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ORDERED NEARNESS SEMIGROUPS

MEHMET ALİ ÖZTÜRK*, G. MUHIUDDIN AND YOUNG BAE JUN

ABSTRACT. A semigroup is an algebraic structure that consists of a set and a binary operation that is associative, meaning that the order in which the operations are performed does not affect the outcome. For example, addition and multiplication are associative operations. The term "semigroup" was first used in its modern sense by Harold Hilton in his book on finite groups in 1908 [6]. Semigroups have since then been studied extensively in mathematics, and they have numerous applications in different fields, such as computer science, physics, and economics. Nearenness semigroup is a generalization of semigroups. Near set theory, which is a generalization of rough set theory, is based on the determination of universal sets according to the available information about the objects. Nearenness semigroups extend the concept of nearness from set theory to semigroups. İnan and Öztürk applied the notion of near sets defined by J. F. Peters to the semigroups [10].

Our objective in this paper is to establish the definition of ordered semigroups on weak near approximation spaces. In addition, we investigated certain characteristics of these ordered nearness semigroups.

1. INTRODUCTION

Set theory is a crucial tool for mathematicians and engineers as it serves as the foundation for their studies. However, when traditional set theory falls short, researchers develop new techniques to address the uncertainties, imprecision, and absoluteness found in the real world. One such approach is rough set theory, which focuses on managing indistinguishable elements with different values in decision properties [21]. In recent years, many researchers have explored the applications of rough set theory.

J.F. Peters introduced Near set theory in 2002 as a generalization of rough set theory. This theory determines universal sets based on the available information about objects and helps identify and differentiate similarities in the perceived properties of different objects. To represent the properties of physical objects, a real-valued search function is used. Peters proposed an indistinguishability relation in [24] based on object properties, which describe object proximity. Near set theory has more recently been adopted as a generalized approximation theory for studying the proximity of similar non-empty sets, as explored in [22, 23, 25].

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*Corresponding author.

Near set theory can be employed to represent elements in algebraic structures as tangible objects. Typically, algebraic structures comprise non-empty abstract points, which are not always applicable to real-world problems. Even though researchers analyzing algebraic structures typically use abstract elements, these elements may prove inadequate for some studies. Instead, near set theory incorporates perceptual objects, which are non-abstract points with distinct features (e.g. color or stage of maturation in an apple). In algebraic structures based on weak nearness approximation spaces, upper approximations of subsets of perceptual objects serve as the primary tool. The approach to nearness in these algebraic structures focuses on non-abstract points, with upper approximations of perceptual objects used to discern nearness in binary operations. This highlights a key difference between classical algebraic structures and nearness algebraic structures.

Let G be a non-empty set. If elements of G have one and only one property, then upper and lower approximation's and itself of this set are equal to each other for $r = n (n \in \mathbb{Z}^+)$. That is, $N_r(B)_* G = G = N_r(B)^* G$. This is the basic property of classical algebraic structures. Therefore, we think that the nearness algebraic structure must be studied in which has property of $G \subsetneq N_r(B)^* G$ since many perceptual objects have more than one property in real life. In 2012, İnan and Öztürk analyzed the concept of nearness groups and investigated their basic properties $([8])$. After, in [9] and [19] the nearness semigroups and nearness rings were established and their basic properties were investigated respectively $($and other algebraic approaches of near sets in see [16], [18], [20], and [26]$)$.

The study of ordered semigroups dates back to the 1950s when it was initiated by Alimov [1], Chehata [3], and Vinogradov [27]. Over time, various researchers have explored different types of ideals in ordered semigroups, as illustrated in publications such as [2], [4], [7], [11], [12], [13], [14], and [15].

The primary focus of this paper is to present the definition of ordered semigroups on weak nearness approximation spaces. Furthermore, the paper examines certain properties of these ordered nearness semigroups.

2. Preliminaries

A partially ordered semigroup (or in short an ordered semigroup) is a semigroup $(S, \cdot)$ with a partial ordered relation " $\leq$ " such that for any $x, y, z \in S$ $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ ([1], [3], [27]). Let $(S, \cdot, \leq)$ be an ordered semigroup and A, B be non-empty subsets of S. We denote

$$AB = \{ab \mid a \in B, b \in B\}$$

and

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Lemma 2.1. [12] *Let S be a po-semigroup. Then, we have*

1) $A \subseteq (A]$ $\forall A \subseteq S$.
2) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$.
3) $(A](B] \subseteq (AB]$ $\forall A, B \subseteq S$.
4) $((A]] = (A]$ $\forall A \subseteq S$.

A non-empty subset A of S is called a left (resp. right) ideal of S if

1) $SA \subseteq A$ (resp. $AS \subseteq A$),
2) $a \in A$, $x \leq a$ for $x \in S$ implies $x \in A$.

If A is both a left ideal and a right ideal of S, then A is called an ideal of S [11].

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ where the nearness approximation space is defined with a set of perceived objects $\mathcal{O}$, set of probe functions $\mathcal{F}$ representing object features, indiscernibility relation $\sim_{B_r}$ defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighbour-hoods) $N_r(B)$, and overlap function ν_{N_r}.

Definition 2.1. ([10]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and let "$\cdot$" be a binary operation defined on $\mathcal{O}$.

A subset S of the set of perceptual objects $\mathcal{O}$ is called a near semigroup on nearness approximation space or shortly nearness semigroup, provided the following properties are satisfied:

1) For all $x, y \in S$, $x \cdot y \in N_r(B)^*(S)$;
2) For all $x, y, z \in S$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^*(S)$.

Definition 2.2. ([10]) Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ be a nearness approximation space and let "$\cdot$" be a binary operation defined on $\mathcal{O}$. Let $X \subseteq \mathcal{O}$ and $B_r \subseteq \mathcal{F}$, $r \leq | B |$. A indiscernibility relation $\sim_{B_r}$on $\mathcal{O}$ is called a complete indiscernibility relation $\sim_{B_r}$ on perceptual objects $\mathcal{O}$ if $[x]_{B_r}[y]_{B_r} = [xy]_{B_r}$ for all $x, y \in X$.

Definition 2.3. ([18]) Let $\mathcal{O}$ be a set of sample objects, $\mathcal{F}$ a set of the probe functions, $\sim_{B_r}$ an indiscernibility relation, and $N_r(B)$ a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is called a weak nearness approximation space.

Theorem 2.2. ([18]) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space and $A, B \subset \mathcal{O}$, then the following statements hold;*

1) $N_r(B)_* A \subseteq A \subseteq N_r(B)^* A$,
2) $N_r(B)^*(A \cup B) = N_r(B)^* A \cup N_r(B)^* B$,
3) $N_r(B)^*(A \cap B) \subseteq N_r(B)^* A \cap N_r(B)^* B$,
4) $A \subseteq B$ *implies* $N_r(B)^* A \subseteq N_r(B)^* B$.

For detailed information about weak nearness approximation space and ordered semigroup, you can refer to [22], [18], and [5].

3. Ordered nearness semigroups

When Definition 2.3 is considered, Definition 2.1, and Definition 2.2 can be restated as follow for weak nearness approximation space.

Definition 3.1. Let $S \subseteq \mathcal{O}$, where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ is weak nearness approximation spaces. If the following properties are satisfied, then S is called a semigroup on weak nearness approximation spaces $\mathcal{O}$, or, in short, a nearness semigroup.

$i)$ $xy \in N_r(B)^* S$ for all $x, y \in S$,
$ii)$ $(xy)z = x(yz)$ property holds in $N_r(B)^* S$ for all $x, y \in S$.

Definition 3.2. Let $(S, \cdot)$ be a semigroup on $\mathcal{O}$, $B_r \subseteq \mathcal{F}$where $r \leq | B |$and $B \subseteq \mathcal{F}$, $\sim_{B_r}$be a indiscernibility relation on $\mathcal{O}$. Then, $\sim_{B_r}$ is called a congruence indiscernibility relation on nearness semigroup S, if $x \sim_{B_r} y$ where $x, y \in S$ implies $xa \sim_{B_r} ya$, and $ax \sim_{B_r} ay$, for all $a \in S$.

Lemma 3.1. *Let S be a nearness semigroup. If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then $[x]_{B_r}[y]_{B_r} \subseteq [xy]_{B_r}$, for all $x, y \in S$.*

Proof. Let $z \in [x]_{B_r}[y]_{B_r}$. In this case, $z = ab; a \in [x]_{B_r}, b \in [y]_{B_r}$. From here $x \sim_{B_r} a$,and $y \sim_{B_r} b$, and so, we have $xy \sim_{B_r} ay$, and $ay \sim_{B_r} ab$ by hypothesis. Thus, we get $xy \sim_{B_r} ab \Rightarrow z = ab \in [xy]_{B_r}$. □

Definition 3.3. Let S be a nearness semigroup. Then, $\sim_{B_r}$ is called a complete congruence indiscernibility relation on S if $[x]_{B_r}[y]_{B_r} = [xy]_{B_r}$ for all $x, y \in S$.

Definition 3.4. Let $(S, \cdot)$ be a nearness semigroup. $(S, \cdot, \leq)$ is called an ordered nearness semigroup if $(S, \leq)$ is an ordered set and it satisfies:

$$x \leq y \implies z \cdot x \leq z \cdot y \text{ and } x \cdot z \leq y \cdot z \text{ for all } x, y, z \in S.$$

Example 3.5. Let $\mathcal{O} = \{o, x, y, z, r, s, t, u, v, w\}$ be a set of perceptual objects, $B = \{\psi_1, \psi_2, \psi_3\} \subseteq \mathcal{F}$ be a set of probe functions, and $S = \{y, z, r, s\}$ be a subset of $\mathcal{O}$. Values of the probe functions

$$\begin{aligned} &\psi_1 : \mathcal{O} \longrightarrow D_1 = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}, \\ &\psi_2 : \mathcal{O} \longrightarrow D_2 = \{\delta_1, \delta_2, \delta_3, \delta_5\}, \\ &\psi_3 : \mathcal{O} \longrightarrow D_3 = \{\delta_1, \delta_2, \delta_3, \delta_5\} \end{aligned}$$

are given in Table 1.

	o	x	y	z	r	s	t	u	v	w
ψ_1	δ_1	δ_2	δ_3	δ_1	δ_4	δ_2	δ_4	δ_5	δ_4	δ_5
ψ_2	δ_5	δ_2	δ_2	δ_3	δ_1	δ_3	δ_2	δ_5	δ_1	δ_5
ψ_3	δ_2	δ_3	δ_1	δ_1	δ_5	δ_3	δ_5	δ_2	δ_5	δ_2

$Table\ 1$

Now, let's find the nearness equivalence classes of $\mathcal{O}$ for $r = 1$ according to the relation $\sim_{B_r}$.

$$\begin{aligned} [o]_{\psi_1} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(o) = \delta_1\} \\ &= \{o, z\} = [z]_{\psi_1}, \\ [x]_{\psi_1} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(x) = \delta_2\} \\ &= \{x, s\} = [s]_{\psi_1}, \\ [y]_{\psi_1} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(y) = \delta_3\} \\ &= \{y\}, \\ [r]_{\psi_1} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(r) = \delta_4\} \\ &= \{r, t, v\} = [t]_{\psi_1} = [v]_{\psi_1}, \\ [u]_{\psi_1} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(w) = \delta_5\} \\ &= \{u, w\} = [w]_{\psi_1}. \end{aligned}$$

Hence, we get that $\xi_{\psi_1} = \left\{[o]_{\psi_1}, [x]_{\psi_1}, [y]_{\psi_1}, [r]_{\psi_1}, [u]_{\psi_1}\right\}$.

$$\begin{aligned} [o]_{\psi_2} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(o) = \delta_5\} \\ &= \{o, u, w\} = [u]_{\psi_2} = [w]_{\psi_2}, \\ [x]_{\psi_2} &= \{x' \in \mathcal{O} \mid \varphi_1(x') = \varphi_1(x) = \delta_2\} \\ &= \{x, y, t\} = [y]_{\psi_2} = [t]_{\psi_2}, \\ [z]_{\psi_2} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(z) = \delta_3\} \\ &= \{z, s\} = [s]_{\psi_2}, \\ [r]_{\psi_2} &= \{x' \in \mathcal{O} \mid \psi_1(x') = \psi_1(r) = \delta_1\} \\ &= \{r, v\} = [v]_{\psi_2}. \end{aligned}$$

Thus, we have that $\xi_{\psi_2} = \left\{[o]_{\psi_2}, [x]_{\psi_2}, [z]_{\psi_2}, [r]_{\psi_2}\right\}$.

$$\begin{aligned}
[o]_{\psi_3} &= \{x' \in \mathcal{O} \mid \psi_3(x') = \psi_3(o) = \delta_2\} \\
&= \{o, u, w\} = [u]_{\psi_3} = [w]_{\psi_3}, \\
[x]_{\psi_3} &= \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_3(x) = \delta_3\} \\
&= \{x, s, t\} = [s]_{\psi_3} = [t]_{\psi_3}, \\
[y]_{\psi_3} & \{x' \in \mathcal{O} \mid \varphi_3(x') = \varphi_3(y) = \delta_1\} \\
&= \{y, z\} = [z]_{\psi_3}, \\
[r]_{\psi_3} &= \{x' \in \mathcal{O} \mid \psi_3(x') = \psi_3(r) = \delta_5\} \\
&= \{r, t, v\} = [t]_{\psi_3} = [v]_{\psi_3}.
\end{aligned}$$

So, we obtain that $\xi_{\psi_3} = \left\{[o]_{\psi_3}, [x]_{\psi_3}, [y]_{\psi_3}, [u]_{\psi_3}\right\}$.

Therefore, for $r = 1$, a set of partitions of $\mathcal{O}$ is $N_1(B) = \{\xi_{\psi_1}, \xi_{\psi_2}, \xi_{\psi_3}\}$. Then, we can write

$$\begin{aligned}
N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \varnothing} [x]_{\varphi_i} \\
&= \{o, z\} \cup \{x, s\} \cup \{y\} \cup \{r, t, v\} \cup \{x, y, t\} \cup \{z, s\} \\
&\cup \{r, v\} \cup \{x, s, t\} \cup \{y, z\} \cup \{r, t, v\} \\
&= \{o, x, y, z, r, s, t, v\}.
\end{aligned}$$

Let "$\cdot$" be a binary operation of perceptual objects on $\mathcal{O}$ as given in Table 2.

$\cdot$	o	x	y	z	r	s	t	u	v	w
o	o	x	o	o	o	o	o	u	o	o
x	x	x	y	z	z	z	t	u	v	w
y	o	y	y	o	o	o	t	o	o	o
z	o	y	z	o	o	o	o	u	v	o
r	o	o	o	o	r	s	o	o	v	w
s	o	s	o	o	s	s	o	u	v	o
t	o	x	o	z	t	t	o	t	y	w
u	u	x	y	z	r	r	s	u	v	o
v	o	x	v	v	o	o	s	v	o	o
w	w	w	o	o	r	o	o	u	o	w

$Table$ 2

Then, "$\cdot$" be an operation of perceptual objects on $S \subseteq \mathcal{O}$ as in Table 3, and the order relation "$\leq$" as below.

$\cdot$	y	z	r	s
y	y	o	o	o
z	z	o	o	o
r	o	o	r	s
s	o	o	s	s

$Table$ 3

$\leq := \{(o,o), (o,r), (o,s), (y,y), (y,z), (y,r), (y,s), (y,o), (z,z), (z,s), (z,o), (r,r), (s,s)\}$.

The covering relation $<$ and Hasse diagram of $N_1(B)^* S$ are given by:

$$< := \{(o,r), (o,s), (y,z), (y,r), (y,s), (y,o), (z,s), (z,o)\}$$

and

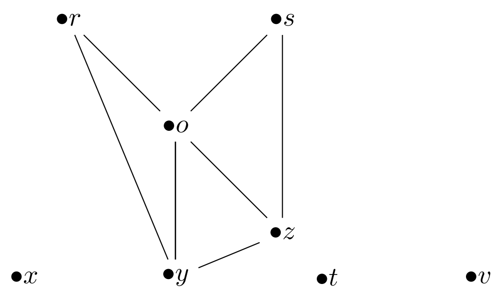

FIGURE 1. Hasse diagram of $N_1(B)^* S$

In this case, S is an ordered nearness semigroup.

Theorem 3.2. *Let $(S, \cdot, \leq)$ be an ordered nearness semigroup and A, B be non-empty subsets of S. In this case, the following statements hold.*

1) $N_r(B)^* A \subseteq N_r(B)^* (A]$.
2) $N_r(B)^* (N_r(B)^* (A]] = N_r(B)^* (A]$.
3) $N_r(B)^* (A \cap B] \subseteq (N_r(B)^* (A]) \cap (N_r(B)^* (B])$.
4) $N_r(B)^* (A \cup B] = (N_r(B)^* (A]) \cup (N_r(B)^* (B])$.
5) *If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then*

$$(N_r(B)^* (A])(N_r(B)^* (B]) \subseteq N_r(B)^* (AB].$$

6) *If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then*

$$N_r(B)^* ((N_r(B)^* (A])(N_r(B)^* (B])] \subseteq N_r(B)^* (AB].$$

Proof. 1) Let $x \in N_r(B)^* A$. Then $[x]_{B_r} \cap A \neq \varnothing$. There is at least one $a \in S$ such that $a \in [x]_{B_r} \cap A$. Hence, $a \in [x]_{B_r}$ and $a \in A \Longrightarrow a \in [x]_{B_r}$ and $a \leq a \Longrightarrow a \in [x]_{B_r}$ and $a \in (A] \Longrightarrow a \in [x]_{B_r} \cap (A] \Longrightarrow [x]_{B_r} \cap A \neq \varnothing \Longrightarrow x \in N_r(B)^* (A]$.

2) Let $x \in N_r(B)^* (N_r(B)^* (A]]$. Thus, we have $[x]_{B_r} \cap N_r(B)^* (A] \neq \varnothing$. Therefore, there exist $y \in S$ such that $y \in [x]_{B_r} \cap N_r(B)^* (A]$. Thus, $y \in [x]_{B_r}$ and $y \in N_r(B)^* (A] \Longrightarrow y \in [x]_{B_r}$ and $[y]_{B_r} \cap N_r(B)^* (A] \neq \varnothing$, and so we get $y \in [x]_{B_r}$, $z \in [y]_{B_r}$, and $z \in (A]$ for $y, z \in S$. In this case, since $y \in [x]_{B_r}$and $z \in [y]_{B_r}$, we have $x \sim_{B_r} y$ and $y \sim_{B_r} z \Longrightarrow x \sim_{B_r} z$. Hence, we obtain $z \in [x]_{B_r} \cap (A] \Longrightarrow [x]_{B_r} \cap A \neq \varnothing$ $\Longrightarrow x \in N_r(B)^* (A]$.

3) Let $x \in N_r(B)^* (A \cap B]$. We have $[x]_{B_r} \cap (A \cap B] \neq \varnothing$. Thus, $y \in [x]_{B_r} \cap (A \cap B], y \in S \Longrightarrow y \in [x]_{B_r}$ and $y \in (A \cap B] \Longrightarrow y \in [x]_{B_r}$ and $s \leq y,\ y \in A \cap B$ for $s \in S$. We get that $y \in [x]_{B_r}$, $s \leq y,\ y \in A$ for $s \in S$, and $y \in [x]_{B_r}$, $s \leq y,\ y \in B$ for $s \in S$. Therefore, we obtain $y \in [x]_{B_r}$, $y \in (A]$ and $y \in [x]_{B_r}$, $y \in (B]$. Thus, $y \in [x]_{B_r} \cap (A]$ and $y \in [x]_{B_r} \cap (B] \Longrightarrow [x]_{B_r} \cap (A] \neq \varnothing$ and $[x]_{B_r} \cap (B] \neq \varnothing \Longrightarrow x \in N_r(B)^* (A]$ and $x \in N_r(B)^* (B] \Longrightarrow x \in (N_r(B)^* (A]) \cap (N_r(B)^* (B])$.

4) The proof can be made same as (3).

5)Let $x \in (N_r(B)^* (A])(N_r(B)^* (B])$. Then, we have $x = yz$; $y \in N_r(B)^* (A], z \in N_r(B)^* (B]$. Thus, $y \in N_r(B)^* (A] \Rightarrow [y]_{B_r} \cap (A] \neq \varnothing \Rightarrow \exists a \in [y]_{B_r} \cap (A] \Rightarrow a \in [y]_{B_r}$ and $a \in (A]$. Likewise, $z \in N_r(B)^* (B] \Rightarrow [z]_{B_r} \cap (B] \neq \varnothing \Rightarrow \exists b \in [z]_{B_r} \cap (B] \Rightarrow b \in [z]_{B_r}$ and $b \in (B]$. Since $ab \in [y]_{B_r}[z]_{B_r} \subseteq [yz]_{B_r}$ by Lemma 3.1, we get $ab \in [yz]_{B_r}$. Moreover, since $ab \in (A](A] \subseteq (AB]$ by Lemma 2.1.(3), we get $ab \in [yz]_{B_r}$ and $ab \in (AB]$. Thus, $ab \in [yz]_{B_r} \cap (AB] \Rightarrow [yz]_{B_r} \cap (AB] \neq \varnothing$, and so $yz = x \in N_r(B)^* (AB]$.

6) We have $N_r(B)^*((N_r(B)^*(A])(N_r(B)^*(B])] \subseteq N_r(B)^*(N_r(B)^*(AB]])$ by (5), and so from (2), we get $N_r(B)^*((N_r(B)^*(A])(N_r(B)^*(B])] \subseteq N_r(B)^*(AB]$. □

Corollary 3.3. *Let $(S, \cdot, \leq)$ be an ordered nearness semigroup. If A, B be non-empty subsets of S, then*

$$(N_r(B)^* A)(N_r(B)^* B) \subseteq N_r(B)^*(AB).$$

Remark. *Since the definition of sub-nearness semigroups (upper-near sub-nearness semigroup) in ordered nearness semigroups is the same as in nearness semigroups, there is no need to give it here.*

Definition 3.6. Let $(S, \cdot, \leq)$ be an ordered nearness semigroup and I be a nonempty subset of S. I is called a left (resp. right) nearness ideal of S if

1) $SI \subseteq N_r(B)^* I$ (resp. $IS \subseteq N_r(B)^* I$),

2) $a \in I$, $x \leq a$ for $x \in S$ implies $x \in I$.

If I is both a left nearness ideal and a right nearness ideal of S, then I is called a nearness ideal of S.

Example 3.7. Let's take ordered nearness semigroup S in Example 3.5. Let $I = \{y, z, s\}$ be a subset of S. Then, we can write

$$\begin{aligned} N_1(B)^* I &= \bigcup_{[x]_{\varphi_i} \cap I \neq \varnothing} [x]_{\varphi_i} \\ &= \{o, z\} \cup \{x, s\} \cup \{y\} \cup \{x, y, t\} \cup \{z, s\} \cup \{x, s, t\} \cup \{y, z\} \\ &= \{o, x, y, z, s, t, v\}. \end{aligned}$$

Then, "$\cdot$" be an operation of perceptual objects on $I \subseteq S$ as in Table 4, and the order relation "$\leq$" as below.

$\cdot$	y	z	s
y	y	o	o
z	z	o	o
s	o	o	s

Table 4

$$\leq := \{(o, o), (y, y), (y, z), (y, r), (y, o), (y, s), (z, z), (z, s), (z, o), (s, s)\}.$$

The covering relation $<$ and Hasse diagram of $N_1(B)^* I$ are given by:

$$< := \{(y, o), (y, z), (y, s), (o, s), (z, o), (z, s)\}$$

and

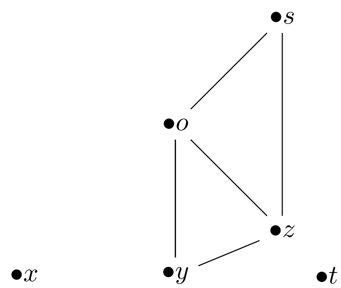

FIGURE 2. Hasse diagram of $N_1(B)^* I$

Then, I is a nearness ideal of S.

Theorem 3.4. *Let $(S, \cdot, \leq)$ be an ordered nearness semigroup. In this case, the following statements hold.*

1) *If I is a right (resp. left) nearness ideal of S, then $N_r(B)^*(I] = N_r(B)^* I$.*

2) *If I is a right (resp. left) nearness ideal of S, then $(I]$ is a right (resp. left) nearness ideal of S.*

3) *If I, J are right (resp. left) nearness ideals of S and $(N_r(B)^* I) \cap (N_r(B)^* J) \subseteq N_r(B)^*(I \cap J)$, then $I \cap J = \varnothing$ or $I \cap J$ is a right (resp. left) nearness ideal of S.*

4) *If I, J are right (resp. left) nearness ideals of S, then $I \cup J$ is a right (resp. left) nearness ideal of S.*

Proof. 1) Let $x \in N_r(B)^*(I]$. We have $[x]_{B_r} \cap (I] \neq \varnothing$. Thus, $s \in [x]_{B_r} \cap (I], s \in S \Longrightarrow s \in [x]_{B_r}$ and $s \in (I] \Longrightarrow s \in [x]_{B_r}$ and $s \leq a$, $a \in I$. Since I is a right nearness ideal of S, we get that $s \in [x]_{B_r}$, $s \in I$. Thus, $s \in [x]_{B_r} \cap I \Longrightarrow [x]_{B_r} \cap I \neq \varnothing \Longrightarrow x \in N_1(B)^* I$. Therefore, we have $N_r(B)^*(I] = N_r(B)^* I$ by Theorem 3.2.(1).

2) We have

$$\begin{aligned}(I]S &\subseteq (N_r(B)^*(I])(N_r(B)^* S) \text{ (by Theorem 2.2.(1))}\\ &= (N_r(B)^* I)(N_r(B)^* S) \text{ (by (1))}\\ &\subseteq N_r(B)^*(IS) \text{ (by Corollary 3.3)}\\ &\subseteq N_r(B)^*(I) \text{ (} I \text{ is a right nearness ideal of } S \text{)}\\ &= N_r(B)^*(I] \text{ (by (1)).}\end{aligned}$$

Therefore, we get $(I]S \subseteq N_r(B)^*(I]$. Let $x \in (I]$, $s \leq x$ for $s \in S$. Then, $x \in (I] \Longrightarrow x \leq a, a \in I$. Thus, since $s \leq x$ and $s \leq x$, we we get $s \leq a, a \in I$, and so $s \in (I]$. Hence $(I]$ is a right nearness ideal of S.

3) Suppose it is $\varnothing \neq I \cap J \subseteq S$. Let $x \in (I \cap J)$ and $s \in S$. Hence, we have $x \in I$ and $x \in J \Longrightarrow xs \in N_r(B)^* I$ and $xs \in N_r(B)^* J$ Therefore, we obtain $xs \in N_r(B)^*(I \cap J)$ by hypothesis, e. i. , $(I \cap J)S \subseteq N_r(B)^*(I \cap J)$.

Now, let $x \in I \cap J$, $s \leq x$ for $s \in S$. From here, $x \in I$ and $x \in J$; $s \leq x$ for $s \in S$. Since I is a right nearness ideal of S, we get that $s \in I \cap J$, and so $I \cap J$ is a right nearness ideal of S.

4) The proof can be made same as (3). □

We can prove that the following Theorem from Theorem 3.2 and 3.4.

Theorem 3.5. *Let S be an ordered nearness semigroup. If I, J are right (resp. left) nearness ideals of S, then $IJ, (IJ]$ are right (resp. left) nearness ideals of S.*

Definition 3.8. Let $(S, \cdot, \leq)$ be an ordered nearness semigroup and I be a nonempty subset of S. I is called an upper-near right (resp. left) nearness ideal of S if

1) $(N_r(B)^* I)S \subseteq N_r(B)^* I$ (resp. $S(N_r(B)^* I) \subseteq N_r(B)^* I$),
2) $a \in N_r(B)^* I$, $x \leq a$ for $x \in S$ implies $x \in N_r(B)^* I$.

Theorem 3.6. *Let S be an ordered nearness semigroup. If I is a right (resp. left) nearness ideal of S and $N_r(B)^* (N_r(B)^* I) = N_r(B)^* I$, then I is a upper-near right (resp. left) nearness ideal of S.*

Proof. Since I is a right nearness ideal of S, we have $\varnothing \neq I \subseteq S$, and $IS \subseteq N_r(B)^* I$. From Theorem 2.2.(1) and Corollary 3.3, we get

$$(N_r(B)^* I)S \subseteq (N_r(B)^* I)(N_r(B)^* S) \subseteq N_r(B)^* (IS).$$

On the other hand, from Theorem 2.2.(4), we have that

$$N_r(B)^* (IS) \subseteq N_r(B)^* (N_r(B)^* I) = N_r(B)^* I$$

by hypothesis. In this case,

$$(N_r(B)^* I)S \subseteq N_r(B)^* I$$

is obtained. Since I is a right nearness ideal of S, $a \in I$, $x \leq a$ for $x \in S$ implies $x \in I$. Thus, we get $a \in N_r(B)^* I$, $x \leq a$ for $x \in S$ implies $x \in N_r(B)^* I$ by Theorem 2.2.(1). Hence, I is an upper-near right nearness ideal of S. □

4. Conclusion

The objective of this paper is to investigate the definition of ordered nearness semigroups. It revolves around the problem of categorizing semigroups based on nearness relations. Furthermore, the paper discusses the conditions for an ordered semigroup to be regarded as a nearness semigroup, and it explains the properties of nearness ideals of ordered nearness semigroups. The study also highlights the possibility of exploring other types of nearness semigroups, such as ordered gamma nearness semigroups, regular (or intra-regular) ordered nearness semigroups, and more.

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MEHMET ALİ ÖZTÜRK
DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, ADIYAMAN UNIVERSITY, 02040 ADIYAMAN, TÜRKİYE.
Email address: maozturk@adiyaman.edu.tr

G. MUHIUDDIN
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF TABUK, P.O. BOX 741, TABUK 71491, SAUDI ARABIA
Email address: chishtygm@gmail.com

YOUNG BAE JUN
DEPARTMENT OF MATHEMATICS EDUCATION, GYEONGSANG NATIONAL UNIVERSITY, JINJU 52828, REPUBLIC OF KOREA
Email address: skywine@gmail.com