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DIRICHLET PROBLEM WITH ROUGH BOUNDARY VALUES

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ABSTRACT. Let D be a connected bounded domain in $\mathbb{R}^n$, $n \geq 2$, S be its boundary, which is closed and smooth. Consider the Dirichlet problem

$$\Delta u = 0 \text{ in } D, \quad u|_S = f,$$

where $f \in L^1(S)$ or $f \in H^{-\ell}$, where $H^{-\ell}$ is the dual space to the Sobolev space $H^\ell := H^\ell(S)$, $\ell \geq 0$ is arbitrary.

The aim of this paper is to prove that the above problem has a solution for an arbitrary $f \in L^1(S)$ and this solution is unique and to prove similar result for rough (distributional) boundary values. These results are new. The method of its proof, based on the potential theory, is also new. Definition of the $L^1(S)$-boundary value and a distributional boundary value of a harmonic in D function is given. For $f \in L^1(S)$ the difficulty comes from the fact that the product of an $L^1(S)$ function times the kernel of the potential on S is not absolutely integrable.

We prove that an arbitrary $f \in H^{-\ell}$, $\ell > 0$, can be the boundary value of a harmonic function in D.

1. INTRODUCTION

Let D be a connected bounded domain in $\mathbb{R}^n$, $n \geq 2$, S be its boundary, which is closed, connected and smooth.

The aim of this paper is to prove that an arbitrary $f \in L^1(S)$ can be the boundary value of a harmonic in D function. The boundary function $f \in L^1(S)$ determines uniquely the harmonic function in D.

Moreover, we prove that an arbitrary $f \in H^{-\ell}$ can be the boundary value of a harmonic function in D. The $H^{-\ell}$ is the dual space to the Sobolev space $H^\ell := H^\ell(S)$, $\ell \geq 0$ is arbitrary. Therefore, the boundary value f of a harmonic function in D can be very rough, a distribution in the sense that it belongs to $H^{-\ell}$, $\ell \geq 0$ is an arbitrary positive number.

The results of this work extends the earlier author's result [9].

There is a large literature on the Dirichlet problem for the equation $\Delta u = 0$ in D going back to 1828, see references. There are three basic directions of research: non-smooth domains, non-smooth coefficients and non-smooth boundary values. This paper deals with smooth domains, simplest elliptic operator, the Laplacean, and the boundary values in $L^1(S)$ or in $H^{-\ell}$, $\ell \geq 0$. In the published papers and books the boundary values of harmonic functions were assumed to be smoother than $L^1(S)$. In [2] the boundary

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conditions in $L^1(S)$ are not considered at all. In [4] and in [5] the boundary values are in $L^p(S)$, $p > 1$. We deal with smooth domains. In this case the integral equation of the potential theory is of Fredholm type in $L^1(S)$ and the corresponding integral operator A is compact in $L^1(S)$. However, the product of an $L^1(S)$ function and the kernel of the operator of the double layer potential on S is not absolutely integrable. This brings the question:

How does one define the integral of such a product?

We give a definition of such a product in the style of distribution theory, [7], [8]. This opens a new direction in the theory of singular integral equations on the space $L^1(S)$ and on the spaces $H^{-\ell}$, $\ell \geq 0$. The author plans to work on this topic in the future.

All the functions in this paper are real-valued. This is not a restriction, because the Laplace equation has real-valued coefficients.

We first consider the case $h \in L^1(S)$ in Section 2. Our arguments in this work are based on a limiting procedure. To our knowledge, in this paper the $L^1(S)$-boundary values of harmonic functions in $D \subset \mathbb{R}^n$, $n > 2$, are considered for the first time. In Section 3 the same problem is studied using single layer potentials. In Section 4 of this paper we consider $h \in H^{-\ell}$, $\ell \geq 0$.

2. Double layer potentials

The problem we study is a classical one:

$$\Delta u = 0 \text{ in } D, \quad u|_S = f \in L^1(S). \tag{2.1}$$

This problem was studied in many papers and books for a long time for $f \in H^\mu(S)$, the space of Hoelder-continuous functions [1], and for $f \in L^p(S)$, $p > 1$, [4], [5]. The notation $H^\ell := H^\ell(S)$ will be used in Section 4 for the Sobolev spaces.

One of the methods to solve problem (2.1) is based on the potential theory. Let us look for the solution in the form of the double-layer potential

$$u(x) = \int_S \frac{\partial g(x,s)}{\partial N} h(s) ds, \quad N := N_s. \tag{2.2}$$

Here $g = \frac{1}{(n-2)\sigma_n r^{n-2}}$, $r := r_{xy} = |x - y|$, $x, y \in \mathbb{R}^n$, $A(t,s) := \frac{\partial g(t,s)}{\partial N}$, $N = N_s$ is the unit normal to S at the point s, N is directed out of D, $h = h(s)$ is the unknown function. We could assume S to be $C^{1,a}-$ smooth, $a \in (0, 1]$, but this is not important for this work.

In our case the operator

$$Ah = \int_S A(t,s) h(s) ds \tag{2.3}$$

is well defined as an operator in $L^1(S)$ and is compact in this space, see Lemma 1 below and papers [3], [11] about the compactness test in $L^1(S)$. Our arguments are essentially the same for $n \geq 2$. Therefore, without loss of generality, we assume below that $n = 3$, $g(x,y) = \frac{1}{4\pi|x-y|}$, $A(t,s) = \frac{\partial g(t,s)}{\partial N} = O(\frac{1}{|t-s|})$ for smooth S, see [10], Chapter 11.

If one looks for the solution to equation (2.1) of the form (2.2) and $f \in C^1(S)$, then the integral equation for h is:

$$Bh := -\frac{h(t)}{2} + \int_S A(t,s) h(s) ds = f \ a.e. \tag{2.4}$$

where a.e. is almost everywhere with respect to the Lebesgue measure on S. Equation (2.4) holds everywhere with respect to the Lebesgue measure on S if $f \in H^\mu(S)$. See, for example, [6], [10], where the derivation of equation (2.4) under the assumption $f \in H^\mu(S)$ is given.

It is well known that the set $C^1(S)$ is dense in $L^1(S)$ in the norm of $L^1(S)$. Equation (2.4) holds almost everywhere with respect to the Lebesgue's measure on S if $f \in L^1(S)$.

Let us recall the compactness criterion for sets in $L^1(S)$:

Proposition 1. *For a bounded set $M \subset L^1(S)$ to be compact in $L^1(S)$, it is necessary and sufficient that for an arbitrary small $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\sigma| \le \delta$ then for any $h \in M$ one has $\|h(s+\sigma) - h(s)\| < \epsilon$, where $s + \sigma \in S$.*

Here and below the norm is the $L^1(S)$ norm, $\|h\| = \int_S |h(s)|ds$.

Proofs of Proposition 1 can be found in [3], [11].

Lemma 1. *The operator A is compact in $L^1(S)$. The null-space of the operator $B = -\frac{I}{2} + A$ is trivial. Here I is the identity operator.*

Proof. Let us prove that A is bounded in $L^1(S)$. One has

$$\|Af\| = \int_S dt \Big| \int_S A(t,s) f(s) ds \Big| \le \int_S |f(s)| ds \sup_{s \in S} \int_S \frac{c}{|t-s|} dt \le c\|f\|. \tag{2.5}$$

By c we denote various constants independent of s, t.

Since S is assumed smooth, estimate $|\frac{\partial g}{\partial N}| \le \frac{c}{|t-s|}$ holds for $s, t \in S$, and the estimate

$$\sup_{s \in S} \int_S \frac{dt}{|t-s|} \le c. \tag{2.6}$$

is proved easily since the dimension of the boundary S is greater than the order of the singularity of the function we integrate. This implies estimate (2.5). So, A is bounded in $L^1(S)$.

Let us prove that A is compact in $L^1(S)$.

Denote $(Af)(t) := p(t)$. By Proposition 1, it is sufficient to check that

$$\|p(t+q) - p(t)\| \le \epsilon, \quad \text{if} \quad \|q\| \le \delta,$$

and $\delta > 0$ is sufficiently small.

This can be checked using the following estimate:

$$\|p(t+q) - p(t)\| \le \sup_{s \in S} \int_S \Big| \frac{\partial [g(t+q,s) - g(t,s)]}{\partial N} \Big| dt \|f\| := J\|f\|,$$

where J, as we prove below, is arbitrarily small provided that δ is sufficiently small.

Since the set $M := \{f\}$ is bounded, we have $\|f\| \le c$.

Let $S = S_1 \cup S_2$, where $S_1 := \{|t-s| > \epsilon\}$, $S_2 = S \setminus S_1$. Then $J \le J_1 + J_2 + J_3$, where

$$J_1 = \sup_{s \in S} \int_{S_1} \Big| \frac{\partial [g(t+q,s) - g(t,s)]}{\partial N} \Big| dt, \ J_2 = \sup_{s \in S} \int_{S_2} \Big| \frac{\partial g(t+q,s)}{\partial N} \Big| dt, \ J_3 = \sup_{s \in S} \int_{S_2} \Big| \frac{\partial g(t,s)}{\partial N} \Big| dt.$$

Choose an arbitrary small $\eta > 0$. If ϵ is sufficiently small, then $J_2 < \eta$ and $J_3 < \eta$. For a fixed $\epsilon > 0$, one can choose sufficiently small $\delta > 0$ such that $J_1 < \eta$. This is possible because J_1 is continuous with respect to q. Therefore, $J < 3\eta$. Since $\eta > 0$ is an arbitrary small number, A is compact by Proposition 1. Therefore, the operator B is Fredholm.

Let us prove that its null-space is trivial.

If this is done, then, by the Fredholm alternative, the operator B^{-1} exists, is bounded and is defined on all of $L^1(S)$. It maps $L^1(S)$ onto $L^1(S)$.

If $f \in L^1(S)$ and $Bf = 0$, that is, $Af = \frac{f}{2}$, then it follows from the equation $Af = \frac{f}{2}$ that f is smoother than $L^1(S)$, because the operator A is improving smoothness. For this reason we had assumed that S is smooth.

If f is continuous and $Bf = 0$, then $u(x) = 0$ in D by the maximum principle, so $f = 0$. Thus, the null-space of B is trivial. Therefore, the Fredholm-type operator B is boundedly invertible. Lemma 1 is proved. □

Let $f \in L^1(S)$, $f_n \in H^\mu(S)$ and $\lim_{n\to\infty} \|f_n - f\| = 0$. Then $\lim_{n\to\infty} \|h_n - h\| = 0$, where $Bh_n = f_n$, $Bh = f$. The solution u of problem (2.1) is given by formula (2.2), where $h = B^{-1}f$. Therefore, if u_n solves problem (2.1) with f_n in place of f, then

$$\|u_n - u\|_{L^1(D)} \to 0 \text{ as } n \to \infty. \tag{2.7}$$

Let us prove relation (2.7). One has

$$\|u_n - u\|_{L^1(D)} \le \|h_n - h\| \sup_{s\in S} \int_D |\frac{\partial g(x,s)}{\partial N}| dx \le c\|h_n - h\| \to 0, \quad n \to \infty, \tag{2.8}$$

where the integral is a bounded function of s since the dimension of the domain of integration is greater than the order of singularity of the function we integrate, see [10], Chapter 11.

Let us formulate our results.

Theorem 1. *Problem* (2.1) *with* $f \in L^1(S)$ *has a solution* $u \in C^\infty(D) \cap L^1(\bar{D})$. *This solution is unique. It can be obtained as* $u(x) = \lim_{n\to\infty} u_n(x)$, $u_n(x)$ *is given by formula* (2.2) *with* $h_n = B^{-1}f_n$ *in place of* h *and* $f_n \in H^\mu(S)$, $\lim_{n\to\infty} \|f_n - f\| = 0$ *is an arbitrary sequence of functions belonging to* $H^\mu(S)$ *and converging in* $L^1(S)$ *to* f.

3. Single layer potentials

Let us look for the solution to problem (2.1) of the form $u(x) = Q_x\phi := \int_S g(x,s)\phi(s)ds$. The boundary value of u as $x \to t \in S$ for smooth ϕ is $Q\phi := \int_S g(t,s)\phi(s)ds$. The operator $Q : H^\ell \to H^{\ell+1}$ is an isomorhism, see [10], p.163. Therefore, equation $Q\phi = f$ is uniquely solvable in $H^{\ell-1}$, $\phi \in H^{\ell-1}$, for any $f \in H^\ell$. We have $\phi = Q^{-1}f$, $u = Q_xQ^{-1}f$.

The case when f is rough, when it belongs to $H^{-\ell}$, $\ell > 0$, is treated in Section 4.

4. Rough boundary values

In this section the technique we develop differs from the one in Section 1. If $f \in H^{-\ell} := H^{-\ell}(S)$, $\ell > 0$, $S \in C^\infty$, we first should define the meaning of the boundary condition (2.1) for $f \in H^{-\ell}$. We look for u of the form

$$u(x) = \int_S g(x,s)\phi(s)ds := Q_x\phi, \tag{4.1}$$

where the integral is understood as a distribution $\phi \in H^{-\ell}$ acting on a infinitely smooth function $g(x,s)$, $x \notin S$. Let $t \in S$ and $x = t + \delta N_t$, where N_t is the unit normal to S, pointing into D, at the point t. The direction of N is chosen here so that the point $t + \delta N_t \in D$. If $\delta \to 0$, then the boundary values are the interior boundary values.

We define the boundary value of u as

$$\lim_{\delta\to 0}((\phi, g(t+\delta N_t, s)), \psi) = \lim_{\delta\to 0}(\phi, \int_S g(t+\delta N_t, s)\psi(t)dt) = (\phi, Q\psi), \tag{4.2}$$

where $\psi \in H^\ell$ is arbitrary, (ϕ, f) is the value of ϕ on the test function f,

$$Q\psi := \int_S g(t,s)\psi(t)dt, \tag{4.3}$$

and $(\lim_{\delta\to 0}(\phi, g(t+\delta N_t, s)), \psi) := (Q\phi, \psi)$ is the value of the distribution $Q\phi \in H^{-\ell+1}$ on an element $\psi \in H^{\ell-1}$, $\ell \ge 0$. The operator Q is an isomorphism of H^ℓ onto $H^{\ell+1}$, see [10], p. 163, for any $\ell \in (-\infty, \infty)$ if S is smooth, $S \in C^{\infty}$. We define the boundary value $Q\phi$ by the relation (4.2), that is,

$$(Q\phi, \psi) = (\phi, Q\psi), \tag{4.4}$$

where the expressions $(Q\phi, \psi)$ and $(\phi, Q\psi)$ denote, respectively, the values of the distribution $Q\phi \in H^{-\ell+1}$ on the element $\psi \in H^{\ell-1}$ and the values of the distribution $\phi \in H^{-\ell}$ on the element $Q\psi \in H^{-\ell+1}$, $\ell \geq 0$.

The problem (2.1) can be written as an equation for ϕ:

$$(\phi, Q\psi) = (f, \psi), \qquad \forall \psi \in H^{\ell},\ \ell > 0, \tag{4.5}$$

where $f \in H^{-\ell}$ is given, $\ell \geq 0$. The set $H^{\ell+1}$ is dense in the set H^{ℓ}.

Lemma 2. *In equation* (4.5) *the ϕ is uniquely defined by f.*

Proof. Suppose the contrary: ϕ_1 and ϕ_2 satisfy equation (4.5), and let $\phi := \phi_1 - \phi_2$. Then

$$(\phi, Q\psi) = 0 \qquad \forall \psi \in H^{\ell}. \tag{4.6}$$

When ψ runs through all of H^{ℓ}, then $Q\psi$ runs through all of $H^{\ell+1}$. Since $H^{\ell+1}$ is dense in H^{ℓ}, equation (4.6) implies $\phi = 0$. Consequently, $\phi_1 = \phi_2$.
Lemma 2 is proved. □

Since Q is an isomorphism of H^{ℓ} onto $H^{\ell+1}$, it follows that the unique solution ϕ to equation (4.5) exists for every $f \in H^{-\ell}$, and $f = Q\phi$. If $f \in H^{-\ell}$, then $\phi = Q^{-1}f \in H^{-\ell-1}$. Equation (4.5) can be writen as

$$(\phi, Q\psi) = (f, Q^{-1}Q\psi) = (Q^{-1}f, Q\psi) \quad \forall \psi \in H^{\ell}. \tag{4.7}$$

Recall that $(Q\phi, \psi) = (\phi, Q\psi)$, because $(\phi, \psi) = \int_S \phi\psi ds$ (and is not $\int_S \phi\overline{\psi} ds$).

When ψ runs through all of $H^{\ell-1}$, the $Q\psi$ runs through all of H^{ℓ}, $\ell \geq 0$. This and equation (4.7) imply

$$\phi = Q^{-1}f. \tag{4.8}$$

We have proved the following theorem.

Theorem 2. *For any $f \in H^{-\ell}$, $\ell \geq 0$, there exists a unique $\phi \in H^{-\ell-1}$ such that $u = Q_x\phi = \int_S g(x,s)\phi(s)ds$ solves problem* (2.1) *and $\phi = Q^{-1}f$.*

Remark 1. One can prove that $QH^{-\ell} = H^{-\ell+1}$, $\ell \geq 0$, using the fact that $QH^{\ell} = H^{\ell+1}$. The dual space for $H^{\ell+1}$ is $(H^{\ell+1})' = H^{-\ell-1}$, and $(QH^{\ell})' = (H^{\ell+1})' = Q^{-1}H^{-\ell}$. Therefore, $Q^{-1}H^{-\ell} = H^{-\ell-1}$. So, $H^{-\ell} = QH^{-\ell-1}$. This is equivalent to $QH^{-\ell} = H^{-\ell+1}$, $\ell \geq 0$.

5. Differential equations with right-hand sides

In this Section we outline the idea for treating the problem:

$$\Delta u = -f, \quad u|_S = h. \tag{5.1}$$

Here f, h are known, $f \in L^2(D)$, u is unknown. We look for u of the form

$$u = G_x f + Q_x \phi, \tag{5.2}$$

where $G_x f := \int_D g(x,y)f(y)dy$ and $Q_x\phi$ was defined in Section 3.

When $x \to t \in S$, one gets:

$$h(t) - G_t f = Q\phi, \tag{5.3}$$

and ϕ is found by the formula analogous to (4.8):

$$\phi = Q^{-1}(h(t) - G_t f). \tag{5.4}$$

The existence and continuity properties of Q^{-1} we have discussed in Remark 1 and Theorem 2.

6. CONCLUSION

The history of the Dirichlet problem goes back to 1828. The results in this paper are new. Theorem 1 yields existence and uniqueness of the solution to the Dirichlet problem with the boundary values in $L^1(S)$. It also gives a method for calculating of the solution to the Dirichlet problem with $f \in L^1(S)$ as a limit of the solutions to this problem with $f_n \in H^\mu(S)$. It is assumed that S is a smooth connected boundary of a bounded domain $D \subset \mathbb{R}^n$, $n \geq 2$.

Theorem 2 yields existence and uniqueness of the solution to problem (2.1) for any $f \in H^{-\ell}, \ell \geq 0$ and the representation of the solution as a potential $u = \int_S g(x,s)\phi(s)ds$, $\phi \in H^{-\ell+1}$, $\ell \geq 0$, $\phi = Q^{-1}f$, where Q is an isomorphism of H^ℓ onto $H^{\ell+1}$ defined by formula (4.3).

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