ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 7, Number 2 (2024), 95-99
DOI: 10.62072/acm.2024.070202
ISSN: 2582-0818

NON-HOMOGENEOUS QUINARY CUBIC EQUATION $(x^3 - y^3) = (z^3 - w^3) + 72t^2$

N. THIRUNIRAISELVI*, SHARADHA KUMAR AND M.A. GOPALAN

ABSTRACT. This article is discussed for finding non-zero different solutions in integers to the non-homogeneous cubic equation with five unknowns represented by $(x^3 - y^3) = (z^3 - w^3) + 72t^2$. Various choices of integer solutions to the above equation are obtained through employing linear transformations and simplification. Some special results based on the solutions are also discussed.

1. INTRODUCTION

As equation of degree three are numerous and have wide area for research [1-3]. For the collection of different problems , one may refer [4-23]. This problem aims in solving the quinary cubic equation $(x^3-y^3) = (z^3-w^3)+72t^2$ for various choices of integer solutions through employing linear transformations and simplification. Some special results based on the solutions are also discussed

2. METHOD OF ANALYSIS

Consider

$$(x^3 - y^3) = (z^3 - w^3) + 72t^2 \tag{2.1}$$

Taking

$$x = c + 1, y = c - 1, z = a + 1, w = a - 1, a, c \neq 0 \tag{2.2}$$

in (2.1) leads to

$$c^2 = a^2 + 12t^2 \tag{2.3}$$

Solving (2.3) through different ways for getting the values of a,c,t and using (2.2) ,one gets different sets of integer solutions to (2.1) . The above process is illustrated below :

2.1. **Way 1.** (2.3) is satisfied by

$$t = 2pq, a = 12p^2 - q^2, c = 12p^2 + q^2 \tag{2.4}$$

2020 *Mathematics Subject Classification.* 11D25.

Key words and phrases. Non-homogeneous cubic; Quinary cubic; Integer solutions.
Received: April 17, 2024. Accepted: June 13, 2024. Published: June 30, 2024.

*Corresponding author.

In view of (2.2) , one obtains

$$x = 12p^2 + q^2 + 1, y = 12p^2 + q^2 - 1,$$
$$z = 12p^2 - q^2 + 1, w = 12p^2 - q^2 - 1 \tag{2.5}$$

(2.5) and (2.4) satisfy (2.1).
Special Properties from the solution are illustrated below:

(1) Each expressions is a square multiple of 6
$y+z, x+w, x^2-y^2+z^2-w^2, 3(x^2-y^2-z^2+w^2), 6(x+y)+3(z+w)+36t$
(2) $12t^2 = x^2 - 2x - w^2 - 2w = x^2 - 2x - z^2 + 2z = y^2 + 2y - z^2 + 2z = y^2 + 2y - w^2 - 2w$
(3) $(x+y)^2 - (z+w)^2 = 48t^2$

2.2. **Way 2.** Express (2.3)as the simultaneous equations as in Table I below:

TABLE 1. Simultaneous equations

choices	I	II	III	IV	v
$c+a$	$2t^2$	$3t^2$	$4t$	$6t$	$12t$
$c-a$	6	4	$3t$	$2t$	t

Solving each choices above , we get c,a,t .
From (2.2) ,the integer solutions to (2.1) are correspondingly obtained . For brevity and simplicity , the respective solutions are given below:
Choice I
$x = k^2 + 4, y = k^2 + 2, z = k^2 - 2, w = k^2 - 4, t = k$
Choice II
$x = 6k^2 + 3, y = 6k^2 + 1, z = 6k^2 - 1, w = 6k^2 - 3, t = 2k$
Choice III
$x = 7k + 1, y = 7k - 1, z = k + 1, w = k - 1, t = 2k$
Choice IV
$x = 4k + 1, y = 4k - 1, z = 2k + 1, w = 2k - 1, t = k$
Choice V
$x = 13k + 1, y = 13k - 1, z = 11k + 1, w = 11k - 1, t = 2k$

2.3. **Way 3.** Write (2.3) as

$$a^2 + 12t^2 = c^2 * 1 \tag{2.6}$$

Assume

$$c = p^2 + 12q^2 \tag{2.7}$$

1 on the R.H.S. of (2.6) can be written as

$$1 = \frac{(2 + i\sqrt{12})(2 - i\sqrt{12})}{16} \tag{2.8}$$

Putting the values of (2.7) and (2.8) in (2.6) and simplifying , consider

$$a + i\sqrt{12}t = \frac{(2 + i\sqrt{12})(p + i\sqrt{12}q)^2}{4} \tag{2.9}$$

Evaluating the real and imaginary parts in (2.9) and changing by p by 2P,q by 2Q the corresponding integer values to a,c,t are given by

$$a = 2P^2 - 24Q^2 - 24PQ, c = 4P^2 + 48Q^2, t = P^2 - 12Q^2 + 4PQ \tag{2.10}$$

Considering (2.2), the corresponding integer solutions to (2.1) are given by

$$\begin{aligned} x = 4P^2 + 48Q^2 + 1, y = 4P^2 + 48Q^2 - 1 \\ z = 2P^2 - 24Q^2 - 24PQ + 1, w = 2P^2 - 24Q^2 - 24PQ - 1 \end{aligned} \tag{2.11}$$

Thus ,x,y,z,w,t values are given by (2.11) and (2.10) and they satisfy (2.1).
Some special fascinating relations from the solutions are presented :

(1) $\dfrac{(x+y) \pm (z+w)}{2 \pm 2t}$ is written as difference of two squares.
(2) $6(x+y) + 3(z+w) + 36t$ is a square multiple of 6
(3) $2(x+y) - (z+w+12t)$ represents area of pythagorean triangle.

Note 1
In addition to (2.8) , integer 1 in (2.6) may also be written as below

$$1 = \frac{(1 + i2\sqrt{12})(1 - i2\sqrt{12})}{49}$$

$$1 = \frac{(r^2 - 12s^2 + i\sqrt{12}rs)(r^2 - 12s^2 - i\sqrt{12}rs)}{(r^2 + 12s^2)^2}$$

Following the same procedure as above , another two distinct sets of integer solutions to (2.1) are generated.

2.4. **Way 4.** Rewrite (2.3) as

$$c^2 - 12t^2 = a^2 * 1 \tag{2.12}$$

Assume

$$a = p^2 - 12q^2 \tag{2.13}$$

Consider integer 1 from (2.12) as

$$1 = \frac{(4 + \sqrt{12})(4 - \sqrt{12})}{4} \tag{2.14}$$

Following the same method as in Way 3 , the solutions to (2.12) are as follows

$$a = 4P^2 - 12q^2, c = 8P^2 + 24q^2 + 24Pq, t = 2P^2 + 6q^2 + 8Pq \tag{2.15}$$

From (2.2) , the respective integer solutions to (2.1) are as follows

$$\begin{aligned} x = 8P^2 + 24q^2 + 24Pq + 1, y = 8P^2 + 24q^2 + 24Pq - 1, \\ z = 4P^2 - 12q^2 + 1, w = 4P^2 - 12q^2 - 1 \end{aligned} \tag{2.16}$$

Thus , the values of x,yz,w,t given by (2.16) and (2.15) satisfy (2.1).
Note 2
Other than (2.14) ,the integer 1 on the right hand side (2.12) can be written as
$1 = (7 + 2\sqrt{12})(7 - 2\sqrt{12})$
Following the above procedure , another set of different solution to (2.1) are found.

3. Conclusion

In this article different ways for finding distinct set of non-zero integer solutions are found for the considered third degree equation with five unknowns $(x^3 - y^3) = (z^3 - w^3) + 72t^2$.Researchers may attempt to find more sets of non-zero different solutions to the problem or may try to solve similar types of problem with multiple variables.

References

[1] L.E. Dickson. History of Theory of Numbers, Chelsea Publishing company, NewYork, 2(1952).

[2] L.J. Mordell. Diophantine equations. Academic press, New York, 1969.

[3] M.A. Gopalan, G. Sangeetha. On the ternary cubic Diophantine equation $y^2 = Dx^2 + z^3$, Archimedes J.Math, 1(1) (2011), 07–14.

[4] M.A. Gopalan, B. Sivakami. Integral solutions of the ternary cubic equation $4x^2 - 4xy + 6y^2 = ((k+1)^2 + 5)w^3$. Impact J.Sci.Tech, 6(1) (2012), p 15-22.

[5] M.A. Gopalan, B. Sivakami On the ternary cubic Diophantine equation $2xz = y^2(x + z)$, Bessel J.Math 2(3)(2012), 171–177.

[6] S. Vidyalakshmi, T.R. Usharani, M.A. Gopalan Integral solutions of non-homogeneous ternary cubic equation $ax^2 + by^2 = (a + b)z^3$, Diophantus J.Math 2(1)(2013), 31–38.

[7] M.A. Gopalan, K. Geetha. On the ternary cubic Diophantine equation $x^2 + y^2 - xy = z^3$, Bessel J.Math., 3(2)(2013), 119–123.

[8] M.A. Gopalan, S. Vidhyalakshmi, A.Kavitha. Observations on the ternary cubic equation $x^2 + y^2 + xy = 12z^3$, Antartica J.Math 10(5)2013, 453–460.

[9] M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi. Lattice points on the non-homogeneous cubic equation $x^3 + y^3 + z^3 + (x + y + z) = 0$, Impact J.Sci.Tech, 7(1)(2013), 21–25.

[10] M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi. Lattice points on the non-homogeneous cubic equation $x^3 + y^3 + z^3 + (x + y + z) = 0$, Impact J.Sci.Tech, 7(1)(2013), 51–55.

[11] M.A. Gopalan, S. Vidhyalakshmi, S. Mallika On the ternary non-homogenous cubic equation $x^3 + y^3 - 3(x + y) = 2(3k^2 - 2)z^3$, Impact J.Sci.Tech, 7(1)(2013), 41–45.

[12] S. Vidhyalakshmi, M.A. Gopalan, S. Aarthy Thangam On the ternary cubic Diophantine equation $4(x^2 + x)+5(y^2+2y) = -6+14z^3$, International Journal of Innovative Research and Review (JIRR), 2(3)(2014), 34–39.

[13] M.A. Gopalan, N. Thiruniraiselvi and V. Kiruthika. On the ternary cubic diophantine equation $7x^2-4y^2 = 3Z^3$, IJRSR, 6(9)(2015), 6197–6199.

[14] M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, J. Maheswari. On ternary cubic diophantine equation $3(x^2 + y^2) - 5xy + x + y + 1 = 12z^3$, International Journal of Applied Research, 1(8)(2015), 209–212.

[15] R. Anbuselvi, K. Kannaki. On ternary cubic diophantine equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$, IJSR, 5(9)(2016), 369–375.

[16] G. Janaki, C. Saranya. Integral solutions of the ternary cubic equation $3(x^2+y^2)-4xy+2(x+y+1) = 972z^3$, IRJET, 04(3)(2017), 665–66.

[17] S. Vidhyalakshmi, T.R. Usha Rani, M.A. Gopalan, V. Kiruthika On the cubic equation with four unknowns $x^3 + y^3 = 14zw^2$, IJSRP, 5(3)(2015), 1–11.

[18] M.A. Gopalan, S. Vidhyalakshmi, G. Sumathi. On the homogeneous cubic equation with four unknowns $X^3 + Y^3 = 14Z^3 - 3W^2(X + Y)$, Discovery, 2(4)(2012), 17–19.

[19] S. Vidhyalakshmi, M.A. Gopalan, A. Kavitha. Observation on homogeneous cubic equation with four unknowns$X^3 + Y^3 = 14Z^3 - 3W^2(X + Y)$, IJMER, 3(3)(2013), 1487–1492.

[20] M.A. Gopalan, S. Vidhyalakshmi, E. Premalatha, C. Nithya. On the cubic equation with four unknowns $x^3 + y^3 = 31(k^2 + 3s^2)zw^2$, IJSIMR, 2(11)(2014), 923–926.

[21] M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi. On the cubic equation with four unknowns $x^2 + 4z^3 = y^3 + 4w^3 + 6(x - y)^3$, International Journal of Mathematics Trends and Technology, 20(1)(2015), 75–84.

[22] R. Anbuselvi, K.S. Araththi. On the cubic equation with four unknowns $x^3 + y^3 = 24zw^2$, IJERA, 7(11)(2017), 01–06.

[23] A.Vijayasankar, G.Dhanalakshmi, Sharadha Kumar, M.A.Gopalan. On The Of Integral Solutions To The Cubic Equation With Four Unknowns $x^3 + y^3 + (x + y)(x - y)^2 = 16zw^2$, International Journal For Innovative Research In Multidisciplinary Field, 6(5)(2020), 337–345.

N. Thiruniraiselvi

Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Samayapuram, Trichy- 621 112, Tamil Nadu, India., ORCID ID:0000-0003-4652-3846

Email address: drntsmaths@gmail.com

Sharadha Kumar

Department of Mathematics, National College, Affiliated to Bharathidasan University, Trichy-620 001, Tamilnadu, India, ORCID ID:0000-0002-0509-6158

Email address: sharadhak12@gmail.com

M.A. GOPALAN
DEPARTMENT OF MATHEMATICS, SHRIMATI INDIRA GANDHI COLLEGE, AFFILIATED TO BHARATHIDASAN UNIVERSITY, TRICHY-620 002, TAMILNADU, INDIA, ORCID ID:0000-0003-1307-2348
Email address: mayilgopalan@gmail.com