satisfied. If $x \in E$, then $a * x \in a * E \subseteq E$, i.e., $a \leq_e x$, and so $x \in \overrightarrow{f}(a)$. Hence $\overrightarrow{f}(a)$ is a superset of E. Let $x, y \in X$ be such that $x * y \in \overrightarrow{f}(a)$ and $x \in \overrightarrow{f}(a)$. Then $a \leq_e x * y$ and $a \leq_e x$, which imply from (4.9) that $a \leq_e y$. Thus $y \in \overrightarrow{f}(a)$, and therefore $\overrightarrow{f}(a)$ is an eBE-filter of (X, E). □

Theorem 4.11. *Let (X, E) be a generalized pseudo eBE-algebra. Then the right f-section of a fixed element a in (X, E) is an eBE-filter of (X, E) if and only if the next assertion is valid.*

$$(\forall x, y \in X)\,(a \leq_e x \circledast y,\, a \leq_e x \;\Rightarrow\; a \leq_e y)\,. \tag{4.10}$$

Proof. It is similar to the proof of Theorem 4.10. □

Definition 4.12. A generalized pseudo eBE-algebra (X, E) is said to be *distributive* if it satisfies any one of the following conditions:

$$(\forall x, y, z \in X)(x * (y \circledast z) = (x * y) \circledast (x * z)). \tag{4.11}$$

$$(\forall x, y, z \in X)(x \circledast (y * z) = (x \circledast y) * (x \circledast z)). \tag{4.12}$$

Example 4.13. Let $X = \{1, a, b, c, d\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d
1	b	c	b	c	b
a	b	c	b	c	b
b	1	a	b	c	d
c	1	a	b	c	1
d	b	c	b	c	b

$\circledast$	1	a	b	c	d
1	c	c	c	c	c
a	b	b	b	b	b
b	a	a	c	c	a
c	1	1	b	b	1
d	c	c	c	c	c

Then it is routine to verify that $(X, *, \circledast, E)$, where $E = \{b, c\}$, is a distributive generalized pseudo eBE-algebra.

It is clear that every distributive pseudo eBE-algebra is a distributive generalized pseudo eBE-algebra, but the converse may not be true as shown in the following example.

Example 4.14. We can observe that the distributive generalized pseudo eBE-algebra $(X, *, \circledast, E)$ with $E = \{b, c\}$ in Example 4.13 does not satisfies (peBE3). Thus it is not a distributive pseudo eBE-algebra.

In the following example, we know that if (X, E) is a generalized pseudo eBE-algebra, then E is not an eBE-filter of (X, E) in general.

Example 4.15. Let $X = \{1, a, b, c, d\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d
1	1	b	b	c	d
a	1	b	b	c	d
b	1	b	b	c	d
c	1	b	b	b	1
d	1	b	b	b	1

$\circledast$	1	a	b	c	d
1	b	b	b	c	c
a	1	1	1	d	d
b	1	1	1	d	d
c	1	1	1	1	1
d	b	b	b	b	b

Then it is routine to verify that $(X, *, \circledast, E)$, where $E = \{1, b\}$, is a (distributive) generalized pseudo eBE-algebra. But $E = \{1, b\}$ is not an eBE-filter of (X, E) since $b \circledast a = 1 \in E$ and $b \in E$ but $a \notin E$.

Theorem 4.12. *Let (X, E) be a distributive generalized pseudo eBE-algebra. If E is an eBE-filter of (X, E), then the right f-section of a fixed element a in (X, E) is an eBE-filter of (X, E).*

Proof. Assume that E is an eBE-filter of (X, E). It is clear that the right f-section is a superset of E. Let $x, y \in X$ be such that $x \in \overrightarrow{f}(a)$ and $x * y \in \overrightarrow{f}(a)$. Then $a \leq_e x$ and $a \leq_e x * y$, which imply that $a \circledast x \in E$ and $a \circledast (x * y) \in E$. Using the distributivity, we have

$$(a \circledast x) * (a \circledast y) = a \circledast (x * y) \in E.$$

Since E is an eBE-filter of (X, E), it follows from Theorem 4.2 that $a \circledast y \in E$, that is, $y \in \overrightarrow{f}(a)$. Therefore $\overrightarrow{f}(a)$ is an eBE-filter of (X, E). □

The following example shows that if (X, E) is not distributive in Theorem 4.12, then the right f-section of a fixed element a in (X, E) is not an eBE-filter of (X, E).

Example 4.16. Let $X = \{1, a, b, c, d, e\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e
1	e	d	e	d	d	e
a	e	e	d	d	d	e
b	1	d	e	c	d	e
c	e	e	d	d	d	e
d	1	a	b	c	e	e
e	1	b	a	c	d	e

$\circledast$	1	a	b	c	d	e
1	d	d	d	d	d	d
a	d	d	d	d	d	d
b	c	d	d	c	d	d
c	e	e	e	e	e	e
d	c	b	a	1	e	d
e	c	b	a	1	e	d

Let $E = \{d, e\}$. Then it is routine to verify that $(X, *, \circledast, E)$ is a generalized pseudo eBE-algebra. But it is not distributive since

$$a * (1 \circledast b) = a * d = d \neq e = e \circledast d = (a * 1) \circledast (a * b).$$

Given a fixed element $b \in X$, the right f-section of b is $\overrightarrow{f}(b) = \{a, b, d, e\}$, and it is not an eBE-filter of (X, E) since $a \circledast c = d \in \overrightarrow{f}(b)$ and $a \in \overrightarrow{f}(b)$ but $c \notin \overrightarrow{f}(b)$.

An example illustrating Theorem 4.12 is given below.

Example 4.17. Let $X = \{1, a, b, c, d, e\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e
1	d	c	b	c	d	e
a	d	c	b	c	d	e
b	1	a	c	c	d	d
c	1	a	b	c	d	e
d	1	a	b	c	d	e
e	1	a	c	c	d	d

$\circledast$	1	a	b	c	d	e
1	c	d	e	d	c	b
a	d	d	e	d	d	e
b	1	1	d	d	d	d
c	1	1	e	d	d	e
d	a	1	e	d	c	b
e	a	1	d	d	c	c

Then it is routine to verify that $(X, *, \circledast, E)$ is a distributive generalized pseudo eBE-algebra where $E = \{c, d, \}$. We can observe that E is an eGE-filter of (X, E), and the right f-section of a fixed element $b \in X$ is $\overrightarrow{f}(b) = \{b, c, d, e\}$ which is also an eBE-filter of (X, E).

Given two elements a and b in a generalized pseudo eBE-algebra (X, E), consider the following set:

$$P(a, b) := \{x \in X \mid a * (b \circledast x) \in E\}, \tag{4.13}$$

which is called the *eBE-upper set* of a and b.

It is clear that $a, b \in P(a, b)$ and $P(a, b)$ is a superset of E. Also, we have

$$\begin{aligned} P(a, b) &= \{x \in X \mid a * (b \circledast x) \in E\} \\ &= \{x \in X \mid b \circledast (a * x) \in E\} \\ &= \{x \in X \mid b \circledast (a \circledast x) \in E\} \\ &= \{x \in X \mid b * (a \circledast x) \in E\} \\ &= P(b, a). \end{aligned}$$

The following example is given to show that $P(a, b)$ is not an eBE-filter of a generalized pseudo eBE-algebra (X, E) for some $a, b \in X$.

Example 4.18. Let $X = \{1, a, b, c, d, e\}$ and define binary operations "$*$" and "$\circledast$" as follows:

$*$	1	a	b	c	d	e
1	d	d	d	c	d	d
a	d	d	d	c	d	d
b	d	d	d	c	d	d
c	a	a	a	e	e	e
d	a	a	a	e	e	e
e	d	d	d	d	d	d

$\circledast$	1	a	b	c	d	e
1	e	e	d	c	d	e
a	d	e	d	c	d	e
b	d	e	e	c	d	e
c	1	a	a	d	d	e
d	b	a	1	d	d	e
e	e	e	e	d	d	e

Then it is routine to verify that $(X, *, \circledast, E)$ is a generalized pseudo eBE-algebra where $E = \{d, e\}$. If we take $1, a \in X$, then $P(1, a) = \{1, a, b, d, e\}$ which is a superset of E. But it is not an eBE-filter of (X, E) since $d \circledast c = d \in P(1, a)$ and $d \in P(1, a)$ but $c \notin P(1, a)$ as $1 * (a \circledast c) = 1 * c = c \notin E$.

Proposition 4.13. *Let (X, E) be a generalized pseudo eBE-algebra and let a be a fixed element of X. Then*

$$(\forall b \in X)(\overrightarrow{f}(a) \subseteq P(a, b)). \tag{4.14}$$

$$(\forall b, x \in X)(a \leq_e b \Rightarrow b \in P(a, x)). \tag{4.15}$$

Proof. Let $b, x \in X$ be such that $x \in \overrightarrow{f}(a)$. Then $a \leq_e x$, and so

$$a * (b \circledast x) = b \circledast (a * x) \in b \circledast E \subseteq E$$

by (peBE2) and (peBE4). Hence $x \in P(a, b)$, and thus (4.14) is valid. Now, let $b, x \in X$ be such that $a \leq_e b$. Then $a \circledast b \in E$, which implies from (peBE2) that $x*(a \circledast b) \in x*E \subseteq E$. Hence $b \in P(x, a) = P(a, x)$. □

We find and present the conditions under which eBE-upper set becomes a eBE-filter.

Theorem 4.14. *Let (X, E) be a distributive generalized pseudo eBE-algebra. If E is an eBE-filter of (X, E), then the eBE-upper set of $a \in X$ and $b \in X$ is an eBE-filter of (X, E).*

Proof. Let $a, b \in X$. Recall that $P(a, b)$ is a superset of E. Let $x, y \in X$ be such that $x \in P(a, b)$ and $x * y \in P(a, b)$. Then $a * (b \circledast x) \in E$ and $a * (b \circledast (x * y)) \in E$. By the distributivity of (X, E), we get

$$(a * (b \circledast x)) * (a * (b \circledast y)) = a * ((b \circledast x) * (b \circledast y)) = a * (b \circledast (x * y)) \in E.$$

Since E is an eBE-filter of (X, E), it follows from Theorem 4.2 that $a * (b \circledast y) \in E$. Hence $y \in P(a, b)$, and therefore $P(a, b)$ is an eBE-filter of (X, E). □

We will use the eBE-upper set to characterize the eBE-filter.

Theorem 4.15. *Let F be a nonempty set in a generalized pseudo eBE-algebra (X, E). Then F is an eBE-filter of (X, E) if and only if F is a superset of the eBE-upper set $P(a, b)$ for every $a, b \in F$.*

Proof. Assume that F is an eBE-filter of (X, E). For every $a, b \in F$, let $x \in P(a, b)$. Then $a * (b \circledast x) \in E \subseteq F$. Since $a, b \in F$ and F is an eBE-filter of (X, E), it follows that $x \in F$. Hence $P(a, b) \subseteq F$.

Conversely, suppose that F is a superset of the eBE-upper set $P(a, b)$ for every $a, b \in F$. Note that $E \subseteq P(a, b) \subseteq F$. Let $x, y \in X$ be such that $x \in F$ and $x \circledast y \in F$. Since $(x \circledast y) * (x \circledast y) \in E$ by (peBE1), we have $y \in P(x \circledast y, x) \subseteq F$. Therefore F is an eBE-filter of (X, E). □

Theorem 4.16. *In a generalized pseudo eBE-algebra (X, E), any eBE-filter can be represented as the union of eBE-upper sets, that is, if F is an eBE-filter of (X, E), then $F = \bigcup_{a,b \in F} P(a, b)$.*

Proof. Let F be an eBE-filter of (X, E) and let $x \in F$. Then $x * (y \circledast x) = y \circledast (x * x) \in y \circledast E \subseteq E$ for all $y \in E$, and so $x \in P(x, y)$, which shows that $F \subseteq P(x, y)$. Since $E \subseteq F$, it follows that

$$F \subseteq \bigcup_{a \in F, b \in E} P(a, b) \subseteq \bigcup_{a,b \in F} P(a, b).$$

If $x \in \bigcup_{a,b \in F} P(a, b)$, then there exist $y, z \in F$ such that $x \in P(y, z) \subseteq F$ by Theorem 4.15. Hence $\bigcup_{a,b \in F} P(a, b) \subseteq F$. This completes the proof. □

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CONFLICT OF INTEREST

All authors declare no conflicts of interest in this paper.

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