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FUZZY FILTERS IN ORDERED SEMIRINGS

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ABSTRACT. We introduce the notion of ideal, prime ideal, filter, fuzzy ideal, fuzzy prime ideal, fuzzy filter of an ordered semiring and study their properties and relations between them. We characterize the prime ideals and filters of an ordered semiring with respect to fuzzy ideals and fuzzy filters respectively. We proved a fuzzy subset μ is a fuzzy filter of an ordered semiring M if and only if $\mu_{\beta,}^{MT} : X \to [0,1]$ is a fuzzy filter of an ordered semiring M. M and N be ordered semirings and $\phi : M \to N$ be an onto homomorphism. If f is a ϕ homomorphism invariant fuzzy filter of M then $\phi(f)$ is a fuzzy filter of N.

1. INTRODUCTION

Semiring is an algebraic structure as, a common generalization of a ring and a distributive lattice. Semiring was first introduced by American mathematician Vandiver[13] in 1934 but non trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by German mathematician Richard Dedekind in 19th century. Semiring is a universal algebra with two associative binary operations called addition and multiplication where one of them is distributive over the other. Semiring was used in the areas of theoretical computer science as well as in the solutions of graph theory and optimization theory and in particular for studying automata, coding theory and formal languages as the basic algebraic structure. Semiring theory has many applications in other branches.

The fuzzy set theory was developed by L. A. Zadeh [14] in 1965. The fuzzification of algebraic structure was introduced by A. Rosenfeld [11] and he introduced the notion of fuzzy subgroups in 1971. K.L. N. Swamy and U. M. Swamy [12] studied fuzzy prime ideals in rings in 1988. Applying the concept of fuzzy sets to the theory of ring, Y. B. Jun and C. Y. Lee [2] introduced the notion of fuzzy ideals in ring and studied the properties of fuzzy ideals of ring. D. Mandal [5] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. N. Kuroki [4] studied fuzzy interior ideals in semigroups. In 1988, Zhang [15] studied prime $L-$fuzzy ideals in rings where L is completely distributive lattice. The concept of $L-$fuzzy ideal and normal $L-$fuzzy ideal in semirings were studied by Jun, Neggers and Kim [3]. Madeline [9] studied Properties of quasi-filters of ordered semigroups. Nuttapong Wattanasiripong [10] discussed some algebraic properties

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of tripolar fuzzy pure ideals in in ordered semigroups. Ali N. A. Koam[1] defined and studied ordered quasi-Γ-ideals and ordered bi-Γ-ideals in ordered $\Gamma-$semirings by defining the relation $\leq$ in ordered $\Gamma-$semiring S as $a \leq b$ if $a + x = b$ for any $a, b, x \in S$. Mondal [8] studied the characterization of simple ordered and developed a relation between a prime ordered ideal and filter. Murali Krishna Rao [6, 7] studied the properties of fuzzy filters and T-fuzzy ideals in $\Gamma-$semirings.

In this paper, we introduce the notion of a fuzzy filter and a fuzzy prime ideal in an ordered semiring and study their properties. We prove that if $\phi : M \to N$ be an onto homomorphism and f is a ϕ homomorphism invariant fuzzy filter of an ordered semiring M then $\phi(f)$ is a fuzzy filter of an ordered semiring N, characterize the prime ideals and filters of an ordered semiring with respect to fuzzy ideals and fuzzy filters respectively.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. A set S together with two associative binary operations called addition and multiplication (denoted by + and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Example 2.2. Let M be the set of all natural numbers. Then $(M,\ max,\ min)$ is a semiring.

Definition 2.3. Let M be a semiring. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of M then M is said to be semiring with unity.

Definition 2.4. An element a of a semiring S is called a regular element if there exists an element b of S such that $a = aba$.

Definition 2.5. A semiring S is called a regular semiring if every element of S is a regular element.

Definition 2.6. An element a of a semiring S is called a multiplicatively idempotent (an additively idempotent) element if $aa = a(a + a = a)$.

Definition 2.7. An element b of a semiring M is called an inverse element of a of M if $ab = ba = 1$.

Definition 2.8. A non-empty subset A of a semiring M is called

(i) a subsemiring of M if A is an additive subsemigroup of M and $AA \subseteq A$.
(ii) a left(right) ideal of M if A is an additive subsemigroup of M and $MA \subseteq A(AM \subseteq A)$.
(iii) an ideal if A is an additive subsemigroup of $M,\ MA \subseteq A$ and $AM \subseteq A$.
(iv) a $k-$ideal if A is a subsemiring of $M, AM \subseteq A, MA \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.

Definition 2.9. A semiring M is called a division semiring if for each non-zero element of M has multiplication inverse.

Definition 2.10. A semiring M is called an ordered semiring if it admits a compatible relation $\leq$. i.e. $\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $a + c \leq b + d, c + a \leq d + b$
(ii) $ac \leq bd$
(iii) $ca \leq db$, for all $a, b, c, d \in M$

Definition 2.11. An ordered semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0x = x0 = 0,$ for all $x \in M.$

An ordered semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$

Definition 2.12. A non zero element a in an ordered semiring M is said to be a zero divisor if there exists non zero element $b \in M$, such that $ab = ba = 0.$

Definition 2.13. An ordered semiring M with unity 1 and zero element 0 is called an integral ordered semiring if it has no zero divisors.

Definition 2.14. An ordered semiring M is said to be totally ordered semiring M if any two elements of M are comparable.

Definition 2.15. In an ordered semiring M

(i) the semigroup $(M, +)$ is said to be positively ordered , if $a \leq a + b$ and $b \leq a + b,$ for all $a, b \in M.$
(ii) the semigroup $(M, +)$ is said to be negatively ordered, if $a + b \leq a$ and $a + b \leq b,$ for all $a, b \in M.$
(iii) the semigroup $(M, \cdot)$ is said to be positively ordered, if $a \leq ab$ and $b \leq ab,$ for all $a, b \in M.$
(iv) the semigroup $(M, \cdot)$ is said to be negatively ordered if $ab \leq a$ and $ab \leq b$ for all $a, b \in M.$

Definition 2.16. A non-empty subset A of an ordered semiring M is called a subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $ab \in A$ for all $a, b \in A.$

Definition 2.17. Let M be an ordered semiring. A non-empty subset I of M is called a left (right) ideal of an ordered semiring M if I is closed under addition, $MI \subseteq I$ $(IM \subseteq I)$ and if for any $a \in M,\ b \in I,\ a \leq b \Rightarrow a \in I$. I is called an ideal of M if it is both a left ideal and a right ideal of $M.$

Definition 2.18. A non-empty subset A of an ordered semiring M is called a $k-$ideal if A is an ideal and $x \in M,\ x + y \in A, y \in A$ then $x \in A.$

Definition 2.19. Let M and N be ordered semirings. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a + b) = f(a) + f(b)$
(ii) $f(ab) = f(a)f(b)$, for all $a, b \in M.$

Definition 2.20. Let M be an ordered semiring. A mapping $f : M \to M$ is called an endomorphism if

(i) f is an onto ,
(ii) $f(a + b) = f(a) + f(b),$
(iii) $f(ab) = f(a)f(b)$, for all $a, b \in M.$

Definition 2.21. Let M be a non-empty set. A mapping $f : M \to [0,1]$ is called a fuzzy subset of a semiring M. If f is not a constant function then f is called a non-empty fuzzy subset

Definition 2.22. The complement of a fuzzy subset μ of a semiring M is denoted by μ^c and is defined as $\mu^c(x) = 1 - \mu(x),$ for all $x \in M$.

Definition 2.23. Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset μ of S is called a $\phi-$invariant if $\phi(x) = \phi(y) \Rightarrow \mu(x) = \mu(y)$.

Definition 2.24. Let M be an ordered semiring. A fuzzy subset μ of M is called a fuzzy subsemiring of M if

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$, for all $x, y \in M$.

Definition 2.25. Let μ be a non-empty fuzzy subset of an ordered semiring M. Then μ is called a fuzzy left (right) ideal of M if

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) \geq \mu(y)\big(\mu(x)\big)$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y),$ for all $x, y \in M$.

3. Fuzzy filters in an ordered semiring

In this section, we introduce the notion of ideal, prime ideal, filter, fuzzy ideal, fuzzy prime ideal and fuzzy filter in an ordered semiring and we study some of their properties.

Definition 3.1. Let M be an ordered semiring. A subsemiring P of M is called a prime ideal of M if

(i) $a \leq b, a \in M, b \in P \Rightarrow a \in P$
(ii) $ab \in P, a, b \in M \Rightarrow a \in P$ or $b \in P$

Definition 3.2. Let μ be a non-empty fuzzy subset of an ordered semiring M. Then μ is called a fuzzy prime ideal of M if

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) = max\{\mu(x), \mu(y)\}$
(iii) $x \leq y \Rightarrow \mu(x) \geq \mu(y),$ for all $x, y \in M$.

Definition 3.3. Let M be an ordered semiring. A subsemiring F of M is called a filter of M if

(i) $a \leq b,\ a \in F, b \in M \Rightarrow b \in F$
(ii) $ab \in F \Rightarrow a \in F$ and $b \in F,$ for any $a, b \in M$.

Definition 3.4. Let M be an ordered semiring. A fuzzy subsemiring μ of M is called a fuzzy filter of M if

(i) $\mu(x + y) \leq \max\{\mu(x), \mu(y)\}$
(ii) $\mu(xy) = \min\{\mu(x), \mu(y)\}$
(iii) $x \leq y \Rightarrow \mu(x) \leq \mu(y),$ for all $x, y \in M$.

Example 3.5. Let $M = \{0, 1, 2, 3\}$. Then $(M, +)$ is a semigroup when $+$ is defined as $a + b = max\{a, b\}$. $(M, \cdot)$ is a semigroup when $\cdot$ is defined as cayley table:

Cayley table for the binary operation " $\cdot$ ".

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	0
2	0	2	1	0
3	0	3	0	0

Then M is an ordered semiring with respect to usual order.
(i). Define a fuzzy subset μ of M as

$$\mu(x) = \begin{cases} 0.9 & \text{if } x = 0 \\ 0.3 & \text{otherwise} \end{cases}$$

Then μ is a fuzzy prime ideal of M.
(ii). Define a fuzzy subset μ of M as

$$\mu(x) = \begin{cases} 0.4 & \text{if } x = 0 \\ 0.8 & \text{otherwise} \end{cases}$$

Then μ is a fuzzy filter of M.

Definition 3.6. Let R and M be ordered semirings. Then a mapping f from R to M is called a homomorphism of ordered semirings R and S if

(i) $f(a+b) = f(a) + (b)$
(ii) $f(ab) = f(a)f(b)$
(iii) $a \leq b \Rightarrow f(a) \leq f(b),$ for all $a, b \in R$.

Proof of the following theorems are trivial, so we omit the proofs.

Theorem 3.1. *If μ and μ' be a fuzzy subset and complement of μ respectively of an ordered semiring M then the following are equivalent for all $x, y \in M$.*

(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
$\mu(xy) \geq max\{\mu(x), \mu(y)\}$
$x \leq y \Rightarrow \mu(x) \geq \mu(y$
(ii) $\mu'(x+y) \leq max\{\mu'(x), \mu'(y)\}$
$\mu'(xy) \leq min\{\mu'(x), \mu'(y)\}$
$x \leq y \Rightarrow \mu'(x) \leq \mu'(y$

Theorem 3.2. *μ is a fuzzy filter of an ordered semiring M if and only if for any $t \in [0, 1]$, $\phi \neq \mu_t$ is a filter of an ordered semiring M.*

Theorem 3.3. *μ is a fuzzy filter of an ordered semiring M if and only if μ' is a fuzzy prime ideal of an ordered semiring M.*

Theorem 3.4. *Let M be an ordered semiring and $\phi \neq F \subseteq M$. F is a filter of M if and only if the characteristic set χ_F is a fuzzy filter of an ordered semiring M.*

Let M be an ordered semiring , $a \in M$ andμ be a fuzzy filter of M. The set $\{x \in M \mid \mu(a) \leq \mu(x)\}$ is denoted by $F_{\mu(a)}$.

Theorem 3.5. *Let M be an ordered semiring and μ be a fuzzy filter of M. Then $F_{\mu(a)}$ is a filter of an ordered semiring M.*

Proof. Let μ be a fuzzy filter of an ordered semiring M and $b, c \in F_{\mu(a)}$.

Then $\mu(a) \leq \mu(b)$ and $\mu(a) \leq \mu(c)$

$\Rightarrow \mu(a) \leq \max\{\mu(b), \mu(c)\} \leq \mu(b+c)$

$\Rightarrow \mu(a) \leq \mu(b+c)$

Therefore $b + c \in F_{\mu(a)}$.

Now $\mu(a) \leq \max\{\mu(b), \mu(c)\} = \mu(bc)$

$\Rightarrow \mu(a) \leq \mu(bc)$,

Therefore $bc \in F_{\mu(a)}$.

Suppose $bc \in F_a$ and $b, c \in F_{\mu(a)}$.

$\Rightarrow \mu(a) \leq \mu(bc) = \min\{\mu(b), \mu(c)\}$

$\Rightarrow \mu(a) \leq \mu(b)$ and $\mu(a) \leq \mu(c)$

Therefore $b, c \in F_{\mu(a)}$.

Let $x \in F_a$,$x \leq y$ and $y \in M$

$\Rightarrow \mu(a) \leq \mu(x) \leq \mu(y)$

$\Rightarrow \mu(a) \leq \mu(y)$.

Therefore $y \in F_{\mu(a)}$.

Hence $F_{\mu(a)}$ is a filter of an ordered semiring M. □

Theorem 3.6. *Let μ and γ be fuzzy filters of an ordered semiring M. Then $\mu \cap \gamma$ is a fuzzy filter of an ordered semiring M.*

Proof. Let μ and γ be fuzzy filters of an ordered semiring M, $x, y \in M$ and Then

$$\begin{aligned}
\mu \cap \gamma(x+y) &= \min\{\mu(x+y), \gamma(x+y)\} \\
&\leq \min\Big\{ \max\{\mu(x), \mu(y)\}, \max\{\gamma(x), \gamma(y)\}\Big\} \\
&= \max\Big\{ \min\{\mu(x), \gamma(x)\}, \min\{\mu(y), \gamma(y)\}\Big\} \\
&= \max\Big\{\mu \cap \gamma(x), \mu \cap \gamma(y)\Big\}
\end{aligned}$$

$$\begin{aligned}
\mu \cap \gamma(xy) &= \min\{\mu(xy), \gamma(xy), \} \\
&= \min\Big\{ \min\{\mu(x), \mu(y)\}, \min\{\gamma(x), \gamma(y)\}\Big\} \\
&= \min\Big\{ \min\{\mu(x), \gamma(x)\}, \min\{\mu(y), \gamma(y)\}\Big\} \\
&= \min\Big\{\mu \cap \gamma(x), \mu \cap \gamma(y)\Big\}
\end{aligned}$$

If $x \leq y$ then $\mu(x) \leq \mu(y)$ and $\gamma(x) \leq \gamma(y)$

$$\begin{aligned}
\Rightarrow \mu \cap \gamma(x) &= \min\Big\{\mu(x), \gamma(x)\Big\} \\
&\leq \min\Big\{\mu(y), \gamma(y)\Big\} \\
&= \mu \cap \gamma(y).
\end{aligned}$$

Hence $\mu \cap \gamma$ is a fuzzy filter of an ordered semiring M. □

Corollary 3.7. *Let μ and γ be fuzzy filters of an ordered semiring M. Then $\mu \cup \gamma$ is a fuzzy filter of an ordered semiring M.*

Definition 3.7. Let μ be a fuzzy subset of $S \times S$ and γ be a fuzzy subset of S. Then μ is said to be a fuzzy relation on γ if

$$\mu(x, y) \leq min\{\gamma(x), \gamma(y)\}, \text{ for all } x, y \in S.$$

Definition 3.8. Let γ be a fuzzy subset on a set S. Then μ_γ is said to be strongest fuzzy relation on γ if

$$\mu_\gamma(x, y) = min\{\gamma(x), \gamma(y)\}, \text{ for all } x, y \in S.$$

Theorem 3.8. *Let μ_γ be the strongest fuzzy relation on ordered semiring M. Then γ is a fuzzy filter of M if and only if μ_γ is a fuzzy filter of an ordered semiring $M \times M$.*

Proof. Let γ be a fuzzy filter of an ordered semiring M, $(x_1, x_2), (y_1, y_2) \in M \times M$. Then

$$\begin{aligned}
\mu_\gamma\big\{(x_1, x_2) + (y_1, y_2)\big\} &= \mu_\gamma\big\{(x_1 + y_1), (x_2 + y_2)\big\} \\
&= \min\big\{\gamma(x_1 + y_1), \gamma(x_2 + y_2)\big\} \\
&\leq min\Big\{max\{\gamma(x_1), \gamma(y_1)\}, maxn\{\gamma(x_2), \gamma(y_2)\}\Big\} \\
&= \min\Big\{max\{\gamma(x_1), \gamma(x_2)\}, max\{\gamma(y_1), \gamma(y_2)\}\Big\} \\
&= \max\Big\{min\{\gamma(x_1), \gamma(x_2)\}, min\{\gamma(y_1), \gamma(y_2)\}\Big\} \\
&= \max\Big\{\mu_\gamma(x_1, x_2), \mu_\gamma(y_1, y_2)\Big\}
\end{aligned}$$

$$\begin{aligned}
\mu_\gamma\big\{(x_1, x_2)(y_1, y_2)\big\} &= \mu_\gamma\big\{(x_1 y_1), (x_2 y_2)\big\} \\
&= \min\big\{\gamma(x_1 y_1), \gamma(x_2 y_2)\big\} \\
&= \min\Big\{min\{\gamma(x_1), \gamma(y_1)\}, min\{\gamma(x_2), \gamma(y_2)\}\Big\} \\
&= \min\Big\{min\{\gamma(x_1), \gamma(x_2)\}, min\{\gamma(y_1), \gamma(y_2)\}\Big\} \\
&= \min\Big\{\mu_\gamma(x_1, x_2), \mu_\gamma(y_1, y_2)\Big\}.
\end{aligned}$$

$$\begin{aligned}
\text{Suppose } (x_1, x_2) \leq (y_1, y_2) &\Rightarrow x_1 \leq y_1,\ x_2 \leq y_2 \\
&\Rightarrow \gamma(x_1) \leq \gamma(y_1), \gamma(x_2) \leq \gamma(y_2) \\
\mu_\gamma(x_1, x_2) &= min\{\gamma(x_1), \gamma(x_2)\} \\
&\leq min\{\gamma(y_1), \gamma(y_2)\} \\
&= \mu_\gamma(y_1, y_2).
\end{aligned}$$

Hence μ_γ is a fuzzy filter of an ordered semiring $M \times M$.

Conversely suppose that μ_γ is a fuzzy filter of an ordered semiring M. $(x_1, x_2), (y_1, y_2) \in M \times M$. Then

$$\begin{aligned} min\{\gamma(x_1+y_1), \gamma(x_2+y_2)\}\} &= \mu_\gamma\big(x_1+y_1), (x_2+y_2)\big\} \\ &= \mu_\gamma\big\{(x_1,x_2)+(y_1,y_2)\big\} \\ &\leq \max\Big\{\mu_\gamma(x_1,x_2), \mu_\gamma(y_1,y_2)\Big\} \\ &= \max\Big\{\min\{\gamma(x_1),\gamma(x_2)\}, min\{\gamma(y_1),\gamma(y_2)\}\Big\}. \end{aligned}$$

Now put $x_1 = x, x_2 = 1, y_1 = y, y_2 = 1$ then we get

$$\begin{aligned} \min\{\gamma(x+y),\gamma(1)\} &\leq \max\Big\{\min\{\gamma(x),\gamma(1)\}, \min\{\gamma(y),\gamma(1)\}\Big\} \\ \Rightarrow\ \gamma(x+y) &\leq \max\{\gamma(x),\gamma(y)\},\ \text{since } \gamma(x)\leq\gamma(1),\ \text{for all } \ x\in M. \end{aligned}$$

$$\begin{aligned} \min\big\{\gamma\{(x_1y_1), \gamma(x_2y_2)\}\big\} &= \mu_\gamma\big\{(x_1y_1),(x_2y_2)\big\} \\ &= \mu_\gamma\big\{(x_1,x_2)(y_1,y_2)\big\} \\ &= max\Big\{\mu_\gamma(x_1,x_2), \mu_\gamma(y_1,y_2)\Big\} \\ &= max\Big\{\min\{\gamma(x_1),\gamma(x_2)\}, \min\{\gamma(y_1),\gamma(y_2)\}\Big\}. \end{aligned}$$

Now put $x_1 = x, x_2 = 1$,$y_1 = y, y_2 = 1,$ then we get

$$\begin{aligned} \min\{\gamma(xy),\gamma(1)\} &= \max\Big\{\min\{\gamma(x),\gamma(1)\}, min\{\gamma(y),\gamma(1)\}\Big\} \\ &= \max\Big\{\gamma(x),\gamma(y)\Big\}. \end{aligned}$$

Therefore $\gamma(xy) = \max\{\gamma(x),\gamma(y)\}$.

Suppose $x \leq y, x, y \in M$. Then

$$\begin{aligned} &(x,1)\leq(y,1) \\ \Rightarrow&\mu_\gamma(x,1)\leq\mu_\gamma(y,1) \\ \Rightarrow&\min\{\gamma(x),\gamma(1)\}\leq min\{\gamma(y),\gamma(1)\} \\ \Rightarrow&\gamma(x)\leq\gamma(y). \end{aligned}$$

Hence γ is a fuzzy filter of an ordered semiring M. □

Definition 3.9. Let μ and γ be fuzzy subsets of X. The cartesian product of μ and γ is defined by

$$(\mu\times\gamma)(x,y) = \min\{\mu(x),\gamma(y)\},\ \text{for all } x,y\in X.$$

Theorem 3.9. *Let μ and γ be fuzzy filters of an ordered semiring M. Then $\mu\times\gamma$ is a fuzzy filter of an ordered semiring $M \times M$.*

Proof. Let μ and γ be fuzzy filters of an ordered semiring M and $(x_1,x_2),(y_1,y_2)\in M\times M$. Then

$$\begin{aligned}(\mu\times\gamma)\Big((x_1,x_2)+(y_1,y_2)\Big) &= \mu\times\gamma\Big(x_1+y_1,x_2+y_2\Big)\\ &= min\{\mu(x_1+y_1),\gamma(x_2+y_2)\}\\ &\le min\Big\{max\{\mu(x_1),\mu(y_1)\},\max\{\gamma(x_2),\gamma(y_2)\}\Big\}\\ &= min\Big\{max\{\mu(x_1),\gamma(x_2)\},\max\{\mu(y_1),\gamma(y_2)\}\Big\}\\ &= max\Big\{min\{\mu(x_1),\gamma(x_2)\},\min\{\mu(y_1),\gamma(y_2)\}\Big\}\\ &= max\Big\{(\mu\times\gamma)(x_1,x_2),(\mu\times\gamma)(y_1,y_2)\Big\}\end{aligned}$$

$$\begin{aligned}(\mu\times\gamma)\Big((x_1,x_2)(y_1,y_2)\Big) &= \mu\times\gamma\Big(x_1y_1,x_2y_2\Big)\\ &= \min\{\mu(x_1y_1),\gamma(x_2y_2)\}\\ &= \min\Big\{min\{\mu(x_1),\mu(y_1)\},\min\{\gamma(x_2),\gamma(y_2)\}\Big\}\\ &= \min\Big\{min\{\mu(x_1),\gamma(x_2)\},\min\{\mu(y_1),\gamma(y_2)\}\Big\}\\ &= \min\Big\{(\mu\times\gamma)(x_1x_2),(\mu\times\gamma)(y_1y_2)\Big\}\end{aligned}$$

If $(x_1,x_2)\le(y_1,y_2)$ then $x_1\le y_1$ and $x_2\le y_2$

$$\begin{aligned}(\mu\times\gamma)(x_1,x_2) &= \min\{\mu(x_1),\gamma(x_2)\}\\ &\le \min\{\mu(y_1),\gamma(y_2)\}\\ &= (\mu\times\gamma)(y_1,y_2).\end{aligned}$$

Therefore $\mu\times\gamma$ is a fuzzy filter of an ordered semiring $M\times M$. □

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Definition 3.10. Let μ be a fuzzy subset of X and $\alpha\in[0,1-\sup\{\mu(x)\mid x\in X\}]$. The mapping $\mu^T_\alpha : X\to[0,1]$ is called a fuzzy translation of μ if $\mu^T_\alpha(x)=\mu(x)+\alpha$

Definition 3.11. Let μ be a fuzzy subset of X and $\beta\in[0,1]$. Then a mapping $\mu^M_\beta : X\to[0,1]$ is called a fuzzy multiplication of μ if $\mu^M_\beta(x)=\beta\mu(x)$.

Definition 3.12. Let μ be a fuzzy subset of X and $\alpha\in[0,1-\sup\{\mu(x)\mid x\in X\}],\ \beta\in[0,1]$. Then mapping $\mu^{MT}_{\beta,\alpha} : X\to[0,1]$ is called a magnified translation of μ if $\mu^{MT}_{\beta,\alpha}(x)=\beta\mu(x)+\alpha$, for all $x\in X$.

Theorem 3.10. *A fuzzy subset μ is a fuzzy filter of an ordered semiring M if and only if $\mu^{MT}_{\beta,\alpha}$ is a fuzzy filter of an ordered semiring M*

Proof. Suppose μ is a fuzzy filter of an ordered semiring M and $x,y\in M$.

$$\begin{aligned}\mu^T_\alpha(x+y) &= \mu(x+y)+\alpha\\ &\le \max\{\mu(x),\mu(y)\}+\alpha\\ &= \max\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \max\{\mu^T_\alpha(x),\mu^T_\alpha(y)\}\end{aligned}$$

$$\mu_\alpha^T(xy) = \mu(xy) + \alpha$$
$$= \min\{\mu(x), \mu(y)\} + \alpha$$
$$= \min\{\mu(x) + \alpha, \mu(y) + \alpha\}$$
$$= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$$

Let $x \leq y$.Then $\mu(x) \leq \mu(y)$
$\Rightarrow \mu(x) + \alpha \leq \mu(y) + \alpha$
$\Rightarrow \mu_\alpha^T(x) \leq \mu_\alpha^T(y)$.

Hence μ_α^T is a fuzzy filter of an ordered semiring M.

Conversely suppose that μ_α^T is a fuzzy filter of an ordered semiring $M,\ x, y \in M$.

$$\mu(x + y) + \alpha = \mu_\alpha^T(x + y)$$
$$\leq \max\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$$
$$= \max\{\mu(x) + \alpha, \mu(y) + \alpha\}$$
$$= \max\{\mu(x), \mu(y)\} + \alpha$$

Therefore $\mu(x + y) \leq \max\{\mu(x), \mu(y)\}$.

$$\mu(xy) + \alpha = \mu_\alpha^T(xy)$$
$$= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$$
$$= \min\{\mu(x) + \alpha, \mu(y) + \alpha\}$$
$$= \min\{\mu(x), \mu(y)\} + \alpha$$

Therefore $\mu(xy) = \min\{\mu(x), \mu(y)\}$.

Let $x \leq y$.Then $\mu_\alpha^T(x) \leq \mu_\alpha^T(y)$.
$\Rightarrow \mu(x) + \alpha \leq \mu(y) + \alpha$
$\Rightarrow \mu(x) \leq \mu(y)$.

Hence μ is a fuzzy filter of an ordered semiring M. □

Theorem 3.11. *A fuzzy subset μ is a fuzzy filter of an ordered semiring M if and only if μ_β^M is a fuzzy filter of an ordered semiring M.*

Proof. Suppose μ is a fuzzy filter of an ordered semiring M and $x, y \in anM$. Then

$$\mu_\beta^M(x + y) = \beta\mu(x + y)$$
$$= \beta \max\{\mu(x), \mu(y)\}$$
$$= \max\{\beta\mu(x), \beta\mu(y)\}$$
$$= \max\{\mu_\beta^M(x), \mu_\beta^M(y)\}.$$

$$\begin{aligned}\mu_\beta^M(xy) &= \beta\mu(x\gamma y)\\ &\leq \beta\min\{\mu(x),\mu(y)\}\\ &= \min\{\beta\mu(x),\beta\mu(y)\}\\ &= \min\{\mu_\beta^M(x),\mu_\beta^M(y)\}.\end{aligned}$$

$$\begin{aligned}&\text{Let } x \leq y. \text{Then } \mu(x) \leq \mu(y)\\ &\Rightarrow \beta\mu(x) \leq \beta\mu(y)\\ &\Rightarrow \mu_\beta^M(x) \leq \beta_\beta^M\mu(y).\end{aligned}$$

Hence μ_β^M is a fuzzy filter of an ordered semiring M.

Conversely, suppose that μ_β^M is a fuzzy filter of an ordered semiring M and $x, y \in M$. Then

$$\begin{aligned}\mu_\beta^M(x+y) &\leq \max\{\mu_\beta^M(x),\mu_\beta^M(y)\}\\ \Rightarrow\ \beta\mu(x+y) &\leq \max\{\beta\mu(x),\beta\mu(y)\}\\ &= \beta\max\{\mu(x),\mu(y)\}\end{aligned}$$

$$\text{Therefore } \mu(x+y) \leq \max\{\mu(x),\mu(y)\}.$$

$$\begin{aligned}\mu_\beta^M(xy) &\leq \min\{\mu_\beta^M(x),\mu_\beta^M(y)\}\\ &= \beta\min\{\mu(x),\mu(y)\}\\ \beta\mu(xy) &= \beta\min\{\mu(x),\mu(y)\}\end{aligned}$$

$$\text{Therefore } \mu(xy) = \min\{\mu(x),\mu(y)\}.$$

$$\begin{aligned}&\text{Let } x \leq y. \text{Then } \mu_\beta^M(x) \leq \beta_\beta^M\mu(y)\\ &\Rightarrow \beta\mu(x) \leq \beta\mu(y)\\ &\Rightarrow \mu(x) \leq \mu(y).\end{aligned}$$

Hence μ is a fuzzy filter of an ordered semiring M. □

Theorem 3.12. *A fuzzy subset μ is a fuzzy filter of an ordered semiring M if and only if $\mu_\beta^{MT} : X \to [0,1]$ is a fuzzy filter of an ordered semiring M.*

Proof. Suppose μ is a fuzzy filter of an ordered semiring M.
$\Leftrightarrow$ μ_β^M is a fuzzy filter of an ordered semiring M, by Theorem 3.11
$\Leftrightarrow$ $\mu_{\beta,\alpha}^{MT}$ is a fuzzy filter of an ordered semiring M, by Theorem 3.10
□

Definition 3.13. Let a function $\phi : M \to N$ be a homomorphism of ordered semirings M, N and μ be a fuzzy subset of M. Then μ is said to be ϕ homomorphism invariant if $\phi(a) \leq \phi(b)$ then $\mu(a) \leq \mu(b)$, for $a, b \in M$.

Theorem 3.13. *Let M and N be ordered semirings and $\phi : M \to N$ be an onto homomorphism. If f is a ϕ homomorphism invariant fuzzy filter of M then $\phi(f)$ is a fuzzy filter of N.*

Proof. Let M and N be ordered semirings, $\phi : M \to N$ be an onto homomorphism, f be a homomorphism ϕ invariant fuzzy filter of M and $a \in M$. Suppose $x \in N, t \in \phi^{-1}(x)$

and $x = \phi(a)$. Then $a \in \phi^{-1}(x) \Rightarrow \phi(t) = x = \phi(a)$,

$$\begin{aligned}\phi(ab) =& \phi(a)\phi(b)\\ =& xy\\ \phi f(xy) =& f(ab)\\ =& \min\{f(a), f(b)\}\\ =& \min\{\phi(f(x)), \phi(f(y))\}\end{aligned}$$

since f is a ϕ invariant, $f(t) = f(a) \Rightarrow \phi(f)(x) = \inf_{t\in\phi^{-1}(x)} f(t) = f(a)$. Hence $\phi(f)(x) = f(a)$. Let $x, y \in N$. Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y \Rightarrow \phi(a+b) = x+y \Rightarrow \phi(f)(x+y) = f(a+b) \leq \max\{f(a), f(b)\} = \min\{\phi(f)(x), \phi(f)(y)\}$.
Let $x, y \in N$ and $x \leq y$.
Then there exist $a, b \in M$ such that $\phi(a) = x, \phi(b) = y$
and $\phi(f)(x) = f(a), \phi(f)(y) = f(b)$.

$$\begin{aligned}\text{Therefore } x \leq y \Rightarrow& \phi(a) \leq \phi(b)\\ \Rightarrow& f(a) \leq f(b)\\ \Rightarrow& \phi(f)(x) \leq \phi(f)(y).\end{aligned}$$

□

Theorem 3.14. *Let $f : M \to N$ be a homomorphism of ordered semirings and η be a fuzzy filter of N. If $\eta \circ f = \mu$ then μ is a fuzzy filter of M.*

Proof. Let $f : M \to N$ be a homomorphism of ordered semirings, η be a fuzzy filter of $N, \eta \circ f = \mu$ and $x, y \in M$.

$$\begin{aligned}\mu(x+y) &= \eta(f(x+y)) = \eta(f(x) + f(y))\\ &\leq \max\{\eta(f(x)), \eta(f(y))\}\\ &= \max\{\mu(x), \mu(y)\}\\ \mu(xy) &= \eta(f(xy)) = \eta(f(x)f(y))\\ &= \min\{\eta(f(x)), \eta(f(y))\}\\ &= \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Suppose $x, y \in M$ and $x \leq y$. Since $f : M \to N$ be a homomorphism, we have

$$\begin{aligned}& f(x) \leq f(y)\\ \Rightarrow& \eta(f(x)) \leq \eta(f(y))\\ \Rightarrow& \mu(x) \leq \mu(y)\end{aligned}$$

Hence μ is a fuzzy filter of an ordered semiring M. □

Definition 3.14. Let M and N be two ordered semirings and f be a function from M into N. If μ is a fuzzy ideal of N then the pre-image of μ under f is the fuzzy subset of M, defined by $f^{-1}(\mu)(x) = \mu(f(x))$ for all $x \in M$.

Theorem 3.15. *Let $f : M \to N$ be an onto homomorphism of ordered semirings. If μ is a fuzzy filter of N then $f^{-1}(\mu)$ is a fuzzy filter of M.*

Proof. Suppose $f : M \to N$ is an onto homomorphism of ordered semirings and μ is a fuzzy filter of N and $x_1, x_2 \in M$.

$$\begin{aligned} f^{-1}(\mu)(x_1+x_2) &= \mu(f(x_1+x_2) = \mu\big(f(x_1)+f(x_2)\big) \\ &\leq \max\{\mu(f(x_1)), \mu(f(x_2))\} = \max\{f^{-1}(\mu)(x_1)), f^{-1}(\mu)(x_2))\} \\ f^{-1}(\mu)(x_1x_2) &= \mu(f(x_1x_2)) = \min\{\mu(f(x_1)), \mu(f(x_2))\} \\ &= \min\{f^{-1}(\mu)(x_1)), f^{-1}(\mu)(x_2))\} \end{aligned}$$

$$\begin{aligned} &\text{Let } x, y \in M, \text{ and } x \leq y. \\ &\Rightarrow f(x) \leq f(y) \\ &\Rightarrow \mu(f(x)) \leq \mu(f(y)) \\ &\Rightarrow f^{-1}(\mu)(x) \leq f^{-1}(\mu)(y) \end{aligned}$$

Hence $f^{-1}(\mu)$ is a fuzzy filter of an ordered semiring M. □

4. Conclusions and/or Discussions

We introduced the notion of a fuzzy filter and a fuzzy prime ideal of an ordered semiring and studied their properties and relations between them. We characterized filters in an ordered semiring using fuzzy filters. We proved the following results: Let μ and γ be fuzzy filters of an ordered semiring M. Then $\mu \times \gamma$ is a fuzzy filter of an ordered semiring $M \times M$. Let M and N be ordered semirings and $\phi : M \to N$ be an onto homomorphism. If f is a ϕ homomorphism invariant fuzzy filter of M then $\phi(f)$ is a fuzzy filter of N. Let $f : M \to N$ be an onto homomorphism of ordered semirings. If μ is a fuzzy filter of N then $f^{-1}(\mu)$ is a fuzzy filter of M. One can extend this work by studying the other ordered algebraic structures.

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