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INVARIANT SMOOTH EXTENSIONS

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ABSTRACT. We study continuous extension operators for smooth functions from $[0, \infty)$ to $\mathbb{R}$. If $\mathcal{A}$ is a topological vector space of smooth functions in $\mathbb{R}$, let us denote by $\mathcal{A}[0, \infty)$ the space of restrictions of functions of $\mathcal{A}$ to $[0, \infty)$. We show that when $\mathcal{A}$ is any of the standard test function spaces $\mathcal{D}$, $\mathcal{S}$, or $\mathcal{K}$ then there is a continuous linear operator E from $\mathcal{A}[0, \infty)$ to $\mathcal{A}$ that satisfies that $E(\varphi)(t) = \varphi(t)$ for $t \geq 0$ and that satisfies the invariant condition $E\{\phi(\lambda x); t\} = E\{\phi(x); \lambda t\}$, for $\lambda \geq 0$. However, we show that when $\mathcal{A}$ is $\mathcal{E}$, the space of all smooth functions, then such an operator E does not exist.

1. INTRODUCTION

An interesting and much studied problem in analysis is the possibility of extending a function defined on a closed set to the whole space. Of particular interest is the case when the extension is *smooth.* The case of the extension from a closed set in $\mathbb{R}^n$ was studied in the 1930's by Whitney in his influential article [8] and several hundred papers have been written on the subject since then. For our purposes we would like to highlight the extension result for a half space given by Seeley in 1964 [7] that gives a *continuous* linear extension procedure from functions smooth in $[0, \infty) \times \mathbb{R}^n$ to smooth functions in the whole $\mathbb{R} \times \mathbb{R}^n$.

In this note we consider the following extension problem. Let $\mathcal{A}$ be a topological vector space of smooth functions in $\mathbb{R}$; $\mathcal{A}$ can be any of the standard spaces of test functions, namely, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{E}$, or $\mathcal{K}$. Let us denote by $\mathcal{A}[0, \infty)$ the space of restrictions of functions of $\mathcal{A}$ to $[0, \infty)$. Is there a continuous linear operator E from $\mathcal{A}[0, \infty)$ to $\mathcal{A}$ that satisfies that $E(\varphi)(t) = \varphi(t)$ for $t \geq 0$ and certain additional conditions?

Interestingly, imposing some very natural extra conditions implies that no such operator exists. For example, if φ is a test function in $(0, \infty)$ we can extend it to a function $\widetilde{\varphi} = E_n(\varphi)$ defined in $\mathbb{R}$ by putting $\widetilde{\varphi}(t) = 0$ for t outside of the support of φ. The operator E_n is a continuous linear operator from $\mathcal{D}(0, \infty)$ to $\mathcal{D}(\mathbb{R})$. We can extend E_n to a continuous linear operator from $C^k[0, \infty)$ to $C^k(\mathbb{R})$ for $k < \infty$, but such extension does not exist for $\mathcal{E} = C^\infty$ or for $\mathcal{D}$. In fact the transpose of E_n is the restriction of distributions to the *open* set $(0, \infty)$, and the transpose of an extension of E_n would be a restriction to the closed set $[0, \infty)$, but continuous restrictions to closed sets do not exist [3].

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Therefore we consider a milder restriction on the extension operator, namely, we seek invariant operators, those that satisfy

$$E\{\phi(\lambda x);t\} = E\{\phi(x);\lambda t\}, \qquad \lambda \geq 0. \tag{1.1}$$

We show that there are invariant continuous linear extension operators in the spaces $\mathcal{D}$, $\mathcal{S}$, or $\mathcal{K}$ but no such operators exist for the space $\mathcal{E}$ of all smooth functions in $\mathbb{R}$. We present our results in one variable, but corresponding results in a cylinder $\mathbb{R}\times M$, where M is a smooth manifold, follow immediately.

2. Smooth extensions

Our notation for spaces of smooth functions and their duals is the standard [5]; the reader is referred to this for all the necessary facts about distributions.

We start with the following result.

Proposition 2.1. *Let $f \in \mathcal{K}'(\mathbb{R})$ with $\operatorname{supp} f \subset [0,\infty)$ and moments $\{\mu_n\}_{n=0}^{\infty}$. Define the operator $R = R_f$, on $\mathcal{K}[0,\infty)$ as*

$$\Phi(t) = R\{\phi(x);t\} = \langle f(x), \phi(-xt)\rangle, \qquad t \leq 0. \tag{2.1}$$

Then

$$R : \mathcal{K}[0,\infty) \longrightarrow \mathcal{K}(-\infty,0], \tag{2.2}$$

and in fact R is a continuous operator. Furthermore,

$$\Phi^{(q)}\left(0^-\right) = (-1)^q \mu_q \phi^{(q)}(0), \qquad q \geq 0. \tag{2.3}$$

Proof. We need to prove that $\Phi(t)$ is a smooth function in $(-\infty,0]$. The fact that all derivatives exist when $t<0$ is clear because differentiation of distributional evaluations is valid,

$$\Phi^{(q)}(t) = \left\langle f(x), (-x)^q \phi^{(q)}(-xt)\right\rangle, \quad t<0. \tag{2.4}$$

On the other hand, the moment asymptotic expansion [5, Chp. 3] gives the strong development

$$\Phi^{(q)}(t) = \sum_{n=0}^{N} \frac{(-1)^{n+q}\mu_n \phi^{(n+q)}(0)}{n!} t^n + O\left(|t|^{N+1}\right), \tag{2.5}$$

as $t \to 0^-$. Therefore, $\Phi^{(q)}(0^-)$ exists for all q and (2.3) holds.

The fact that $\phi \in \mathcal{K}[0,\infty)$ and (2.4) give that Φ satisfies the same sort of bounds as $t \to -\infty$, and this yields that $\Phi \in \mathcal{K}(-\infty,0]$ and the continuity of the operator. □

In particular if the moments of f satisfy

$$\mu_n = (-1)^n, \tag{2.6}$$

then the function Ψ given by

$$\Psi(t) = \begin{cases} \phi(t), & t \geq 0, \\ \Phi(t), & t \leq 0, \end{cases} \tag{2.7}$$

provides a smooth extension of ϕ to the whole real line.

The existence of distributions $f \in \mathcal{K}'(\mathbb{R})$ with $\operatorname{supp} f \subset [0,\infty)$ that satisfy (2.6) follows from the more general result [1, 2] that for *any* arbitrary sequence $\{\mu_n\}_{n=0}^{\infty}$ of complex numbers there exist smooth functions $\varphi \in \mathcal{S}(0,\infty)$ with

$$\int_0^{\infty} \varphi(t)\, t^n dt = \mu_n, \quad n \geq 0. \tag{2.8}$$

Of course, $\mathcal{S}(0,\infty)$ is a subspace of $\mathcal{K}'(\mathbb{R})$, so that existence in this latter space follows.

Proposition 2.2. *Suppose* $f\in\mathcal{K}'(\mathbb{R})$ *with* $\operatorname{supp} f\subset[0,\infty)$ *has moments* $\mu_n=(-1)^n$. *Then the operator* E *that sends* ϕ *to* Ψ *is a continuous extension operator from* $\mathcal{K}[0,\infty)$ *to* $\mathcal{K}(\mathbb{R})$.

Proof. Follows at once from our analysis. ☐

Notice also that both the operator R and E are *invariant* with respect to dilations, namely

$$R\{\phi(\lambda x);t\}=R\{\phi(x);\lambda t\}, \qquad \lambda\geq 0, \tag{2.9}$$
$$E\{\phi(\lambda x);t\}=E\{\phi(x);\lambda t\}, \qquad \lambda\geq 0. \tag{2.10}$$

3. Other spaces

If we choose f in an appropriate way then E will not only send $\mathcal{K}[0,\infty)$ to $\mathcal{K}(\mathbb{R})$ but it will also send $\mathcal{S}[0,\infty)$ to $\mathcal{S}(\mathbb{R})$ or maybe $\mathcal{D}[0,\infty)$ to $\mathcal{D}(\mathbb{R})$.

Proposition 3.1. *Suppose* $f\in\mathcal{K}'(\mathbb{R})$ *with* $\operatorname{supp} f\subset[0,\infty)$ *has moments* $\mu_n=(-1)^n$. *If all the distributional values*[1] $f^{(n)}(0)\ (\mathfrak{dist})$ *exist and vanish then the operator* E *sends* $\mathcal{S}[0,\infty)$ *to* $\mathcal{S}(\mathbb{R})$.

Proof. If $\phi\in\mathcal{S}[0,\infty)$ then the asymptotic expansion

$$\langle f(x),\phi(-xt)\rangle=t^{-1}\left\langle f\left(t^{-1}x\right),\phi(t)\right\rangle=o\left(|t|^{-N}\right), \quad \text{as } t\to\infty, \tag{3.1}$$

follows because $f^{(n)}(0)=0$ for all n. ☐

The existence of such $f\in\mathcal{K}'(\mathbb{R})$ with null distributional values for all derivatives at zero actually follows because the moment problem (2.8) has solutions in $\mathcal{S}(a,\infty)$ for any $a\in\mathbb{R}$. This is a stronger condition that yields the ensuing.

Proposition 3.2. *Suppose* $f\in\mathcal{K}'(\mathbb{R})$ *has moments* $\mu_n=(-1)^n$ *.If* $\operatorname{supp} f\subset[a,\infty)$ *for some* $a>0$ *then* E *sends* $\mathcal{D}[0,\infty)$ *to* $\mathcal{D}(\mathbb{R})$.

Proof. In fact, if $\operatorname{supp}\phi\subset[0,b]$ then $\langle f(x),\phi(-xt)\rangle=0$ if $|t|>b/a$. ☐

4. A converse

We also have a converse result.

Proposition 4.1. *Let* $E:\mathcal{K}[0,\infty)\longrightarrow\mathcal{K}(\mathbb{R})$ *be a continuous extension operator that satisfies the invariance condition (2.9). Then there exists* $f\in\mathcal{K}'(\mathbb{R})$ *with* $\operatorname{supp} f\subset[0,\infty)$ *and moments* $\mu_n=(-1)^n$, *such that (2.7) holds, where* R *is given by (2.1).*

Proof. Indeed, let R be the restriction of E to $(-\infty,0]$ and let

$$f(x)=R^t\{\delta(t+1);-x\}. \tag{4.1}$$

so that $f\in\mathcal{K}'(\mathbb{R})$ with $\operatorname{supp} f\subset[0,\infty)$ and moments $\mu_n=(-1)^n$. Also if $\lambda\geq 0$,

$$\begin{aligned}\langle f(x),\phi(-\lambda x)\rangle&=\left\langle R^t\{\delta(t+1);-x\},\phi(-\lambda x)\right\rangle\\&=\langle\delta(t+1),R\{\phi(\lambda x);t\}\rangle\\&=\langle\delta(t+1),R\{\phi(x);\lambda t\}\rangle\\&=R\{\phi(x);\lambda\},\end{aligned}$$

[1] Distributional point values were introduced by Łojasiewicz [6]. See [5] for their properties.

as required. □

It is possible to find invariant extension operators for the spaces $\mathcal{K}$, $\mathcal{S}$, and $\mathcal{D}$. Is it possible to find them for all the typical spaces of test functions? From the Proposition 4.1 we immediately obtain the ensuing negative result.

Proposition 4.2. *There are no extension operators from* $\mathcal{E}[0,\infty)$ *to* $\mathcal{E}(\mathbb{R})$ *that satisfy the invariance condition (2.9).*

Proof. If such an operator exists then $f \in \mathcal{E}'[0,\infty)$ would have moments $\mu_n = (-1)^n$, and that is impossible because there is at most one solution of any moment problem in $\mathcal{E}'(\mathbb{R})$ [4] and the one with moments $(-1)^n$ is $\delta(x+1)$, whose support is $\{-1\}$. □

The construction of [7] gives a continuous linear extension from $\mathcal{E}[0,\infty)$ to $\mathcal{E}(\mathbb{R})$, but that operator is not invariant with respect to dilations.

5. Conclusions

We have established the intrinsic connection between the existence of continuous invariant extension operators E from a space $\mathcal{A}[0,\infty)$ to $\mathcal{A}(\mathbb{R})$ and the existence of solutions of certain moment problems in the space $\mathcal{A}'[0,\infty)$. In particular, such operators exist for some standard spaces of test functions, namely $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{K}$, but they do not exist for the space $\mathcal{E}$ of all smooth functions in $\mathbb{R}$.

These results promise to have interesting consequences to study the extension problems for sectionally analytic functions and the solution of Wiener-Hopf integral equations in spaces of distributions, because such problems and the ones studied presently are related by the use of the holomorphic Fourier transform.

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