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MULTIVARIATE APPROXIMATION BY PARAMETRIZED LOGISTIC ACTIVATED MULTIDIMENSIONAL CONVOLUTION TYPE OPERATORS

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ABSTRACT. In this work we introduce for the first time the multivariate parametrized logistic activated convolution type operators in three kinds. We present their approximation properties, that is the quantitative convergence to the unit operator via the multivariate modulus of continuity. We continue with the multivariate global smoothness preservation of these operators.

We present extensively the related multivariate iterated approximation, as well as, the multivariate simultaneous approximation and their combinations.

Using differentiability into our research, we are producing higher speeds of approximation, multivariate simultaneous global smoothness preservation is also studied.

1. ABOUT CONTENTS

In Section 2 we give the Preliminaries of our theory. In Section 3 come the Basics, the introduction of our multivariate activated convolution type operators with properties. In Section 4 come the main multivariate approximation results. We also include there the multivariate global smoothness preservation by our operators. We further study the differentiation of these operators, as well as, we introduce their iterates and give their basic properties. Next, we present the convergence of our operators under differentiability achieving higher rates of approximation. It follows the multivariate simultaneous differential approximation and in detail the multivariate simultaneous global smoothness preservation, as well as the multivariate iterated approximation. We finish with the combination of multivariate simultaneous and iterated approximations.

We are motivated and ispired by [1], [2].

2. PRELIMINARIES

The following come from [1], see Chapters 1,2.

A Richards's curve is

$$\varphi(x) = \frac{1}{1+e^{-\mu x}}; \quad x \in \mathbb{R}, \text{ with parameter } \mu > 0, \tag{1}$$

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with great interest when $0 < \mu < 1$.

The function $\varphi(x)$ is strictly increasing on $\mathbb{R}$, and it is a sigmoid function, in particular this is a generalized logistic function [5], [6].

See that

$$\lim_{x \to +\infty} \varphi(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} \varphi(x) = 0. \tag{2}$$

We consider the following activation function

$$G(x) = \frac{1}{2} (\varphi(x+1) - \varphi(x-1)), \ x \in \mathbb{R}, \tag{3}$$

with $G(x) > 0$, all $x \in \mathbb{R}$.

We have that

$$\varphi(0) = \frac{1}{2} \quad \text{and} \quad \varphi(x) = 1 - \varphi(-x), \tag{4}$$

and

$$G(-x) = G(x), \ \forall x \in \mathbb{R}, \tag{5}$$

so that G is an even function.

It is

$$G(0) = \frac{e^{\mu} - 1}{2(e^{\mu} + 1)}. \tag{6}$$

Let $x \geq 1$, then $G'(x) < 0$ and $G(x)$ is strictly decreasing.

When $0 < x < 1$, again $G'(x) < 0$, and $G(x)$ is strictly decreasing.

Thus $G(x)$ is strictly decreasing on $(0, +\infty)$.

Clearly then $G(x)$ is strictly increasing on $(-\infty, 0)$, and $G'(0) = 0$.

We observe that

$$\begin{aligned} &\lim_{x \to +\infty} G(x) = \tfrac{1}{2} (\varphi(+\infty) - \varphi(+\infty)) = 0, \\ &\text{and} \\ &\lim_{x \to -\infty} G(x) = \tfrac{1}{2} (\varphi(-\infty) - \varphi(-\infty)) = 0. \end{aligned} \tag{7}$$

That is, the x-axis is the horizontal asymptote for G.

Conclusion, G is a bell shape symmetric function with maximum

$$G(0) = \frac{e^{\mu} - 1}{2(e^{\mu} + 1)}. \tag{8}$$

We need.

Theorem 2.1. *It holds*

$$\sum_{i=-\infty}^{\infty} G(x - i) = 1, \ \forall x \in \mathbb{R}. \tag{9}$$

Theorem 2.2. *We have*

$$\int_{-\infty}^{\infty} G(x) \, dx = 1. \tag{10}$$

So $G(x)$ is a density function.

It is clear that

$$\int_{-\infty}^{\infty} G(nx - u) \, du = 1, \ \forall n \in \mathbb{N}, x \in \mathbb{R}. \tag{11}$$

Theorem 2.3. *([3]) Let $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$. Then*

$$\int_{\{u \in \mathbb{R} : |nx - u| \geq n^{1-\alpha}\}} G(nx - u) \, du < \frac{2}{e^{\mu(n^{1-\alpha} - 1)}}, \ \mu > 0. \tag{12}$$

It is $\|x\|_\infty := \max\{|x_1|,...,|x_N|\},\ x \in \mathbb{R}^N$.

Denote by

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in\mathbb{R}^N,\\ \|x-y\|_\infty \le \delta}} |f(x) - f(y)|, \quad \delta > 0, \tag{13}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ bounded and continuous, denoted by $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$. Similarly ω_1 is defined for $f \in C_U(\mathbb{R}^N)$ (uniformly continuous functions).

We have that $f \in C_U(\mathbb{R}^N)$, iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$.

We need the following:

Proposition 2.4. *([3]) It holds ($k \in \mathbb{N}$)*

$$\int_{-\infty}^{\infty} |z|^k G(z)\, dz \le \left(\frac{e^\mu - 1}{(e^\mu + 1)(k+1)}\right) + \frac{2e^\mu k!}{\mu^k} < \infty. \tag{14}$$

3. Basics

We make

Remark. *We introduce*

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^N G(x_i), \ \ x = (x_1,...,x_N) \in \mathbb{R}^N, N \in \mathbb{N}. \tag{15}$$

It has the properties

(i) $Z(x) > 0$, $\forall\, x \in \mathbb{R}^N$; $Z(-x) = Z(x)$,

(ii)

$$\int_{\mathbb{R}^N} Z(x-u)\, du = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} Z(x_1 - u_1, ..., x_N - u_N)\, du_1 ... du_N =$$
$$\prod_{i=1}^N \int_{-\infty}^{\infty} G(x_i - u_i)\, du_i \overset{(11)}{=} 1, \ \ \forall x \in \mathbb{R}^N, \tag{16}$$

hence

(iii)

$$\int_{\mathbb{R}^N} Z(nx - u)\, du = 1, \ \ \forall x \in \mathbb{R}^N, \ n \in \mathbb{N}, \text{ and} \tag{17}$$

(iv) by (10),

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{18}$$

that is Z is a multivariate density function.

(v) Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then, by Theorem 2.3 and (11) we derive that

$$\int_{\left\{u\in\mathbb{R}^N : \|nx-u\|_\infty \ge n^{1-\beta}\right\}} Z(nx - u)\, du < \frac{2}{e^{\mu(n^{1-\beta}-1)}}, \ \ \mu > 0. \tag{19}$$

The last is true, because the condition any $u \in \mathbb{R}^N : \|nx - u\|_\infty \ge n^{1-\beta}$, implies that there exists at least one $u_r : |nx_r - u_r| \ge n^{1-\beta}$, where $r \in \{1,...,N\}$.

Indeed it is

$$\left\{u \in \mathbb{R}^N : \|nx - u\|_\infty \ge n^{1-\beta}\right\} \subset \tag{20}$$
$$\cup_{r=1}^N \left\{\mathbb{R}^{N-1} \cup \left\{u_r \in \mathbb{R} : |nx_r - u_r| \ge n^{1-\beta}\right\}\right\} \subset$$
$$\mathbb{R}^{N-1} \cup \left\{u_{r^*} \in \mathbb{R} : |nx_{r^*} - u_{r^*}| \ge n^{1-\beta}\right\},$$

for some $r^* \in \{1, ..., N\}$.

We also mention a useful related result.

Theorem 3.1. *It holds (*$k \in \mathbb{N}$*)*

$$\int_{\mathbb{R}^N} \|x\|_\infty^k Z(x)\, dx \leq N^k \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(k+1)} \right) + \frac{2e^\mu k!}{\mu^k} \right] < \infty. \tag{21}$$

When $k = 0$*, (21) is again valid.*

Proof. We have that

$$\int_{\mathbb{R}^N} \|x\|_\infty^k Z(x)\, dx \leq \int_{\mathbb{R}^N} \left(\sum_{j=1}^N |x_j| \right)^k Z(x)\, dx \leq$$

(by a convexity argument)

$$\int_{\mathbb{R}^N} N^{k-1} \left(\sum_{j=1}^N |x_j|^k \right) Z(x)\, dx = N^{k-1} \left(\sum_{j=1}^N \int_{\mathbb{R}^N} |x_j|^k Z(x)\, dx \right) = \tag{22}$$

$$N^{k-1} \left(\sum_{j=1}^N \int_{\mathbb{R}^N} |x_j|^k \prod_{i=1}^N G(x_i)\, dx \right) =$$

$$N^{k-1} \left(\sum_{j=1}^N \left[\left(\int_{-\infty}^{\infty} |x_j|^k G(x_j)\, dx_j \right) \prod_{\substack{i=1 \\ i \neq j}}^N \int_{-\infty}^{\infty} G(x_i)\, dx \right] \right) \overset{(10)}{=}$$

$$N^{k-1} \left(\sum_{j=1}^N \int_{-\infty}^{\infty} |x_j|^k G(x_j)\, dx_j \right) \overset{(14)}{\leq}$$

$$N^{k-1} \left(\sum_{j=1}^N \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(k+1)} \right) + \frac{2e^\mu k!}{\mu^k} \right] \right) =$$

$$N^k \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(k+1)} \right) + \frac{2e^\mu k!}{\mu^k} \right], \tag{23}$$

proving the claim. □

We will use the following differentiation result.

Theorem 3.2. *(H. Bauer* [4]*, pp. 103-104) Let* (Ω, A, μ) *be a measure space. Let* U *be an open subset of* $\mathbb{R}^d$*,* $d \geq 1$*, and let* $f : U \times \Omega \to \mathbb{R}$ *be a function with the properties:*

(a) $\omega \to f(x, \omega)$ *is* μ*-integrable for all* $x \in U$*,*

(b) $x \to f(x, \omega)$ *is at each point in* U *partially differentiable with respect to* x_i*,*

(c) there exists a μ*-integrable function* $h \geq 0$ *on* Ω *such that*

$$\left| \frac{\partial f}{\partial x_i}(x, \omega) \right| \leq h(\omega), \text{ for all } (x, \omega) \in U \times \Omega.$$

Then, the function φ *defined on* U *as*

$$\varphi(x) = \int_\Omega f(x, \omega)\, \mu(d\omega)$$

is partially differentiable with respect to x_i *on all of* U. *The mapping* $\omega \to \frac{\partial f}{\partial x_i}(x,\omega)$ *is* μ*-integrable, and we have*

$$\frac{\partial \varphi}{\partial x_i}(x) = \int_{\Omega} \frac{\partial f}{\partial x_i}(x,\omega)\,\mu(d\omega), \text{ all } x \in U.$$

We give

Definition 3.1. Let $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$. We define the following activated logistic parametrized multivariate convolution type operators:

The basic one

$$B_n(f)(x) := \int_{\mathbb{R}^N} f\left(\frac{u}{n}\right) Z(nx-u)\,du, \ \ \forall\, x \in \mathbb{R}^N, \tag{24}$$

the activated Kantorovich type

$$B_n^*(f)(x) := n^N \int_{\mathbb{R}^N} \left(\int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t)\,dt \right) Z(nx-u)\,du, \ \ \forall\, x \in \mathbb{R}^N. \tag{25}$$

Let now $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1 r_2 ... r_N} \geq 0$, such that

$$\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1 r_2 ... r_N} = 1; \ \ u \in \mathbb{R}^N,$$

and

$$\delta_n(f)(u) := \delta_n(f)(u_1, ..., u_N) := \sum_{r=0}^{\theta} w_r f\left(\frac{u}{n} + \frac{r}{n\theta}\right) =$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1 r_2 ... r_N} f\left(\frac{u_1}{n} + \frac{r_1}{n\theta_1}, \frac{u_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{u_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{26}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We define the activated quadrature operators

$$\overline{B_n}(f)(x) := \overline{B_n}(f, x_1, ..., x_N) := \int_{\mathbb{R}^N} \delta_n(f)(u)\, Z(nx-u)\,du = \tag{27}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \delta_n(f)(u_1, ..., u_N) \left(\prod_{i=1}^{N} G(nx_i - u_i) \right) du_1 ... du_N, \ \ \forall x \in \mathbb{R}^N.$$

One can rewrite

$$B_n(f)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f\left(\frac{u_1}{n}, \frac{u_2}{n}, ..., \frac{u_N}{n}\right) \left(\prod_{i=1}^{N} G(nx_i - u_i) \right) du_1 ... du_N, \tag{28}$$

and

$$B_n^*(f)(x) = n^N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left(\int_{\frac{u_1}{n}}^{\frac{u_1+1}{n}} \int_{\frac{u_2}{n}}^{\frac{u_2+1}{n}} ... \int_{\frac{u_N}{n}}^{\frac{u_N+1}{n}} f(t_1, ..., t_N)\, dt_1 ... dt_N \right)$$

$$\left(\prod_{i=1}^{N} G(nx_i - u_i) \right) du_1 ... du_N, \tag{29}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

For some $f \in C_U(\mathbb{R}^N)$ the above operators can exist.

In this work we study the approximation properties of the operators B_n, B_n^* and $\overline{B_n}$, expecially their convergence to the unit operator I.

4. Main Results

We present the following approximation results:

Theorem 4.1. *Let $0 < \beta < 1$, $\mu > 0$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $f \in C_B\left(\mathbb{R}^N\right)$, $n \in \mathbb{N}$, $x \in \mathbb{R}^N$. Then*

$$\left|B_n\left(f\right)\left(x\right) - f\left(x\right)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4\left\|f\right\|_\infty}{e^{\mu\left(n^{1-\beta}-1\right)}} =: \lambda, \tag{30}$$

and

$$\left\|B_n\left(f\right) - f\right\|_\infty \leq \lambda, \tag{31}$$

where $\left\|\cdot\right\|_\infty$ is the supremum norm.

So, for $f \in C_{UB}\left(\mathbb{R}^N\right) := C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)$, we get that $\lim_{n\to\infty} B_n\left(f\right) = f$, pointwise and uniformly.

Proof. Call

$$A_1 := \left\{u \in \mathbb{R}^N : \left\|\frac{u}{n} - x\right\|_\infty < \frac{1}{n^\beta}\right\}, \tag{32}$$

and

$$A_2 := \left\{u \in \mathbb{R}^N : \left\|\frac{u}{n} - x\right\|_\infty \geq \frac{1}{n^\beta}\right\}. \tag{33}$$

That is $A_1 \cup A_2 = \mathbb{R}^N$.

We have that

$$\left|B_n\left(f\right)\left(x\right) - f\left(x\right)\right| \overset{(17)}{=}$$

$$\left|\int_{\mathbb{R}^N} f\left(\frac{u}{n}\right) Z\left(nx-u\right) du - f\left(x\right)\int_{\mathbb{R}^N} Z\left(nx-u\right) du\right| =$$

$$\left|\int_{\mathbb{R}^N} \left(f\left(\frac{u}{n}\right) - f\left(x\right)\right) Z\left(nx-u\right) du\right| \leq \tag{34}$$

$$\int_{\mathbb{R}^N} \left|f\left(\frac{u}{n}\right) - f\left(x\right)\right| Z\left(nx-u\right) du =$$

$$\int_{A_1} \left|f\left(\frac{u}{n}\right) - f\left(x\right)\right| Z\left(nx-u\right) du + \int_{A_2} \left|f\left(\frac{u}{n}\right) - f\left(x\right)\right| Z\left(nx-u\right) du \leq$$

$$\int_{A_1} \omega_1\left(f, \left\|\frac{u}{n} - x\right\|_\infty\right) Z\left(nx-u\right) du + 2\left\|f\right\|_\infty \int_{A_2} Z\left(nx-u\right) du \leq \tag{35}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right)\int_{A_1} Z\left(nx-u\right) du + 2\left\|f\right\|_\infty \frac{2}{e^{\mu\left(n^{1-\beta}-1\right)}} \leq$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right)\int_{\mathbb{R}^N} Z\left(nx-u\right) du + \frac{4\left\|f\right\|_\infty}{e^{\mu\left(n^{1-\beta}-1\right)}} =$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4\left\|f\right\|_\infty}{e^{\mu\left(n^{1-\beta}-1\right)}}.$$

□

We continue with the following.

Theorem 4.2. *All as in Theorem 4.1. Then*

$$\left|B_n^*\left(f\right)\left(x\right) - f\left(x\right)\right| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4\left\|f\right\|_\infty}{e^{\mu\left(n^{1-\beta}-1\right)}} =: \rho, \tag{36}$$

and

$$\left\| B_n^* (f) - f \right\|_\infty \leq \rho. \tag{37}$$

For $f \in C_{UB}\left(\mathbb{R}^N\right)$, *we get that* $\lim_{n\to\infty} B_n^*(f) = f$, *pointwise and uniformly.*

Proof. We notice that

$$\int_{\frac{u}{n}}^{\frac{u+1}{n}} f(t)\,dt = \int_{\frac{u_1}{n}}^{\frac{u_1+1}{n}} \int_{\frac{u_2}{n}}^{\frac{u_2+1}{n}} \ldots \int_{\frac{u_N}{n}}^{\frac{u_N+1}{n}} f(t_1,\ldots,t_N)\,dt_1\ldots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{u_1}{n}, t_2 + \frac{u_2}{n}, \ldots, t_N + \frac{u_N}{n}\right) dt_1\ldots dt_N = \int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{u}{n}\right) dt. \tag{38}$$

Thus, it holds

$$B_n^*(f)(x) = n^N \int_{\mathbb{R}^N} \left(\int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{u}{n}\right) dt \right) Z(nx-u)\,du. \tag{39}$$

We observe that

$$\left| B_n^*(f)(x) - f(x) \right| =$$

$$\left| \int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{u}{n}\right) dt \right) Z(nx-u)\,du - f(x) \int_{\mathbb{R}^N} Z(nx-u)\,du \right| =$$

$$\left| \int_{\mathbb{R}^N} \left(\left(n^N \int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \frac{u}{n}\right) dt \right) - f(x) \right) Z(nx-u)\,du \right| =$$

$$\left| \int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left(f\left(t + \frac{u}{n}\right) - f(x) \right) dt \right) Z(nx-u)\,du \right| \leq$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{u}{n}\right) - f(x) \right| dt \right) Z(nx-u)\,du = \tag{40}$$

$$\int_{A_1} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{u}{n}\right) - f(x) \right| dt \right) Z(nx-u)\,du +$$

$$\int_{A_2} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \frac{u}{n}\right) - f(x) \right| dt \right) Z(nx-u)\,du \leq$$

$$\int_{A_1} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \omega_1\left(f, \|t\|_\infty + \left\| \frac{u}{n} - x \right\|_\infty \right) dt \right) Z(nx-u)\,du +$$

$$2\|f\|_\infty \left(\int_{A_2} Z(nx-u)\,du \right) \leq$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) \int_{A_1} Z(nx-u)\,du + 2\|f\|_\infty \frac{2}{e^{\mu(n^{1-\beta}-1)}} \leq \tag{41}$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4\|f\|_\infty}{e^{\mu(n^{1-\beta}-1)}}.$$

□

It follows:

Theorem 4.3. *All as in Theorem 4.1. Then*

$$\left|\overline{B_n}(f)(x)-f(x)\right| \leq \omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{4\|f\|_{\infty}}{e^{\mu\left(n^{1-\beta}-1\right)}}=\rho, \tag{42}$$

and

$$\left\|\overline{B_n}(f)-f\right\|_{\infty} \leq \rho. \tag{43}$$

For $f \in C_{UB}\left(\mathbb{R}^N\right)$, we get that $\lim_{n\to\infty}\overline{B_n}(f)=f$, pointwise and uniformly.

Proof. We have that

$$\left|\overline{B_n}(f)(x)-f(x)\right| =$$

$$\left|\int_{\mathbb{R}^N}\delta_n(f)(u)\,Z(nx-u)\,du-f(x)\int_{\mathbb{R}^N}Z(nx-u)\,du\right| =$$

$$\left|\int_{\mathbb{R}^N}\left(\delta_n(f)(u)-f(x)\right)Z(nx-u)\,du\right| =$$

$$\left|\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left(f\left(\frac{u}{n}+\frac{r}{n\theta}\right)-f(x)\right)\right)Z(nx-u)\,du\right| \leq \tag{44}$$

$$\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{u}{n}+\frac{r}{n\theta}\right)-f(x)\right|\right)Z(nx-u)\,du =$$

$$\int_{A_1}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{u}{n}+\frac{r}{n\theta}\right)-f(x)\right|\right)Z(nx-u)\,du+$$

$$\int_{A_2}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{u}{n}+\frac{r}{n\theta}\right)-f(x)\right|\right)Z(nx-u)\,du \leq$$

$$\int_{A_1}\left(\sum_{r=0}^{\theta}w_r\omega_1\left(f,\left\|\frac{u}{n}-x\right\|_{\infty}+\frac{1}{n}\left\|\frac{r}{\theta}\right\|_{\infty}\right)\right)Z(nx-u)\,du+$$

$$2\|f\|_{\infty}\int_{A_2}Z(nx-u)\,du \leq$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}+\frac{1}{n}\right)\int_{A_1}Z(nx-u)\,du+2\|f\|_{\infty}\frac{2}{e^{\mu\left(n^{1-\beta}-1\right)}} \leq \tag{45}$$

$$\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{4\|f\|_{\infty}}{e^{\mu\left(n^{1-\beta}-1\right)}}.$$

□

Next, we describe the global smoothness preservation property of our activated multivariate operators.

Theorem 4.4. *Here $f \in C_B\left(\mathbb{R}^N\right)\cup C_U\left(\mathbb{R}^N\right)$. Then*

$$\omega_1\left(B_n(f),\delta\right) \leq \omega_1(f,\delta), \quad \delta>0. \tag{46}$$

If $f \in C_U\left(\mathbb{R}^N\right)$, then $B_n(f) \in C_U\left(\mathbb{R}^N\right)$.

Proof. We have that

$$B_n(f)(x)=\int_{\mathbb{R}^N}f\left(x-\frac{z}{n}\right)Z(z)\,dz. \tag{47}$$

Let $x, y \in \mathbb{R}^N$, then

$$B_n(f)(x) - B_n(f)(y) = \int_{\mathbb{R}^N} \left(f\left(x - \frac{z}{n}\right) - f\left(y - \frac{z}{n}\right)\right) Z(z)\, dz. \tag{48}$$

Thus

$$\left|B_n(f)(x) - B_n(f)(y)\right| \leq \int_{\mathbb{R}^N} \left| f\left(x - \frac{z}{n}\right) - f\left(y - \frac{z}{n}\right)\right| Z(z)\, dz \leq$$

$$\omega_1\left(f, \|x-y\|_\infty\right) \int_{\mathbb{R}^N} Z(z)\, dz \overset{(18)}{=} \omega_1\left(f, \|x-y\|_\infty\right). \tag{49}$$

Let $\|x-y\|_\infty \leq \delta$, $\delta > 0$, then we get (46). $\square$

Remark. *Let f be the projection function onto x_i coordinate, call it $pr_i(x) := x_i$, $i \in \{1, ..., N\}$, where $x = (x_1, ..., x_i, ..., x_N) \in \mathbb{R}^N$. Then, it holds*

$$B_n(pr_i)(x) = \int_{\mathbb{R}^N} \left(x_i - \frac{z_i}{n}\right) Z(z)\, dz = x_i - \frac{1}{n}\int_{\mathbb{R}^N} z_i Z(z)\, dz. \tag{50}$$

Hence

$$\left|B_n(pr_i)(x) - B_n(pr_i)(y)\right| = |x_i - y_i| = |pr_i(x) - pr_i(y)|,$$

proving

$$\omega_1\left(B_n(pr_i), \delta\right) = \omega_1\left(pr_i, \delta\right), \tag{51}$$

any $\delta > 0$.

So (46) is an attained sharp inequality.

Furthermore, it is

$$B_n(pr_i)(x) = x_i - \frac{1}{n}\left(\prod_{\substack{j=1\\ j\neq i}}^{N} \int_{-\infty}^{\infty} G(z_j)\, dz_j\right)\left(\int_{-\infty}^{\infty} z_i G(z_i)\, dz_i\right) \tag{52}$$

$$= x_i - \frac{1}{n}\int_{-\infty}^{\infty} z_i G(z_i)\, dz_i,$$

and

$$\left|B_n(pr_i)(x)\right| \leq |x_i| + \frac{1}{n}\int_{-\infty}^{\infty} |z_i|\, G(z_i)\, dz_i$$

$$\overset{(14)}{\leq} |x_i| + \frac{1}{n}\left[\left(\frac{e^\mu - 1}{2(e^\mu + 1)}\right) + \frac{2e^\mu}{\mu}\right] < \infty. \tag{53}$$

Thus, $B_n(pr_i)(x)$ is well-defined.

Theorem 4.5. *Let $f \in C_B\left(\mathbb{R}^N\right) \cup C_U\left(\mathbb{R}^N\right)$. Then*

$$\omega_1\left(B_n^*(f), \delta\right) \leq \omega_1(f, \delta), \quad \delta > 0. \tag{54}$$

If $f \in C_U\left(\mathbb{R}^N\right)$, then $B_n^*(f) \in C_U\left(\mathbb{R}^N\right)$.

Inequality (54) is an attained sharp inequality by $f(x) = pr_i(x)$, $i \in \{1, ..., N\}$.

Proof. See (39), one can write

$$B_n^*(f)(x) = \int_{\mathbb{R}^N} \left(n^N \int_{\left[0, \frac{1}{n}\right]^N} f\left(t + \frac{u}{n}\right) dt\right) Z(nx - u)\, du = \tag{55}$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{\left[0, \frac{1}{n}\right]^N} f\left(t + \left(x - \frac{z}{n}\right)\right) dt\right) Z(z)\, dz,$$

and

$$B_n^*(f)(y) = \int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} f\left(t + \left(y - \frac{z}{n}\right)\right) dt \right) Z(z)\, dz, \tag{56}$$

where $x, y \in \mathbb{R}^N$.

Thus

$$|B_n^*(f)(x) - B_n^*(f)(y)| =$$

$$\left| \int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left(f\left(t + \left(x - \frac{z}{n}\right)\right) - f\left(t + \left(y - \frac{z}{n}\right)\right)\right) dt \right) Z(z)\, dz \right| \leq$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left| f\left(t + \left(x - \frac{z}{n}\right)\right) - f\left(t + \left(y - \frac{z}{n}\right)\right)\right| dt \right) Z(z)\, dz \leq$$

$$\omega_1\left(f, \|x-y\|_\infty\right) \int_{\mathbb{R}^N} Z(z)\, dz = \omega_1\left(f, \|x-y\|_\infty\right), \tag{57}$$

proving inequality (54).

We do have

$$|B_n^*(pr_i)(x)| \leq \int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left| \left(t_i + \left(x_i - \frac{z_i}{n}\right)\right)\right| dt \right) Z(z)\, dz \leq$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} \left(|t_i| + |x_i| + \frac{|z_i|}{n} \right) dt \right) Z(z)\, dz \leq \tag{58}$$

$$\int_{\mathbb{R}^N} \left(\frac{1}{n} + |x_i| + \frac{|z_i|}{n} \right) Z(z)\, dz = \int_{-\infty}^{\infty} \left(\frac{1}{n} + |x_i| + \frac{|z_i|}{n} \right) G(z_i)\, dz_i =$$

$$\frac{1}{n} + |x_i| + \frac{1}{n} \int_{-\infty}^{\infty} |z_i|\, G(z_i)\, dz_i \overset{(14)}{\leq} \tag{59}$$

$$\frac{1}{n} + |x_i| + \frac{1}{n} \left[\left(\frac{e^\mu - 1}{2(e^\mu + 1)} \right) + \frac{2e^\mu}{\mu} \right] < \infty.$$

Thus, $B_n^*(pr_i)(x)$ is well-defined. □

Theorem 4.6. *Let $f \in C_B\left(\mathbb{R}^N\right) \cup C_U\left(\mathbb{R}^N\right)$. Then*

$$\omega_1\left(\overline{B_n}(f), \delta\right) \leq \omega_1(f, \delta), \quad \delta > 0. \tag{60}$$

If $f \in C_U\left(\mathbb{R}^N\right)$, then $\overline{B_n}(f) \in C_U\left(\mathbb{R}^N\right)$.

Inequality (60) is an attained sharp inequality by $f(x) = pr_i(x)$, $i \in \{1, ..., N\}$.

Proof. Let $x, y \in \mathbb{R}^N$, then

$$\overline{B_n}(f)(x) = \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r f\left(\left(x - \frac{z}{n}\right) + \frac{r}{n\theta} \right) \right) Z(z)\, dz, \tag{61}$$

and

$$\overline{B_n}(f)(y) = \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r f\left(\left(y - \frac{z}{n}\right) + \frac{r}{n\theta} \right) \right) Z(z)\, dz. \tag{62}$$

Hence

$$\left|\overline{B_n}(f)(x) - \overline{B_n}(f)(y)\right| =$$

$$\left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\left(x - \frac{z}{n}\right) + \frac{r}{n\theta} \right) - f\left(\left(y - \frac{z}{n}\right) + \frac{r}{n\theta} \right) \right) \right) Z(z)\, dz \right| \leq$$

$$\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\left(x-\frac{z}{n}\right)+\frac{r}{n\theta}\right)-f\left(\left(y-\frac{z}{n}\right)+\frac{r}{n\theta}\right)\right|\right)Z(z)\,dz\leq \tag{63}$$
$$\omega_1\left(f,\|x-y\|_\infty\right)\int_{\mathbb{R}^N}Z(z)\,dz=\omega_1\left(f,\|x-y\|_\infty\right).$$

That is (60) is true, and it is an attained sharp inequality by $f(x)=pr_i(x)$, $i\in\{1,...,N\}$.

Indeed, we have that

$$\left|\overline{B_n}(pr_i)(x)\right|=\left|\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left[\left(x_i-\frac{z_i}{n}\right)+\frac{r_i}{n\theta_i}\right]\right)Z(z)\,dz\right|\leq$$
$$\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left|\left(x_i-\frac{z_i}{n}\right)+\frac{r_i}{n\theta_i}\right|\right)Z(z)\,dz\leq$$
$$\int_{\mathbb{R}^N}\left(\sum_{r=0}^{\theta}w_r\left[|x_i|+\frac{|z_i|}{n}+\frac{1}{n}\right]\right)Z(z)\,dz= \tag{64}$$
$$\int_{\mathbb{R}^N}\left[|x_i|+\frac{|z_i|}{n}+\frac{1}{n}\right]Z(z)\,dz=$$

(as in (59))

$$\int_{-\infty}^{\infty}\left[|x_i|+\frac{|z_i|}{n}+\frac{1}{n}\right]G(z_i)\,dz_i<\infty.$$

Thus, $\overline{B_n}(pr_i)(x)$ is well-defined. □

We make

Remark. *Let $i\in\mathbb{N}$ be fixed. Assume that $f\in C^{(i)}\left(\mathbb{R}^N\right)$, $N\in\mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_j\in\mathbb{Z}_+$, $j=1,...,N$, and $|\alpha|:=\sum_{j=1}^{N}\alpha_j=l$, where $l=0,1,...,i$.*

We write also $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.

We assume that any partial $f_\alpha\in C_B\left(\mathbb{R}^N\right)$, for all $\alpha:|\alpha|=l$, $l=0,1,...,i$.

By repeated application of Theorem 3.2 we obtain

$$\left(B_n(f)\right)_\alpha(x)=\int_{\mathbb{R}^N}f_\alpha\left(x-\frac{z}{n}\right)Z(z)\,dz= \tag{65}$$
$$\int_{\mathbb{R}^N}f_\alpha\left(\frac{u}{n}\right)Z(nx-u)\,du=\left(B_n(f_\alpha)\right)(x),\ \ \forall\,x\in\mathbb{R}^N.$$

Similarly, we obtain that

$$\left(B_n^*(f)\right)_\alpha(x)=\left(B_n^*(f_\alpha)\right)(x), \tag{66}$$

and

$$\left(\overline{B_n}(f)\right)_\alpha(x)=\left(\overline{B_n}(f_\alpha)\right)(x),\ \ \forall\,x\in\mathbb{R}^N; \tag{67}$$

for all $\alpha:|\alpha|=l$, $l=0,1,...,i$.

So all of our results in this work can be written in the simultaneous approximation context, see Theorems 4.10-4.12.

We make

Remark. ***About activated Iterated Multivariate Convolution***

We have that

$$B_n(f)(x) = \int_{\mathbb{R}^N} f\left(x - \frac{z}{n}\right) Z(z)\, dz,$$

$\forall\, x \in \mathbb{R}^N$, *where* $f \in C_B(\mathbb{R}^N)$.

Let $x_N \to x$, *as* $N \to \infty$, *and*

$$B_n(f)(x_N) - B_n(f)(x) = \int_{\mathbb{R}^N} \left(f\left(x_N - \frac{z}{n}\right) - f\left(x - \frac{z}{n}\right)\right) Z(z)\, dz. \tag{68}$$

We have that

$$f\left(x_N - \frac{z}{n}\right) Z(z) \to f\left(x - \frac{z}{n}\right) Z(z),\ \forall\, z \in \mathbb{R}^N,\ \text{as } N \to \infty.$$

Furthermore, it holds

$$|B_n(f)(x_N) - B_n(f)(x)| \leq \tag{69}$$

$$\int_{\mathbb{R}^N} \left| f\left(x_N - \frac{z}{n}\right) - f\left(x - \frac{z}{n}\right)\right| Z(z)\, dz \to 0, \text{ as } N \to \infty,$$

by dominated convergence theorem, because we have that

$$\left| f\left(x_N - \frac{z}{n}\right)\right| Z(z) \leq \|f\|_\infty Z(z),$$

and $\|f\|_\infty Z(z)$ *is integrable over* $\mathbb{R}^N$, $\forall\, z \in \mathbb{R}^N$.

Hence $B_n(f) \in C_B(\mathbb{R}^N)$.

Furthermore it holds

$$|B_n(f)(x)| \leq \|f\|_\infty \int_{\mathbb{R}^N} Z(z)\, dz = \|f\|_\infty,$$

i.e.

$$\|B_n(f)\|_\infty \leq \|f\|_\infty. \tag{70}$$

So B_n *is a bounded positive linear operator.*

Clearly it holds

$$\left\|B_n^2(f)\right\|_\infty = \|B_n(B_n(f))\|_\infty \leq \|B_n(f)\|_\infty \leq \|f\|_\infty. \tag{71}$$

And for $k \in \mathbb{N}$ *we obtain*

$$\left\|B_n^k(f)\right\|_\infty \leq \left\|B_n^{k-1}(f)\right\|_\infty \leq \left\|B_n^{k-2}(f)\right\|_\infty \leq ... \leq \|f\|_\infty, \tag{72}$$

so the contraction property valid and B_n^k *is a bounded linear operator.*

Remark. *Let* $r \in \mathbb{N}$. *We observe that*

$$B_n^r f - f = \left(B_n^r f - B_n^{r-1} f\right) + \left(B_n^{r-1} f - B_n^{r-2} f\right) + \left(B_n^{r-2} f - B_n^{r-3} f\right)$$
$$+ ... + \left(B_n^2 f - B_n f\right) + (B_n f - f). \tag{73}$$

Then

$$\|B_n^r f - f\|_\infty \leq \left\|B_n^r f - B_n^{r-1} f\right\|_\infty + \left\|B_n^{r-1} f - B_n^{r-2} f\right\|_\infty + \left\|B_n^{r-2} f - B_n^{r-3} f\right\|_\infty$$
$$+ ... + \left\|B_n^2 f - B_n f\right\|_\infty + \|B_n f - f\|_\infty =$$
$$\left\|B_n^{r-1}(B_n f - f)\right\|_\infty + \left\|B_n^{r-2}(B_n f - f)\right\|_\infty + ... + \|B_n(B_n f - f)\|_\infty +$$
$$\|B_n f - f\|_\infty \leq r\|B_n f - f\|_\infty.$$

Therefore

$$\|B_n^r f - f\|_\infty \leq r\|B_n f - f\|_\infty. \tag{74}$$

Let now $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, *and* B_{m_i} *as above.*

$$B_{m_r}\left(B_{m_{r-1}}(...B_{m_2}(B_{m_1} f))\right) - f = ... =$$

$$B_{m_r}\left(B_{m_{r-1}}\left(...B_{m_2}\right)\right)\left(B_{m_1}f-f\right)+B_{m_r}\left(B_{m_{r-1}}\left(...B_{m_3}\right)\right)\left(B_{m_2}f-f\right)+ \quad (75)$$
$$B_{m_r}\left(B_{m_{r-1}}\left(...B_{m_4}\right)\right)\left(B_{m_3}f-f\right)+...+B_{m_r}\left(B_{m_{r-1}}f-f\right)+B_{m_r}f-f.$$

Consequently it holds, as in [1], *Chapter 2,*

$$\left\|B_{m_r}\left(B_{m_{r-1}}\left(...B_{m_2}\left(B_{m_1}f\right)\right)\right)-f\right\|_\infty \leq \sum_{i=1}^{r}\left\|B_{m_i}f-f\right\|_\infty . \quad (76)$$

Next we have

Remark. *It is*

$$B_n^*(f)(x)=n^N\int_{\mathbb{R}^N}\left(\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\left(x-\frac{z}{n}\right)\right)dt\right)Z(z)\,dz,$$

$f\in C_B\left(\mathbb{R}^N\right)$.

Let $x_N\to x$, as $N\to\infty$, and

$$\left|B_n^*(f)(x_N)-B_n^*(f)(x)\right|=$$
$$\left|n^N\int_{\mathbb{R}^N}\left(\int_{\left[0,\frac{1}{n}\right]^N}\left[f\left(t+\left(x_N-\frac{z}{n}\right)\right)-f\left(t+\left(x-\frac{z}{n}\right)\right)\right]dt\right)Z(z)\,dz\right|\leq \quad (77)$$
$$\int_{\mathbb{R}^N}\left(n^N\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\left(x_N-\frac{z}{n}\right)\right)-f\left(t+\left(x-\frac{z}{n}\right)\right)\right|dt\right)Z(z)\,dz\to 0,$$

as $N\to\infty$.

This is true by the bounded convergence theorem, we get that:

$$x_N\to x \quad \rightsquigarrow t+\left(x_N-\frac{z}{n}\right)\to t+\left(x-\frac{z}{n}\right)$$

and

$$f\left(t+\left(x_N-\frac{z}{n}\right)\right)\to f\left(t+\left(x-\frac{z}{n}\right)\right), \text{ and}$$
$$\left|f\left(t+\left(x_N-\frac{z}{n}\right)\right)\right|\leq \|f\|_\infty ,$$

where $\left[0,\frac{1}{n}\right]^N$ is a cube. Thus

$$n^N\int_{\left[0,\frac{1}{n}\right]^N}\left|f\left(t+\left(x_N-\frac{z}{n}\right)\right)-f\left(t+\left(x-\frac{z}{n}\right)\right)\right|dt\to 0, \text{ as } N\to\infty. \quad (78)$$

Therefore it holds

$$n^N\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\left(x_N-\frac{z}{n}\right)\right)dt\to n^N\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\left(x-\frac{z}{n}\right)\right)dt, \text{ as } N\to\infty. \quad (79)$$

And it is

$$\left(n^N\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\left(x_N-\frac{z}{n}\right)\right)dt\right)Z(z)\to\left(n^N\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\left(x-\frac{z}{n}\right)\right)dt\right)Z(z), \quad (80)$$

as $N\to\infty$, $\forall\, z\in\mathbb{R}^N$.

Furthermore we have

$$\left|\left(n^N\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\left(x_N-\frac{z}{n}\right)\right)dt\right)Z(z)\right|\leq \|f\|_\infty Z(z), \quad (81)$$

with $\|f\|_\infty Z(z)$ being integrable over $\mathbb{R}^N$.

Therefore by dominated convergence theorem

$$B_n^*(f)(x_N) \to B_n^*(f)(x), \text{ as } N \to \infty.$$

Hence $B_n^*(f)(x)$ is a bounded and continuous in $x \in \mathbb{R}^N$.

The iterated facts hold for B_n^* as in the $B_n(f)$ case, all the same! See (70) - (72) and all of Remark 4.

Next we observe that: Let $f \in C_B(\mathbb{R}^N)$, and

$$\overline{B_n}(f)(x) = \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r f\left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) Z(z)\, dz.$$

Let $x_N \to x$, as $N \to \infty$. Then

$$\left| \overline{B_n}(f)(x_N) - \overline{B_n}(f)(x) \right| =$$

$$\left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) - f\left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) \right) Z(z)\, dz \right| \leq \tag{82}$$

$$\int_{\mathbb{R}^N} \left| \sum_{r=0}^{\theta} w_r \left(f\left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) - f\left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) \right| Z(z)\, dz \to 0,$$

as $N \to \infty$.

The last comes by the dominated convergence theorem:

$$\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \to \left(x - \frac{z}{n} \right) + \frac{r}{n\theta}$$

and

$$\sum_{r=0}^{\theta} w_r f\left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \to \sum_{r=0}^{\theta} w_r f\left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right),$$

and

$$\left(\sum_{r=0}^{\theta} w_r f\left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) Z(z) \to \left(\sum_{r=0}^{\theta} w_r f\left(\left(x - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right) Z(z),$$

as $N \to \infty$, $\forall\, z \in \mathbb{R}^N$.

Furthermore it holds

$$\left| \sum_{r=0}^{\theta} w_r f\left(\left(x_N - \frac{z}{n} \right) + \frac{r}{n\theta} \right) \right| Z(z) \leq \|f\|_\infty Z(z), \tag{83}$$

which the last one function is integrable over $\mathbb{R}^N$.

Therefore

$$\overline{B_n}(f)(x_N) \to \overline{B_n}(f)(x), \text{ as } N \to \infty.$$

Hence $\overline{B_n}(f)(x)$ is a bounded and continuous in $x \in \mathbb{R}^N$.

Iterated stuff for $\overline{B_n}$: it is all the same as with $B_n(f)$! See (70) - (72) and all of Remark 4.

See the related Theorems 4.13, 4.14 later.

Next we improve greatly the speed of convergence of our activated multivariate operators by using differentiation of functions.

Notation. *Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., N$, and $|\alpha| = \sum_{i=1}^{N} \alpha_i = l$, where $l = 0, 1, ..., m$. We write also $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.*

We denote

$$\omega_{1,m}^{\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=m} \omega_1(f_\alpha, h), \ \ h > 0. \tag{84}$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m} \{\|f_\alpha\|_\infty\}, \tag{85}$$

where $\|\cdot\|_\infty$ is the supremum norm.

Theorem 4.7. *Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$; $\mu > 0$, $x \in \mathbb{R}^N$, $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with $f_\alpha \in C_B(\mathbb{R}^N)$, for all $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., N$, and $|\alpha| = \sum_{i=1}^{N} \alpha_i = l$, where $l = 0, 1, ..., m$. Then*

(i)

$$\Bigg| B_n(f)(x) - f(x) - \sum_{j=1}^{m} \sum_{\left\{\begin{array}{c}\alpha := (\alpha_1, ..., \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, ..., N, \ |\alpha| := \sum_{i=1}^{N} \alpha_i = j\end{array}\right\}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) f_\alpha(x) B_n\left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}\right)\right)(x) \Bigg|$$

$$\leq \left(\frac{N^m}{m! n^{m\beta}}\right) \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) +$$

$$\left(\frac{2\|f_\alpha\|_{\infty,m}^{\max} N^{2m}}{n^m m!}\right)\left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m + 1)}\right) + \frac{2e^\mu m!}{\mu^m}\right] =: \Phi, \tag{86}$$

(ii) assume that $f_\alpha(x) = 0$, for all $\alpha : |\alpha| = j$, $j = 1, ..., m$, we have

$$|B_n(f)(x) - f(x)| \leq \Phi, \tag{87}$$

with the high speed of $n^{-\beta(m+1)}$.

(iii)

$$|B_n(f)(x) - f(x)| \leq$$

$$\sum_{j=1}^{N} \sum_{\left\{\begin{array}{c}\alpha : \\ |\alpha| = j\end{array}\right.} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) |f_\alpha(x)| \frac{1}{n^j} \left\{\prod_{i=1}^{N} \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(\alpha_i + 1)}\right) + \frac{2e^\mu \alpha_i!}{\mu^{\alpha_i}}\right]\right\}$$

$$+\Phi, \tag{88}$$

and

(iv)

$$\|B_n(f) - f\|_\infty \leq$$

$$\sum_{j=1}^{N} \sum_{\left\{\begin{array}{c}\alpha : \\ |\alpha| = j\end{array}\right.} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \|f_\alpha\|_\infty \frac{1}{n^j} \left\{ \prod_{i=1}^{N} \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(\alpha_i + 1)} \right) + \frac{2e^\mu \alpha_i!}{\mu^{\alpha_i}} \right] \right\} + \Phi. \tag{89}$$

We have that $B_n(f) \to f$, as $n \to \infty$, pointwise and uniformly.

Proof. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \mathbb{R}^N$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{90}$$

for all $j = 0, 1, ..., m$.

We have the multivariate Taylor's formula

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m} \frac{g_z^{(j)}(0)}{j!} +$$

$$\frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left(g_z^{(m)}(\theta) - g_z^{(m)}(0) \right) d\theta. \tag{91}$$

Notice $g_z(0) = f(z_0)$. Also for $j = 0, 1, ..., m$, we have

$$g_z^{(j)}(0) = \sum_{\left\{\begin{array}{c}\alpha := (\alpha_1, ..., \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, ..., N, \ |\alpha| := \sum_{i=1}^{N} \alpha_i = j\end{array}\right\}} \left(\frac{j!}{\prod_{i=1}^{N} \alpha_i!} \right) \tag{92}$$

$$\left(\prod_{i=1}^{N} (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0).$$

Furthermore it is

$$g_z^{(m)}(\theta) = \sum_{\left\{\begin{array}{c}\alpha := (\alpha_1, ..., \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, ..., N, \ |\alpha| := \sum_{i=1}^{N} \alpha_i = m\end{array}\right\}} \left(\frac{m!}{\prod_{i=1}^{N} \alpha_i!} \right)$$

$$\left(\prod_{i=1}^{N} (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)), \tag{93}$$

$0 \leq \theta \leq 1$.

So, we treat $f \in C^m(\mathbb{R}^N)$.

Thus, we have for $u, x \in \mathbb{R}^N$ that

$$f\left(\frac{u_1}{n}, ..., \frac{u_N}{n} \right) - f(x) =$$

$$\sum_{j=1}^{m} \sum_{\left\{\begin{array}{c} \alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i\in\mathbb{Z}_+ \\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=j \end{array}\right\}} \left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right) \left(\prod_{i=1}^{N}\left(\frac{u_i}{n}-x_i\right)^{\alpha_i}\right) f_\alpha(x)+R, \quad (94)$$

where

$$R:=m\int_0^1(1-\theta)^{m-1} \sum_{\left\{\begin{array}{c} \alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i\in\mathbb{Z}_+ \\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=m \end{array}\right\}} \left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right) \left(\prod_{i=1}^{N}\left(\frac{u_i}{n}-x_i\right)^{\alpha_i}\right)\left[f_\alpha\left(x+\theta\left(\frac{u}{n}-x\right)\right)-f_\alpha(x)\right]d\theta. \quad (95)$$

We see that

$$|R|\leq m\int_0^1(1-\theta)^{m-1}\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right) \left(\prod_{i=1}^{N}\left|\frac{u_i}{n}-x_i\right|^{\alpha_i}\right)\left|f_\alpha\left(x+\theta\left(\frac{u}{n}-x\right)\right)-f_\alpha(x)\right|d\theta\leq$$

$$m\int_0^1(1-\theta)^{m-1}\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left|\frac{u_i}{n}-x_i\right|^{\alpha_i}\right) \omega_1\left(f_\alpha,\theta\left\|\frac{u}{n}-x\right\|_\infty\right)\right)d\theta\leq(*). \quad (96)$$

Notice here that

$$\left\|\frac{u}{n}-x\right\|_\infty\leq\frac{1}{n^\beta},\ \text{iff}\ \left|\frac{u_i}{n}-x_i\right|\leq\frac{1}{n^\beta},\ \ i=1,...,N. \quad (97)$$

We further see that

$$(*)\leq m\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)\int_0^1(1-\theta)^{m-1}\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{1}{n^\beta}\right)^{\alpha_i}\right)\right)d\theta$$

$$=\left(\frac{\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)}{(m!)\,n^{m\beta}}\right)\left(\sum_{|\alpha|=m}\frac{m!}{\prod_{i=1}^{N}\alpha_i!}\right)=$$

$$\left(\frac{\omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right)}{(m!)\, n^{m\beta}} \right) N^m. \tag{98}$$

Conclusion: When $\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$, we proved that

$$|R| \leq \left(\frac{N^m}{m! n^{m\beta}} \right) \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right). \tag{99}$$

By (95) we have that

$$|R| \leq m \left(\int_0^1 (1-\theta)^{m-1} \sum_{\left\{ \begin{array}{c} \alpha := (\alpha_1, ..., \alpha_N), \alpha_i \in \mathbb{Z}_+ \\ i = 1, ..., N, \ |\alpha| = m \end{array} \right\}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \right.$$

$$\left. \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) 2 \|f_\alpha\|_\infty \right) d\theta \right) =$$

$$2 \sum_{|\alpha| = m} \frac{1}{\prod_{i=1}^N \alpha_i!} \left(\prod_{i=1}^N \left| \frac{u_i}{n} - x_i \right|^{\alpha_i} \right) \|f_\alpha\|_\infty \leq$$

$$\left(\frac{2 \left\| \frac{u}{n} - x \right\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max}}{m!} \right) \left(\sum_{|\alpha| = m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) =$$

$$\frac{2 \left\| \frac{u}{n} - x \right\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}. \tag{100}$$

We proved in general that

$$|R| \leq \frac{2 \left\| \frac{u}{n} - x \right\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}. \tag{101}$$

Next we see that, let

$$U_n := \int_{\mathbb{R}^N} R Z (nx - u)\, du, \tag{102}$$

then

$$|U_n| \leq \int_{\mathbb{R}^N} |R|\, Z (nx - u)\, du =$$

$$\int_{\left\{ u \in \mathbb{R}^N : \|nx - u\|_\infty < n^{1-\beta} \right\}} |R|\, Z (nx - u)\, du +$$

$$\int_{\left\{ u \in \mathbb{R}^N : \|nx - u\|_\infty \geq n^{1-\beta} \right\}} |R|\, Z (nx - u)\, du \overset{\{(99), (101)\}}{\leq}$$

$$\left(\frac{N^m}{m! n^{m\beta}} \right) \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^\beta} \right) + \tag{103}$$

$$\left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \left(\int_{\left\{ u \in \mathbb{R}^N : \|nx - u\|_\infty \geq n^{1-\beta} \right\}} \left\| \frac{u}{n} - x \right\|_\infty^m Z (nx - u)\, du \right) \leq$$

$$\left(\frac{N^m}{m!n^{m\beta}}\right)\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+$$
$$\left(\frac{2\left\|f_\alpha\right\|_{\infty,m}^{\max}N^m}{n^m m!}\right)\int_{\mathbb{R}^N}\left\|nx-u\right\|_\infty^m Z\left(nx-u\right)du=$$
$$\left(\frac{N^m}{m!n^{m\beta}}\right)\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+ \tag{104}$$
$$\left(\frac{2\left\|f_\alpha\right\|_{\infty,m}^{\max}N^m}{n^m m!}\right)\int_{\mathbb{R}^N}\left\|x\right\|_\infty^m Z\left(x\right)dx\overset{(21)}{\leq}$$
$$\left(\frac{N^m}{m!n^{m\beta}}\right)\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+$$
$$\left(\frac{2\left\|f_\alpha\right\|_{\infty,m}^{\max}N^m}{n^m m!}\right)N^m\left[\left(\frac{e^\mu-1}{\left(e^\mu+1\right)\left(m+1\right)}\right)+\frac{2e^\mu m!}{\mu^m}\right]. \tag{105}$$

We have proved that

$$\left|U_n\right|\leq\left(\frac{N^m}{m!n^{m\beta}}\right)\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+$$
$$\left(\frac{2\left\|f_\alpha\right\|_{\infty,m}^{\max}N^{2m}}{n^m m!}\right)\left[\left(\frac{e^\mu-1}{\left(e^\mu+1\right)\left(m+1\right)}\right)+\frac{2e^\mu m!}{\mu^m}\right]. \tag{106}$$

Next, we estimate

$$\int_{\mathbb{R}^N}\left(\prod_{i=1}^N\left|\frac{u_i}{n}-x_i\right|^{\alpha_i}\right)Z\left(nx-u\right)du=$$
$$\prod_{i=1}^N\int_{-\infty}^{\infty}\left|\frac{u_i}{n}-x_i\right|^{\alpha_i}G\left(nx_i-u_i\right)du_i=$$
$$\frac{1}{n^j}\prod_{i=1}^N\int_{-\infty}^{\infty}\left|u_i-nx_i\right|^{\alpha_i}G\left(u_i-nx_i\right)du_i= \tag{107}$$
$$\frac{1}{n^j}\prod_{i=1}^N\left(\int_{-\infty}^{\infty}\left|x\right|^{\alpha_i}G\left(x\right)dx\right)\overset{(14)}{\leq}$$
$$\frac{1}{n^j}\left(\prod_{i=1}^N\left[\left(\frac{e^\mu-1}{\left(e^\mu+1\right)\left(\alpha_i+1\right)}\right)+\frac{2e^\mu\alpha_i!}{\mu^{\alpha_i}}\right]\right).$$

So that, it holds

$$\int_{\mathbb{R}^N}\left(\prod_{i=1}^N\left|\frac{u_i}{n}-x_i\right|^{\alpha_i}\right)Z\left(nx-u\right)du\leq$$
$$\frac{1}{n^j}\left\{\prod_{i=1}^N\left[\left(\frac{e^\mu-1}{\left(e^\mu+1\right)\left(\alpha_i+1\right)}\right)+\frac{2e^\mu\alpha_i!}{\mu^{\alpha_i}}\right]\right\}. \tag{108}$$

By (94), we can write

$$\int_{\mathbb{R}^N}f\left(\frac{u}{n}\right)Z\left(nx-u\right)du-f\left(x\right)-$$

$$\sum_{j=1}^{m} \sum_{\left\{\begin{array}{c}\alpha:=(\alpha_1,...,\alpha_N),\alpha_i\in\mathbb{Z}_+\\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=j\end{array}\right.} \left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)$$

$$\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}\left(\frac{u_i}{n}-x_i\right)^{\alpha_i}\right)Z(nx-u)\,du\right)f_\alpha(x)=\int_{\mathbb{R}^N}RZ(nx-u)\,du. \quad (109)$$

The theorem is proved. □

We continue with the activated multivariate Kantorovich operators under differentiation.

Theorem 4.8. *Let* $0<\beta<1$, $n\in\mathbb{N}:n^{1-\beta}>2$*;* $\mu>0$, $x\in\mathbb{R}^N$, $f\in C^m(\mathbb{R}^N)$, $m,N\in\mathbb{N}$*, with* $f_\alpha\in C_B(\mathbb{R}^N):|\alpha|=l$*,* $l=0,1,...,m$*. Then*

(i)

$$\left|B_n^*(f)(x)-f(x)-\sum_{j=1}^{N}\sum_{\left\{\begin{array}{c}\alpha:\\|\alpha|=j\end{array}\right.}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)f_\alpha(x)\,B_n^*\left(\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i}\right)\right)(x)\right|$$

$$\leq\frac{N^m}{m!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^m\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n}+\frac{1}{n^\beta}\right)+$$

$$\left(\frac{(2N)^m\|f_\alpha\|_{\infty,m}^{\max}}{n^m m!}\right)\left\{1+N^m\left[\left(\frac{e^\mu-1}{(e^\mu+1)(m+1)}\right)+\frac{2e^\mu m!}{\mu^m}\right]\right\}=:\Psi, \quad (110)$$

(ii) assume that $f_\alpha(x)=0$*, for all* $\alpha:|\alpha|=j$*,* $j=1,...,m$*, we have*

$$|B_n^*(f)(x)-f(x)|\leq\Psi, \quad (111)$$

with the high speed of $\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^{m+1}$,

(iii)

$$|B_n^*(f)(x)-f(x)|\leq$$

$$\sum_{j=1}^{N}\sum_{\left\{\begin{array}{c}\alpha:\\|\alpha|=j\end{array}\right.}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)|f_\alpha(x)|\left(\frac{2}{n}\right)^j$$

$$\left(\prod_{i=1}^{N}\left[1+\left(\frac{e^\mu-1}{(e^\mu+1)(\alpha_i+2)}\right)+\frac{2e^\mu(\alpha_i+1)!}{\mu^{(\alpha_i+1)}}\right]\right)+\Psi, \quad (112)$$

and

(iv)

$$\|B_n^*(f)-f\|_\infty\leq$$

$$\sum_{j=1}^{N}\sum_{\left\{\begin{array}{c}\alpha:\\|\alpha|=j\end{array}\right.}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\|f_\alpha\|_\infty\left(\frac{2}{n}\right)^j$$

$$\left(\prod_{i=1}^{N}\left[1+\left(\frac{e^{\mu}-1}{\left(e^{\mu}+1\right)\left(\alpha_i+2\right)}\right)+\frac{2e^{\mu}\left(\alpha_i+1\right)!}{\mu^{\left(\alpha_i+1\right)}}\right]\right)+\Psi. \tag{113}$$

We have that $B_n^*(f)\to f$, as $n\to\infty$, pointwise and uniformly.

Proof. It holds that

$$f\left(t+\frac{u}{n}\right)-f(x)-\sum_{j=1}^{m}\sum_{\left\{\begin{array}{c}\alpha:\\ |\alpha|=j\end{array}\right.}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(t_i+\frac{u_i}{n}-x_i\right)^{\alpha_i}\right)f_{\alpha}(x)=R, \tag{114}$$

where

$$R:=m\int_0^1(1-\theta)^{m-1}\sum_{\left\{\begin{array}{c}\alpha:\\ |\alpha|=m\end{array}\right.}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right) \tag{115}$$

$$\left(\prod_{i=1}^{N}\left(t_i+\frac{u_i}{n}-x_i\right)^{\alpha_i}\right)\left[f_{\alpha}\left(x+\theta\left(t+\frac{u}{n}-x\right)\right)-f_{\alpha}(x)\right]d\theta.$$

We see that

$$|R|\leq m\int_0^1(1-\theta)^{m-1}\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)$$

$$\left(\prod_{i=1}^{N}\left|t_i+\frac{u_i}{n}-x_i\right|^{\alpha_i}\right)\left|f_{\alpha}\left(x+\theta\left(t+\frac{u}{n}-x\right)\right)-f_{\alpha}(x)\right|d\theta\leq \tag{116}$$

$$m\int_0^1(1-\theta)^{m-1}\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left|t_i+\frac{u_i}{n}-x_i\right|^{\alpha_i}\right)\right.$$

$$\left.\omega_1\left(f_{\alpha},\theta\left\|t+\frac{u}{n}-x\right\|_{\infty}\right)\right)d\theta\leq(*)\,.$$

Notice that

$$\left\|\frac{u}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}},\ \text{iff}\ \left|\frac{u_i}{n}-x_i\right|\leq\frac{1}{n^{\beta}},\ \ i=1,...,N.$$

Here we consider $0\leq t_i\leq\frac{1}{n}$, $i=1,...,N$.

We further see that

$$(*)\leq m\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n}+\frac{1}{n^{\beta}}\right)\int_0^1(1-\theta)^{m-1}$$

$$\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{\alpha_i}\right)\right)d\theta=$$

$$\frac{\omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n}+\frac{1}{n^\beta}\right)}{m!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^m\left(\sum_{|\alpha|=m}\frac{m!}{\prod_{i=1}^N \alpha_i!}\right)=$$

$$\frac{\omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n}+\frac{1}{n^\beta}\right)}{m!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^m N^m. \tag{117}$$

Conclusion: When $\left\|\frac{u}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}$, we proved that

$$|R| \leq \frac{N^m}{m!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^m \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n}+\frac{1}{n^\beta}\right). \tag{118}$$

By (115) we have that

$$|R| \leq m\int_0^1 (1-\theta)^{m-1}\sum_{\substack{\alpha:\\|\alpha|=m}}\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)$$

$$\left(\prod_{i=1}^N\left(|t_i|+\left|\frac{u_i}{n}-x_i\right|\right)^{\alpha_i} 2\left\|f_\alpha\right\|_\infty\right)d\theta \leq$$

$$m\int_0^1 (1-\theta)^{m-1}\sum_{\substack{\alpha:\\|\alpha|=m}}\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)$$

$$\left(\prod_{i=1}^N\left(\frac{1}{n}+\left|\frac{u_i}{n}-x_i\right|\right)^{\alpha_i} 2\left\|f_\alpha\right\|_\infty\right)d\theta = \tag{119}$$

$$\sum_{\substack{\alpha:\\|\alpha|=m}}\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)\left(\prod_{i=1}^N\left(\frac{1}{n}+\left|\frac{u_i}{n}-x_i\right|\right)^{\alpha_i}\right) 2\left\|f_\alpha\right\|_\infty \leq$$

$$\left(\frac{2\left(\left\|\frac{u}{n}-x\right\|_\infty+\frac{1}{n}\right)^m\left\|f_\alpha\right\|_{\infty,m}^{\max}}{m!}\right)\left(\sum_{|\alpha|=m}\frac{m!}{\prod_{i=1}^N \alpha_i!}\right)=$$

$$\left(\frac{2\left(\left\|\frac{u}{n}-x\right\|_\infty+\frac{1}{n}\right)^m\left\|f_\alpha\right\|_{\infty,m}^{\max}}{m!}\right)N^m. \tag{120}$$

We proved in general

$$|R| \leq \frac{2\left(\left\|\frac{u}{n}-x\right\|_\infty+\frac{1}{n}\right)^m\left\|f_\alpha\right\|_{\infty,m}^{\max} N^m}{m!}. \tag{121}$$

Next, we see that

$$n^N\int_{\left[0,\frac{1}{n}\right]^N} f\left(t+\frac{u}{n}\right)dt - f(x) - \sum_{j=1}^m\sum_{\substack{\alpha:\\|\alpha|=j}}\left(\frac{1}{\prod_{i=1}^N \alpha_i!}\right)f_\alpha(x) \tag{122}$$

$$n^N \int_{\left[0,\frac{1}{n}\right]^N} \left(\prod_{i=1}^{N} \left(t_i + \frac{u_i}{n} - x_i \right)^{\alpha_i} \right) dt = n^N \int_{\left[0,\frac{1}{n}\right]^N} R dt.$$

So, when $\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$, we get

$$n^N \int_{\left[0,\frac{1}{n}\right]^N} |R|\, dt \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \tag{123}$$

And, in general it holds

$$n^N \int_{\left[0,\frac{1}{n}\right]^N} |R|\, dt \leq \tag{124}$$

$$\frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}.$$

Furthermore, it holds

$$B_n^* (f)(x) - f(x) - \sum_{j=1}^{N} \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) f_\alpha (x) B_n^* \left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i} \right) \right)(x) =$$

$$\int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} R dt \right) Z(nx - u)\, du. \tag{125}$$

Call

$$U_n := \int_{\mathbb{R}^N} \left(n^N \int_{\left[0,\frac{1}{n}\right]^N} R dt \right) Z(nx - u)\, du. \tag{126}$$

Here A_1 is as in (32), and A_2 is as in (33).

We do have, under $\left\| \frac{u}{n} - x \right\|_\infty < \frac{1}{n^\beta}$,

$$|U_n|\,|_{A_1} \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \tag{127}$$

And, furthermore we get that

$$|U_n|\,|_{A_2} \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \int_{A_2} \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m Z(nx - u)\, du. \tag{128}$$

Or, better

$$|U_n|\,|_{A_2} \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{n^m m!} \right) \int_{A_2} \left(\|u - nx\|_\infty + 1 \right)^m Z(u - nx)\, du = \tag{129}$$

$$\left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{n^m m!} \right) \left(\int_{A_2} \left(\|z\|_\infty + 1 \right)^m Z(z)\, dz \right) \leq$$

$$\left(\frac{2^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{n^m m!} \right) \left(\int_{A_2} \left(1 + \|z\|_\infty^m \right) Z(z)\, dz \right) \leq \tag{130}$$

$$\left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left[1 + \int_{A_2} \|z\|_\infty^m Z(z)\, dz \right] \overset{(21)}{\leq}$$

$$\left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m + 1)} \right) + \frac{2 e^\mu m!}{\mu^m} \right] \right\}.$$

We have proved that

$$|U_n|\,|_{A_2} \leq \left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!}\right)$$
$$\left\{1 + N^m \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m+1)}\right) + \frac{2e^\mu m!}{\mu^m}\right]\right\}. \tag{132}$$

Consequently, we derive that

$$|U_n| \leq |U_n|\,|_{A_1} + |U_n|\,|_{A_2} \leq$$
$$\frac{N^m}{m!}\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^m \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta}\right) +$$
$$\left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!}\right)\left\{1 + N^m \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m+1)}\right) + \frac{2e^\mu m!}{\mu^m}\right]\right\}. \tag{133}$$

Finally, we estimate

$$\left|B_n^*\left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}\right)\right)(x)\right| =$$
$$\left|\int_{\mathbb{R}^N}\left(n^N \int_{\left[0,\frac{1}{n}\right]^N}\left(\prod_{i=1}^N \left(t_i + \frac{u_i}{n} - x_i\right)^{\alpha_i}\right) dt\right) Z(nx - u)\, du\right| \leq \tag{134}$$
$$\int_{\mathbb{R}^N}\left(\prod_{i=1}^N \left(\frac{1}{n} + \left|\frac{u_i}{n} - x_i\right|\right)^{\alpha_i}\right) Z(nx - u)\, du =$$
$$\prod_{i=1}^N \int_{-\infty}^{\infty}\left(\frac{1}{n} + \left|\frac{u_i}{n} - x_i\right|\right)^{\alpha_i} G(nx_i - u_i)\, du_i =$$
$$\frac{1}{n^j}\left(\prod_{i=1}^N \int_{-\infty}^{\infty} (1 + |u_i - nx_i|)^{\alpha_i} G(u_i - nx_i)\, du_i\right) =$$
$$\frac{1}{n^j}\left(\prod_{i=1}^N \int_{-\infty}^{\infty} (1 + |x|)^{\alpha_i} G(x)\, dx\right) \leq$$
$$\frac{1}{n^j}\left(\prod_{i=1}^N \int_{-\infty}^{\infty} (1 + |x|)^{\alpha_i + 1} G(x)\, dx\right) \leq \tag{135}$$
$$\frac{1}{n^j}\left(\prod_{i=1}^N 2^{\alpha_i} \int_{-\infty}^{\infty}\left(1 + |x|^{\alpha_i + 1}\right) G(x)\, dx\right) =$$
$$\left(\frac{2}{n}\right)^j \left(\prod_{i=1}^N \left[1 + \int_{-\infty}^{\infty} |x|^{\alpha_i + 1} G(x)\, dx\right]\right) \overset{(14)}{\leq}$$
$$\left(\frac{2}{n}\right)^j \left(\prod_{i=1}^N \left[1 + \left(\frac{e^\mu - 1}{(e^\mu + 1)(\alpha_i + 2)}\right) + \frac{2e^\mu (\alpha_i + 1)!}{\mu^{(\alpha_i + 1)}}\right]\right). \tag{136}$$

We have derived that

$$\left|B_n^*\left(\left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}\right)\right)(x)\right| \leq$$

$$\left(\frac{2}{n}\right)^j \left(\prod_{i=1}^{N}\left[1+\left(\frac{e^{\mu}-1}{(e^{\mu}+1)(\alpha_i+2)}\right)+\frac{2e^{\mu}(\alpha_i+1)!}{\mu^{(\alpha_i+1)}}\right]\right). \tag{137}$$

The theorem is proved. □

We continue with the activated multivariate quadrature operators under differentiation.

Theorem 4.9. *Let* $0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}>2$; $\mu>0$, $x\in\mathbb{R}^N$, $f\in C^m\left(\mathbb{R}^N\right)$, $m,N\in\mathbb{N}$, *with* $f_\alpha\in C_B\left(\mathbb{R}^N\right): |\alpha|=l$, $l=0,1,...,m$. *Then*

(i)

$$\left|\overline{B_n}(f)(x)-f(x)-\sum_{j=1}^{N}\sum_{\begin{cases}\alpha:\\|\alpha|=j\end{cases}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)f_\alpha(x)\,\overline{B_n}\left(\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i}\right)\right)(x)\right|$$

$$\leq\frac{N^m}{m!}\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^m\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+$$

$$\left(\frac{(2N)^m\|f_\alpha\|_{\infty,m}^{\max}}{n^m m!}\right)\left\{1+N^m\left[\left(\frac{e^{\mu}-1}{(e^{\mu}+1)(m+1)}\right)+\frac{2e^{\mu}m!}{\mu^m}\right]\right\}=\Psi, \tag{138}$$

(ii) assume that $f_\alpha(x)=0$, *for all* $\alpha:|\alpha|=j$, $j=1,...,m$, *we have*

$$\left|\overline{B_n}(f)(x)-f(x)\right|\leq\Psi, \tag{139}$$

with the high speed $\left(\frac{1}{n}+\frac{1}{n^{\beta}}\right)^{m+1}$,

(iii)

$$\left|\overline{B_n}(f)(x)-f(x)\right|\leq$$

$$\sum_{j=1}^{N}\sum_{\begin{cases}\alpha:\\|\alpha|=j\end{cases}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)|f_\alpha(x)|\left(\frac{2}{n}\right)^j$$

$$\left(\prod_{i=1}^{N}\left[1+\left(\frac{e^{\mu}-1}{(e^{\mu}+1)(\alpha_i+2)}\right)+\frac{2e^{\mu}(\alpha_i+1)!}{\mu^{(\alpha_i+1)}}\right]\right)+\Psi, \tag{140}$$

and

(iv)

$$\left\|\overline{B_n}(f)-f\right\|_\infty\leq$$

$$\sum_{j=1}^{N}\sum_{\begin{cases}\alpha:\\|\alpha|=j\end{cases}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\|f_\alpha\|_\infty\left(\frac{2}{n}\right)^j$$

$$\left(\prod_{i=1}^{N}\left[1+\left(\frac{e^{\mu}-1}{(e^{\mu}+1)(\alpha_i+2)}\right)+\frac{2e^{\mu}(\alpha_i+1)!}{\mu^{(\alpha_i+1)}}\right]\right)+\Psi. \tag{141}$$

We have that $\overline{B_n}(f)\to f$, *as* $n\to\infty$, *pointwise and uniformly.*

Proof. We have that

$$f\left(\frac{u}{n}+\frac{r}{n\theta}\right)-f(x)-\sum_{j=1}^{m}\sum_{\substack{\alpha:\\|\alpha|=j}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{u_i}{n}+\frac{r_i}{n\theta_i}-x_i\right)^{\alpha_i}\right)f_\alpha(x)=R, \tag{142}$$

where

$$R:=m\int_0^1(1-\theta)^{m-1}\sum_{\substack{\alpha:\\|\alpha|=m}}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right) \tag{143}$$

$$\left(\prod_{i=1}^{N}\left(\frac{u_i}{n}+\frac{r_i}{n\theta_i}-x_i\right)^{\alpha_i}\right)\left[f_\alpha\left(x+\theta\left(\frac{u}{n}+\frac{r}{n\theta}-x\right)\right)-f_\alpha(x)\right]d\theta.$$

We see that

$$|R|\leq m\int_0^1(1-\theta)^{m-1}\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)$$

$$\left(\prod_{i=1}^{N}\left|\frac{u_i}{n}+\frac{r_i}{n\theta_i}-x_i\right|^{\alpha_i}\right)\left|f_\alpha\left(x+\theta\left(\frac{u}{n}+\frac{r}{n\theta}-x\right)\right)-f_\alpha(x)\right|d\theta\leq \tag{144}$$

$$m\int_0^1(1-\theta)^{m-1}\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left|\frac{u_i}{n}+\frac{r_i}{n\theta_i}-x_i\right|^{\alpha_i}\right)\right.$$

$$\left.\omega_1\left(f_\alpha,\theta\left\|\frac{u}{n}+\frac{r}{n\theta}-x\right\|_\infty\right)\right)d\theta\leq(*).$$

Notice that

$$\left\|\frac{u}{n}-x\right\|_\infty\leq\frac{1}{n^\beta},\text{ iff }\left|\frac{u_i}{n}-x_i\right|\leq\frac{1}{n^\beta},\ \ i=1,...,N.$$

We further see that

$$(*)\leq m\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n}+\frac{1}{n^\beta}\right)\int_0^1(1-\theta)^{m-1}$$

$$\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^{\alpha_i}\right)\right)d\theta=$$

$$\frac{\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n}+\frac{1}{n^\beta}\right)}{m!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^m\left(\sum_{|\alpha|=m}\frac{m!}{\prod_{i=1}^{N}\alpha_i!}\right)=$$

$$\frac{\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n}+\frac{1}{n^\beta}\right)}{m!}\left(\frac{1}{n}+\frac{1}{n^\beta}\right)^m N^m. \tag{145}$$

Conclusion: When $\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$, we proved that

$$|R| \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \tag{146}$$

By (143), we obtain that

$$|R| \leq m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} 2 \|f_\alpha\|_\infty \right) d\theta = \tag{147}$$

$$\sum_{\substack{\alpha: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} \right) 2 \|f_\alpha\|_\infty \leq$$

$$\left(\frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty,m}^{\max}}{m!} \right) \left(\sum_{|\alpha|=m} \frac{m!}{\prod_{i=1}^N \alpha_i!} \right) =$$

$$\left(\frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty,m}^{\max}}{m!} \right) N^m.$$

We have extablished in general

$$|R| \leq \frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}. \tag{148}$$

Next, we observe that

$$\sum_{r=0}^{\theta} w_r f\left(\frac{u}{n} + \frac{r}{n\theta} \right) - f(x) - \sum_{j=1}^{m} \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) f_\alpha(x) \left(\sum_{r=0}^{\theta} w_r \left(\prod_{i=1}^N \left(\frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right)^{\alpha_i} \right) \right) = \sum_{r=0}^{\theta} w_r R. \tag{149}$$

So, when $\left\| \frac{u}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}$, we get

$$\left| \sum_{r=0}^{\theta} w_r R \right| \leq \sum_{r=0}^{\theta} w_r |R| \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \tag{150}$$

And, it holds in general

$$\left| \sum_{r=0}^{\theta} w_r R \right| \leq \frac{2 \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!}. \tag{151}$$

Furthermore, it holds

$$\overline{B_n}(f)(x) - f(x) - \sum_{j=1}^{N} \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) f_\alpha(x) \overline{B_n} \left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i} \right) \right)(x) =$$

$$\int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r R \right) Z(nx-u)\, du. \tag{152}$$

Call

$$E_n := \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r R \right) Z(nx-u)\, du. \tag{153}$$

Here A_1 is as in (32), and A_2 is as in (33).

We derive, under $\left\| \frac{u}{n} - x \right\|_\infty < \frac{1}{n^\beta}$,

$$|E_n|\,|_{A_1} \leq \frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right). \tag{154}$$

And, furthermore we get that

$$|E_n|\,|_{A_2} \leq \left(\frac{2 \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \int_{A_2} \left(\left\| \frac{u}{n} - x \right\|_\infty + \frac{1}{n} \right)^m Z(nx-u)\, du. \tag{155}$$

As in the proof of Theorem 4.8, we obtain

$$|E_n|\,|_{A_2} \leq \left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right)$$
$$\left\{ 1 + N^m \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m+1)} \right) + \frac{2e^\mu m!}{\mu^m} \right] \right\}. \tag{156}$$

Consequently, we derive that

$$|E_n| \leq |E_n|\,|_{A_1} + |E_n|\,|_{A_2} \leq$$
$$\frac{N^m}{m!} \left(\frac{1}{n} + \frac{1}{n^\beta} \right)^m \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta} \right) +$$
$$\left(\frac{(2N)^m \|f_\alpha\|_{\infty,m}^{\max}}{n^m m!} \right) \left\{ 1 + N^m \left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m+1)} \right) + \frac{2e^\mu m!}{\mu^m} \right] \right\}. \tag{157}$$

At the end we estimate

$$\left| \overline{B_n} \left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i} \right) \right)(x) \right| =$$
$$\left| \int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(\prod_{i=1}^{N} \left(\frac{u_i}{n} + \frac{r_i}{n\theta_i} - x_i \right)^{\alpha_i} \right) \right) Z(nx-u)\, du \right| \leq$$
$$\int_{\mathbb{R}^N} \left(\sum_{r=0}^{\theta} w_r \left(\prod_{i=1}^{N} \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} \right) \right) Z(nx-u)\, du = \tag{158}$$
$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} \left(\left| \frac{u_i}{n} - x_i \right| + \frac{1}{n} \right)^{\alpha_i} \right) Z(nx-u)\, du$$

(..., as in the proof of Theorem 4.8)

$$\leq \left(\frac{2}{n}\right)^j \left(\prod_{i=1}^N \left[1 + \left(\frac{e^\mu - 1}{(e^\mu + 1)(\alpha_i + 2)}\right) + \frac{2e^\mu (\alpha_i + 1)!}{\mu^{(\alpha_i + 1)}}\right]\right). \tag{159}$$

The theorem is proved. □

Next comes simultaneous multivariate activated approximation.

Theorem 4.10. *Let $i \in \mathbb{N}$ be fixed, with $f \in C^{(i)}(\mathbb{R}^N)$, $N \in \mathbb{N}$. We assume that $f_\alpha \in C_B(\mathbb{R}^N)$, for $\alpha : |\alpha| = l$, $l = 0, 1, ..., i$. Here $0 < \beta < 1$, $\mu > 0$, $n\mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.*

Then

(i)

$$|(B_n(f))_\alpha(x) - f_\alpha(x)| \leq \omega_1\left(f_\alpha, \frac{1}{n^\beta}\right) + \frac{4\|f_\alpha\|_\infty}{e^{\mu(n^{1-\beta}-1)}} =: \lambda_\alpha, \tag{160}$$

and

$$\|(B_n(f))_\alpha - f_\alpha\|_\infty \leq \lambda_\alpha, \tag{161}$$

ii)

$$|(B_n^*(f))_\alpha(x) - f_\alpha(x)| \leq \omega_1\left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4\|f_\alpha\|_\infty}{e^{\mu(n^{1-\beta}-1)}} =: \rho_\alpha, \tag{162}$$

and

$$\|(B_n^*(f))_\alpha - f_\alpha\|_\infty \leq \rho_\alpha, \tag{163}$$

and

iii)

$$\left|\left(\overline{B_n}(f)\right)_\alpha(x) - f_\alpha(x)\right| \leq \omega_1\left(f_\alpha, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4\|f_\alpha\|_\infty}{e^{\mu(n^{1-\beta}-1)}} = \rho_\alpha, \tag{164}$$

and

$$\left\|\left(\overline{B_n}(f)\right)_\alpha - f_\alpha\right\|_\infty \leq \rho_\alpha. \tag{165}$$

Proof. By Theorems 4.1 - 4.3 and Remark 4. □

Next comes simultaneous global smoothness preservation.

Theorem 4.11. *Let $i \in \mathbb{N}$ be fixed, with $f \in C^{(i)}(\mathbb{R}^N)$, $N \in \mathbb{N}$. We assume that $f_\alpha \in C_B(\mathbb{R}^N) \cup C_U(\mathbb{R}^N)$, for $\alpha : |\alpha| = l$, $l = 0, 1, ..., i$.*

Then

$$\omega_1((B_n(f))_\alpha, \delta) \leq \omega_1(f_\alpha, \delta), \quad \delta > 0 \tag{166}$$

$$\omega_1((B_n^*(f))_\alpha, \delta) \leq \omega_1(f_\alpha, \delta), \tag{167}$$

and

$$\omega_1\left(\left(\overline{B_n}(f)\right)_\alpha, \delta\right) \leq \omega_1(f_\alpha, \delta). \tag{168}$$

If $f_\alpha \in C_U(\mathbb{R}^N)$, then $(B_n(f))_\alpha$, $(B_n^*(f))_\alpha$ and $\left(\overline{B_n}(f)\right)_\alpha \in C_U(\mathbb{R}^N)$.

Proof. By Theorems 4.4, 4.5, 4.6 and Remark 4. □

Under simultaneous activated multivariate extended differentiation we derive the following result.

Theorem 4.12. *Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$; $\mu > 0$, $x \in \mathbb{R}^N$, $N \in \mathbb{N}$. Let $f \in C^{(\bar{i})}(\mathbb{R}^N)$, $\bar{i} \in \mathbb{N}$; f_γ denotes a partial derivative of f, $\gamma := (\gamma_1, ..., \gamma_N)$, $\gamma_j \in \mathbb{Z}_+$,*

$j = 1, ..., N$, and $|\gamma| := \sum_{j=1}^{N} \gamma_j = r$, where $r = 0, 1, ..., \bar{i}$. We assume any $f_\gamma \in C_B(\mathbb{R}^N)$, for all $\gamma : |\gamma| = r$, $r = 0, 1, ..., \bar{i}$.

We further assume that a $f_\gamma \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, with $(f_\gamma)_\alpha \in C_B(\mathbb{R}^N) : |\alpha| = l$, $l = 0, 1, ..., m$. Then

(i)

$$\Bigg|(B_n(f))_\gamma(x) - (f_\gamma)(x) - \sum_{j=1}^{m} \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) (f_\gamma)_\alpha(x) B_n\left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}\right)\right)(x)\Bigg| \leq \left(\frac{N^m}{m! n^{m\beta}}\right) \omega_{1,m}^{\max}\left((f_\gamma)_\alpha, \frac{1}{n^\beta}\right) + \left(\frac{2\left\|(f_\gamma)_\alpha\right\|_{\infty,m}^{\max} N^{2m}}{n^m m!}\right)\left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m+1)}\right) + \frac{2e^\mu m!}{\mu^m}\right], \tag{169}$$

(ii)

$$\Bigg|(B_n^*(f))_\gamma(x) - (f_\gamma)(x) - \sum_{j=1}^{N} \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) (f_\gamma)_\alpha(x) B_n^*\left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}\right)\right)(x)\Bigg| \leq \frac{N^m}{m!}\left(\frac{1}{n} + \frac{1}{n^\beta}\right)^m \omega_{1,m}^{\max}\left((f_\gamma)_\alpha, \frac{1}{n} + \frac{1}{n^\beta}\right) + \left(\frac{(2N)^m \left\|(f_\gamma)_\alpha\right\|_{\infty,m}^{\max}}{n^m m!}\right)\left\{1 + N^m\left[\left(\frac{e^\mu - 1}{(e^\mu + 1)(m+1)}\right) + \frac{2e^\mu m!}{\mu^m}\right]\right\} =: (\Psi_\gamma)_\alpha, \tag{170}$$

and

(iii)

$$\Bigg|\left(\overline{B_n}(f)\right)_\gamma(x) - (f_\gamma)(x) - \sum_{j=1}^{N} \sum_{\substack{\alpha: \\ |\alpha|=j}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) (f_\gamma)_\alpha(x) \overline{B_n}\left(\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}\right)\right)(x)\Bigg| \leq (\Psi_\gamma)_\alpha. \tag{171}$$

Proof. By Theorems 4.7-4.9, Remark 4. □

In the final part of this work we present results related to activated iterated approximation. This is a continuation of Remarks 4-4.

Theorem 4.13. *Let $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $r \in \mathbb{N}$, $\mu > 0$, $f \in C_B(\mathbb{R}^N)$. Then*

(I)

$$\|B_n^r f - f\|_\infty \leq r\|B_n f - f\|_\infty \leq r\left[\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4\|f\|_\infty}{e^{\mu(n^{1-\beta}-1)}}\right], \tag{172}$$

(II)

$$\left\| B_n^{*^r}(f) - f \right\|_\infty \leq r \left\| B_n^* f - f \right\|_\infty \leq r \left[\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu(n^{1-\beta}-1)}} \right], \quad (173)$$

and

(III)

$$\left\| \overline{B_n}^r (f) - f \right\|_\infty \leq r \left\| \overline{B_n} f - f \right\|_\infty \leq r \left[\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu(n^{1-\beta}-1)}} \right]. \quad (174)$$

So, the speed of convergence of B_n^r, $B_n^{*^r}$, $\overline{B_n}^r$ to unit I is not worse than the speed of convergence of B_n, B_n^*, $\overline{B_n}$ to I.

Proof. By Theorems 4.1 - 4.3 and (74). ☐

We continue with the following:

Theorem 4.14. *Let $0 < \beta < 1$; $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, with $m_i^{1-\beta} > 2$, $i = 1, ..., r$; $\mu > 0$; $f \in C_B(\mathbb{R}^N)$. Then*

(I)

$$\left\| B_{m_r}\left(B_{m_{r-1}}\left(...B_{m_2}\left(B_{m_1} f\right)\right)\right) - f \right\|_\infty \leq \sum_{i=1}^r \left\| B_{m_i} f - f \right\|_\infty \leq$$

$$\sum_{i=1}^r \left[\omega_1 \left(f, \frac{1}{m_i^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu\left(m_i^{1-\beta}-1\right)}} \right] \leq r \left[\omega_1 \left(f, \frac{1}{m_1^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu\left(m_1^{1-\beta}-1\right)}} \right], \quad (175)$$

(II)

$$\left\| B_{m_r}^*\left(B_{m_{r-1}}^*\left(...B_{m_2}^*\left(B_{m_1}^* f\right)\right)\right) - f \right\|_\infty \leq \sum_{i=1}^r \left\| B_{m_i}^* f - f \right\|_\infty \leq \quad (176)$$

$$\sum_{i=1}^r \left[\omega_1 \left(f, \frac{1}{m_i} + \frac{1}{m_i^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu\left(m_i^{1-\beta}-1\right)}} \right] \leq$$

$$r \left[\omega_1 \left(f, \frac{1}{m_1} + \frac{1}{m_1^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu\left(m_1^{1-\beta}-1\right)}} \right],$$

and

(III)

$$\left\| \overline{B_{m_r}}\left(\overline{B_{m_{r-1}}}\left(...\overline{B_{m_2}}\left(\overline{B_{m_1}} f\right)\right)\right) - f \right\|_\infty \leq \sum_{i=1}^r \left\| \overline{B_{m_i}} f - f \right\|_\infty \leq$$

$$\sum_{i=1}^r \left[\omega_1 \left(f, \frac{1}{m_i} + \frac{1}{m_i^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu\left(m_i^{1-\beta}-1\right)}} \right] \leq \quad (177)$$

$$r \left[\omega_1 \left(f, \frac{1}{m_1} + \frac{1}{m_1^\beta} \right) + \frac{4 \|f\|_\infty}{e^{\mu\left(m_1^{1-\beta}-1\right)}} \right].$$

Clearly, we notice that the speed of convergence to the unit operator of the above activated multiply iterated operator is not worse the speed of operators B_{m_1}, $B_{m_1}^*$, $\overline{B_{m_1}}$ to the unit, respectively.

Proof. By Theorems 4.1- 4.3 and (76). ☐

We finish our work with multivariate simultaneous iterations.

Remark. *Let $i \in \mathbb{N}$ be fixed. Assume that $f \in C^{(i)}\left(\mathbb{R}^N\right)$, with $f_\alpha \in C_B\left(\mathbb{R}^N\right)$, with $\alpha : |\alpha| = l$, $l = 0, 1, ..., i$; $r \in \mathbb{N}$. Then, by (74), we obtain*

$$\left\| B_n^r\left(f_\alpha\right) - f_\alpha \right\|_\infty \leq r \left\| B_n\left(f_\alpha\right) - f_\alpha \right\|_\infty . \tag{178}$$

By (65) and inductively, we obtain

$$\left\| \left(B_n^r\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| \left(B_n\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty , \tag{179}$$

Similarly, we derive that

$$\left\| \left(B_n^{*^r}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| \left(B_n^*\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty , \tag{180}$$

and

$$\left\| \left(\overline{B_n}^r\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty \leq r \left\| \left(\overline{B_n}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty . \tag{181}$$

Let now $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$. Then, based on (76), we find that

$$\left\| \left(B_{m_r}\left(B_{m_{r-1}}\left(...B_{m_2}\left(B_{m_1} f\right)\right)\right)\right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^{r} \left\| \left(B_{m_{i^*}}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty . \tag{182}$$

Similarly, we get that

$$\left\| \left(B_{m_r}^*\left(B_{m_{r-1}}^*\left(...B_{m_2}^*\left(B_{m_1}^* f\right)\right)\right)\right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^{r} \left\| \left(B_{m_{i^*}}^*\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty , \tag{183}$$

and

$$\left\| \left(\overline{B_{m_r}}\left(\overline{B_{m_{r-1}}}\left(...\overline{B_{m_2}}\left(\overline{B_{m_1}} f\right)\right)\right)\right)_\alpha - f_\alpha \right\|_\infty \leq \sum_{i^*=1}^{r} \left\| \left(\overline{B_{m_{i^*}}}\left(f\right)\right)_\alpha - f_\alpha \right\|_\infty . \tag{184}$$

All the above inequalities (178)-(184) prove that our implied multivariate iterated simultaneous approximations do not have a speed worse then our basic simultaneous approximations by the activated convolution operators.

5. Conclusions

Usually, from pure mathematics, we derive techniques for applied mathematics. Here is established the reverse process as a rare event: the activation functions of neural networks are proved to have a great impact in operator theory!

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