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SOME NEW RESULTS ON JU-ALGEBRAS

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ABSTRACT. JU-algebra (or weak KU-algebra) is a generalization of a KU-algebra. The concepts of atoms in JU-algebras are introduced and their important basic properties are registered. In addition to the previous one, an extension of a (weak) KU-algebra A to the JU-algebra $A \cup \{w\}$ was designed.

1. INTRODUCTION

The study of BCK/BCI-algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic. There exist several generalizations of BCK/BCI-algebras, as such BCH–algebras, dual BCK–algebras, dual BCI–algebras, etc. Y. Komori introduced a notion of BCC-algebras which is a generalization of a notion of BCK-algebras. The concept of weak BCC-algebras was introduced in 1991 by R. F. Ye as one weakening of the BCC-algebra.

In 2011 W. A. Dudek, B. Karamdin and S. Ali Bhatti ([5]) for a given (weak) BCC-algebra designed its extension to a weak BCC-algebra. The concept of atoms in logical algebras has been studied by many researchers (for example, [3, 4, 5, 7, 9, 15]). Atoms in BCI-algebras were discussed in [9] by J. Meng and X. L. Xin in 1992. W. A. Dudek and X. Zhanh in 1995 in [3] analyzed the properties of atoms in BCC-algebras. Atoms in weak BCC-algebras was studied by W. A. Dudek, X. H. Zhang and Y. Q. Wang in 2009 ([4]).

KU-algebra was introduced in 2009 in article [11] by C. Prabpayak and U. Leerawat. One generalization of this algebra, called 'pseudo KU-algebra', was introduced in 2015 in [8] by the same authors. Since then, this generalization has been the focus of interest of researchers (for example, see [1, 2, 12, 13, 14]). In the articles [1, 2], the pseudo KU-algebra was renamed to JU-algebra. This name was also used by the author of the articles [13, 14] in which ideals and filters in that logical algebra were discussed.

As is usual in researches of logical algebras, and in this case, in the case of JU-algebras, the researchers try to recognize the crucial properties of these algebraic structures. By standard, they try to register the properties of the corresponding algebraic structure, both of all substructures in them and of their prominent elements.

This article is a continuation of those research. It is designed as follows:

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In the Preliminaries section, some similarities and differences between KU-algebras and JU-algebras are noted. In addition to the previous one, three examples of JU-algebras that are not KU-algebras were constructed.

Section 3 is the main part of the paper and, according to the author's deep conviction, the results presented in this section are novelties that cannot be found in previously published texts covering the domain of JU-algebra.

In the first subsection, the concept of atoms in JU-algebras (Definition 3.1) is introduced and some properties of this concept are shown. Five examples are given that illustrate the properties of atoms in this class of logical algebras. Two criteria were found that enable the recognition of atoms in JU-algebras. One of them (Theorem 3.2) is specific for this type of logical algebras.

The second subsection describes the design of a new JU-algebra by extension of a given KU-algebra (Theorem 3.11), i.e. by an extension of a given JU-algebra (Theorem 3.13). Two examples are given that illustrate the mentioned extensions.

2. Preliminaries

At the beginning of this section, let us inform the reader that logical functions are written and used literally in this paper. Also, all formulas in this paper are written according to the rules of writing formulas in Mathematical Logic. Therefore, all formulas are closed by some quantifier or they should be seen as free formulas.

The concept of JU-algebra as a generalization of KU-algebra, introduced in [11], was determined for the first time in 2015 under the name 'pseudo KU-algebra' ([8]) and reformulated in 2019 into the name JU-algebra ([1, 2]). This researcher also contributed to the research of JU-algebras ([13, 14]).

Definition 2.1. ([2], Definition 2.1) An algebra $(A, \cdot, 1)$ of type $(2, 0)$ with a binary operation "$\cdot$" and a fixed element 1 is said to be JU-algebra if it satisfies the following conditions:

(JU-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((z \cdot x) \cdot (y \cdot x)) = 1)$,

(JU-2) $(\forall x \in A)(1 \cdot x = x)$ and

(JU-3) $(\forall x, y \in A)((x \cdot y = 1 \wedge y \cdot x = 1) \implies x = y)$.

We denote this axiom system by JU.

Remark. *In* [11]*, a KU-algebra is defined as a system* $(A, \cdot, 0)$ *satisfying the following axioms*

(KU-1) $(\forall x, y, z \in A)((x \cdot y) \cdot ((y \cdot z) \cdot (x \cdot z)) = 0)$*,*

(KU-1) $(\forall x \in A)(0 \cdot x = x)$*,*

(KU-3) $(\forall x \in A)(x \cdot 0 = 0)$ *and*

(KU-4) $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

We denote this axiom system by KU. With wKU we denote the axiomatic system KU without axiom (KU-3). If we followed the formation of the concept of 'weak BCC-algebras' from the 'concept of BCC-algebras', then the name 'weak KU-algebra' *could also be used for a JU-algebra by analogy with the previous one. So, JU $\equiv$ wKU $\equiv$ PKU, where the pseudo KU-algebra is denoted by PKU.*

If in the definition of KU-algebras we write 1 *instead of* 0*, then we see that any KU-algebra A is a JU-algebra. Therefore, the concept of JU-algebras is a generalization of the concept of KU-algebras (*[2]*, pp. 136).*

Example 2.2. Let $A = \{1, 2, 3, 4\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	1	2
3	1	3	1	2
4	1	1	1	1

Then $(A, \cdot, 1)$ is a KU-algebra.

By a subalgebra of a JU-algebra A, we mean a non-empty subset S of A which satisfies the condition

$$(\forall\, x, y \in A)((x \in S \wedge y \in S) \implies x \cdot y \in S).$$

Definition 2.3. ([2]) Let A be a JU-algebra. We define a relation $\leqslant$ in A

$$(\forall x, y \in A)(y \leqslant x \iff x \cdot y = 1).$$

According to Lemmas 2.2 and 2.3 in [2], the relation $\leq$ is a partial order in A, left compatible

(a) $(\forall x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$

and right inverse compatible

(b) $(\forall x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$

with the internal operation in A.

If A is a JU-algebra, let us define $\varphi : A \longrightarrow A$ by $(\forall x \in A)(\varphi(x) = x \cdot 1)$. According to (JU-2), the equality $\varphi(1) = 1$ is valid for mapping φ.

The following proposition gives some of the important properties of the JU-algebra.

Proposition 2.1 ([13], Proposition 1, Lemma 3, Proposition 4)**.** *Let A be a JU-algebra. Then:*

(1) $(\forall\, x, y \in A)(x \cdot \varphi(y) = y \cdot \varphi(x))$,
(2) $(\forall\, x, y \in A)(\varphi(x) \cdot \varphi(y) \leqslant y \cdot x)$,
(3) $(\forall\, x, y \in A)(x \cdot (y \cdot x) \leqslant \varphi(y))$,
(4) $(\forall\, x, y \in A)(\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y))$,
(5) $(\forall\, x, y \in A)(x \leqslant y \implies \varphi(x) = \varphi(y))$,
(6) $(\forall\, x \in A)(\varphi^2(x) \leqslant x)$,
(7) $(\forall\, x \in A)(\varphi^3(x) = \varphi(x))$,
(8) $(\forall\, x, y \in A)(\varphi^2(x \cdot y) = \varphi(y \cdot x))$.

It is obvious that the set $Ker(\varphi) =: \{x \in A : \varphi(x) = 1\}$ is a sub-algebra of the JU-algebra A. Indeed, for $x, y \in Ker(\varphi)$ we have $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = 1$, according to Proposition 2.1(4). So, $Ker(\varphi)$ is a KU-algebra. Therefore, every JU-algebra has at least one KU-subalgebra.

Example 2.4. The algebraic construction described in Example 1 in article [2], or Example 2.1 in article [10] is an example of JU-algebra. In paper [12], the algebraic construction $(X, \cdot, 0)$ described in Example (or Example 2 in [2]), is a JU-algebra but not a KU-algebra.

(1) Let $A = \{1, 2, 3, 4\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra.

(2) Let $A = \{1, a, b\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	a	b
1	1	a	b
a	1	1	b
b	b	b	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra.

(3) Let $A = \{1, 2, 3, 4, 5, 6\}$ and define binary operation $\cdot$ as follows:

$\cdot$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	5
6	1	1	2	1	5	1

Then $(A, \cdot, 1)$ is a JU-algebra but it is not a KU-algebra.

Definition 2.5. ([13], Definition 3.1) A non-empty subset J of a JU-algebra A is called a JU-ideal of A if

(J1) $1 \in J$ and

(J2) $(\forall x, y \in A)((x \cdot y \in J \ \wedge \ x \in J) \implies y \in F)$.

Let J be a JU-ideal of a JU-algebra A. Then

(J3) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \in J) \implies x \in J)$.

Example 2.6. Let A be as in Example 2.4(3). The order relation $\leqslant$ in this JU-algebra A is given by $\leqslant = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (1,3), (1,4), (1,6), (2,3), (2,4), (2,6), (3,4), (3,6), (4,6)\}$. The subsets $J_1 = \{1\}$, $J_2 = \{1,2\}$, $J_4 = \{1,2,3,4\}$ and $J_6 = \{1,2,3,4,6\}$ are JU-ideals in A. The subset $K = \{1,2,3\}$ is not a JU-ideal in A because, for example, $3 \in K$ and $3 \cdot 4 = 2 \in K$ but $4 \notin K$ hold. The subset $L = \{1,5\}$ is also not a JU-ideal in A because, for example, $5 \in L$ and $5 \cdot 2 = 5 \in L$ but $2 \notin L$ hold. $\square$

Definition 2.7. ([14], Definition 3.1) A non-empty subset F of a JU-algebra A is called a JU-filter of A if

(F1) $1 \in F$ and

(F2) $(\forall x, y \in A)((x \cdot y \in F \ \wedge \ y \in F) \implies x \in F)$.

Let F be a JU-filter of a JU-algebra A. Then

(F3) $(\forall x, y \in A)((x \leqslant y \ \wedge \ x \in F) \implies y \in F)$.

Example 2.8. Let A be as in Example 2.4(1). The order relation $\leqslant$ in this JU-algebra A is given by

$$\leqslant = \{(1,1), (2,2), (3,3), (4,4), (1,3), (1,4), (3,4)\}.$$

The set $F =: \{1,3,4\}$ is a JU-filter in A.

In order to understand the properties of JU-algebras, some specific properties of this logical structure are listed here. For consistency of exposition, we give proofs of these lemmas since our evidence is somewhat different than it is in the sources from where it was taken.

Lemma 2.2 ([1], Lemma 4(J_{13}))**.** *If in a JU-algebra A holds*

$$(\forall\, x, y \in A)((x \cdot y) \cdot y = 1),$$

then A is a KU-algebra.

Proof. We need to prove that

(9) $(\forall\, x \in A)(x \cdot 1 = 1)$

is a valid formula. If we put $y = 1$, $z = x$ and $x = 1$ in (JU-1), we get

$(1 \cdot x) \cdot ((x \cdot 1) \cdot (1 \cdot 1)) = 1 \iff x \cdot ((x \cdot 1) \cdot 1) = 1$ according to (JU-2)

$\iff x \cdot 1 = 1$ using the hypothesis of this lemma. □

Remark. *First, let us recall that in a KU-algebra A holds $(\forall\, x \in A)(\varphi(x) = 1)$ and, therefore, $(\forall\, x \in A)(\varphi^2(x) = 1)$. On the other hand, it follows from the previous lemma that if $(\forall\, x \in A)(\varphi^2(x) = 1)$ holds in a JU-algebra A, then A is a KU-algebra. Therefore, a JU-algebra A is a KU-algebra if and only if $(\forall\, x \in A)(\varphi^2(x) = 1)$ is a valid formula.*

Lemma 2.3 ([6], Proposition 1.7)**.** *If A is a KU-algebra, then the following properties holds:*

(10) $(\forall\, x, y \in A)(x \cdot (y \cdot x) = 1)$

(11) $(\forall\, x, y \in A)((y \cdot x) \cdot x = 1 \iff x = y \cdot x)$.

Proof. If we put $z = 1$ in (JU-1), we get $(y \cdot 1) \cdot ((1 \cdot x) \cdot (y \cdot x)) = 1$. From here, taking into account (9) and (JU-2), we get $x \cdot (y \cdot x) = 1$.

Let $(y \cdot x) \cdot x = 1$ be valid. From here and from (10), we get $x = y \cdot x$ by (JU-3). This proves the implication of $(y \cdot x) \cdot x = 1 \implies x = y \cdot x$. The validity of the reverse implication is based on the validity of the formula $(\forall\, u \in A)(u \cdot u = 1)$ in any JU-algebra A. □

One important specificity of the JU-algebra is the relation between the sub-algebra $Ker(\varphi)$ and any filter in it.

Proposition 2.4. *For each JU-filter F in a JU-algebra A the following inclusion $Ker(\varphi) \subseteq F$ is valid.*

Proof. Let F be a filter in a JU-algebra A and let $x \in Ker(\varphi)$ be an arbitrary element. This means $x \cdot 1 = 1 \in F$. Thus $x \in F$ according (F2) since $1 \in F$. Hence $Ker(\varphi) \subseteq F$. □

3. THE MAIN RESULTS

This section is the central part of the paper. It consists of two subsections. In the first of them, the concept of atoms in this class of logical algebras is determined. Unlike the usual way (Proposition 3.1) of recognizing atoms in logical algebras, in the case of JU-algebras we have one specificity (Theorem 3.2) that is not found in other types of algebras. In the second part, both the extension of the KU-algebra to the JU-algebra and the extension of the JU-algebra to the JU-algebra are designed by expanding the first one by one element.

3.1. **Concept of atoms in JU-algebras.**

The following definition introduces the concept of atoms in JU-algebras.

Definition 3.1. Let A be a JU-algebra. An element $a \in A$ such that $a \neq 1$ is called an atom of a JU-algebra A if the following holds

(A) $(\forall\, x \in A)(x \leqslant a \implies (x = 1 \lor x = a))$.

The set of all atoms of A is denoted by $L(A)$.

Proposition 3.1. *Let A be a JU-algebra and $a \in A$ such that $1 \neq a$. Then a is an atom in A if the set $\{1, a\}$ is a JU-ideal in A.*

Proof. Let the subset $\{1, a\}$ be a JU-ideal in a JU-algebra A. Then holds

$$(\forall x \in A)(x \leqslant a \wedge a \in \{1, a\}) \implies x \in \{1, a\})$$

by (J3). This means $x = 1$ or $x = a$. □

The claim, presented in the next theorem, is a specificity of this class of logical algebras.

Theorem 3.2. *Every isolated element in a JU-algebra A, i.e. the element that is not comparable to any other element in A, is an atom in A.*

Proof. Let A be a JU-algebra and let an element $(1 \neq) b \in A$ not be comparable with any other element in A. This means that $x \cdot b \neq 1$ and $b \cdot x \neq 1$ are valid. Therefore, for b it holds $(b \neq 1 \wedge x \neq b) \implies x \nleqslant b$. Since the obtained implication is the contraposition of implication (A), we conclude that b is an atom in A. □

Remark. *An additional explanation of the concept of aromas in a JU-algebra can be given using the concept of 'covering'. We say that z cover x if and only if $x \leqslant z$ and there does not exists $y \in A$ such that $x \leqslant y \leqslant z$ and $x \neq y$. Thus, an atom in the JU-algebra A is either an element in A that covers the neutral 1, or is an isolated element, or, if it is not comparable to 1, then there is at least one element that covers it.*

Also, it is not difficult to conclude that:

Theorem 3.3. *Let A be a JU-algebra and a be an atom in A. Then the following holds*

(12) $(\forall x \in A)((a \cdot x) \cdot x = 1 \vee (a \cdot x) \cdot x = a)$.

Proof. Let $x \in A$ be an arbitrary element. From the valid equality $(a \cdot x) \cdot (a \cdot x) = 1$ follows $a \cdot ((a \cdot x) \cdot x) = 1$. This means that $(a \cdot x) \cdot x \leqslant a$. Thus $(a \cdot x) \cdot x = 1 \vee (a \cdot x) \cdot x = a$ because a is an atom in A. □

Corollary 3.4. *For every atom a in a JU-algebra A holds*

$$\varphi^2(a) = 1 \vee \varphi^2(a) = a.$$

However, the converse of Theorem 3.3 is also valid:

Theorem 3.5. *If an element a in a JU-algebra A satisfies the condition (12), then a is an atom in A.*

Proof. Let A be a JU-algebra and let an element $a \in A$ satisfy the condition (12). Let us take $x \leqslant a$. This means that $a \cdot x = 1$. Thus $(a \cdot x) \cdot x = x$. Hence, according to the premise, we have $x = (a \cdot x) \cdot x = 1$ or $x = (a \cdot x) \cdot x = a$. So, the element a is an atom in A. □

Example 3.2. Let A be as in Example 2.4(1). The order relation $\leqslant$ in this JU-algebra A is given by $\leqslant \{(1,1), (2,2), (3,3), (4,4), (1,3), (1,4), (3,4)\}$. The subsets $J_1 = \{1\}$, $J_2 = \{1, 3\}$, $J_3 = \{1, 3, 4\}$ are JU-ideals in A but the subset $K = \{1, 2\}$ is not a JU-ideal in A because, for example, we have $2 \in K$ and $2 \cdot 3 = 2 \in K$ but $3 \notin K$ which contradicts (J2). In this JU-algebra, the element 3 is an atom in A according to Proposition 3.1. Element $2\,(\neq 1)$ is also an atom in A because $(\forall x \in A)((x \neq 1 \wedge x \neq 2) \implies x \nleqslant 2)$ holds. We notice again that the set $\{1, 2\}$ is not a JU-ideal in A.

From here it can be concluded that the inverse of Proposition 3.1 does not have to be valid.

Example 3.3. Let A be as in Example 2.4(2). The order relation $\leqslant$ in this JU-algebra A is given by $\leqslant = \{(1,1),(a,a),(b,b),(1,a)\}$. The subset $J = \{1,a\}$ is a JU-ideal in A. The subset $K = \{1,b\}$ is not a JU-ideal in A because, for example $b \in K$ and $b \cdot a = b \in K$ but $a \notin K$ holds. The element a is an atom in A according to Proposition 3.1. Element $b\,(\neq 1)$ is also an atom in A because $(\forall\, x \in A)((x \neq 1 \,\wedge\, x \neq b) \implies x \nleqslant b)$ holds.

The following example shows that if all non-unit elements in a JU-algebra A are atoms in A, then any subalgebra in A need not be either a JU-ideal or a JU-filter in A.

Example 3.4. Let A be as in Examples 2.4(2) and 3.3. Then A is a JU-algebra but not a KU-algebra. The subsets $S_1 =: \{1,a\}$ and $S_2 =: \{1,b\}$ are subalgebras in A. The set S_1 is a JU-filter and a JU-ideal in A but the subset S_2 is neither a JU-ideal in A (see, Example 3.3) nor a JU-filter in A, because, for example, $a \cdot b = b \in S_2$, $b \in S_2$ and $a \notin S_2$ holds. However, all elements different from 1 are atoms in A.

Example 3.5. Let A be as in Examples 2.4(3) and 2.6. A is a JU-algebra but not a KU-algebra. Element 2 is an atom in A because $\{1,2\}$ is a JU-ideal in A by Example 2.6 and Proposition 3.1. Element 5 is an atom in A because for it, is valid $(\forall x \in A)((x \neq 5 \wedge x \neq 1) \implies x \nleqslant 5)$. The subset $L =: \{1,5\}$ is a subalgebra in A but it is not a JU-ideal in A (See, example 2.6). L is also not a JU-filter in A because, for example, $4 \cdot 5 = 5 \in L$ and $5 \in L$ but $4 \notin L$ are valid.

The previous two examples illustrate the specificity of the atom concept in this class of logical algebras.

Following the determination in the article [4], we introduce the following subset: For any JU-algebra A we consider following subsets

$$G(A) = \{x \in A : \varphi^2(x) = 1 \,\vee\, \varphi^2(x) = x\}.$$

Theorem 3.6. *Let A be a JU-algebra and B be a non-empty subset of A. Then, for the subset $\varphi^2(B) = \{\varphi^2(b) : b \in B\}$ holds $\varphi^2(B) \subseteq G(A)$ and $\varphi^2(A) = G(A)$.*

Proof. For $c \in \varphi^2(B)$ there exists $b \in B$ such that $c = \varphi^2(b)$. Then $\varphi^2(c) = \varphi^4(b) = \varphi^2(b) = c$, by (7). So, $c \in G(A)$. Thus $\varphi^2(B) \subseteq G(A)$. Consequently $\varphi^2(A) \subseteq G(A)$. The inclusion $G(A) \subseteq \varphi^2(A)$ is obvious. □

Theorem 3.7. *Let A be a JU-algebra. Then $L(A) \subseteq G(A) = \varphi(A)$.*

Proof. Let $a \in A$ be an atom in A. Since $\varphi^2(a) \leqslant a$ according to (6), we conclude that $\varphi^2(a) = 1$ or $\varphi^2(a) = a$ in accordance with (A). This shows that $a \in G(A)$. Obviously $G(A) \subseteq \varphi(A)$. Thus $L(A) \subseteq G(A) \subseteq \varphi(A)$.

Conversely, for any $a \in \varphi(A)$ there exists $y \in A$ such that $a = \varphi(y)$. Hence $\varphi^2(a) = \varphi^3(y) = \varphi(y) = a$ by (7). This means $\varphi(A) \subseteq G(A)$. □

Example 3.6. Let A be as in the Example 3.8. A is a JU-algebra but not a KU-algebra. Elements a, b, c are atoms in A, because $x \leqslant a \implies (x = 1 \,\vee\, x = a)$, $x \leqslant c \implies (x = 1 \,\vee\, x = c)$ and $(x \neq 1 \,\wedge\, x \neq b) \implies x \nleqslant b$. In this case, we have $\varphi^2(a) = 1$, $\varphi^2(b) = b$ and $\varphi^2(c) = 1$.

It is easy to see that $Ker(\varphi) \subseteq G(A)$. However, not every element in $Ker(\varphi)$ is an atom in A. Therefore, $L(A) \neq G(A)$.

Let A be a JU-algebra. Let $H(A)$ denote the set $\{x \in A : \varphi^2(x) = x\}$. This set is not empty because $1 \in H(A)$. It is clear that $H(A) \subseteq G(A)$.

Proposition 3.8. *Let A be a JU-algebra. The subset $H(A)$ of A is a JU-subalgebra in A.*

Proof. It is clear that $1 \in H(A)$. Let $x, y \in H(A)$ be arbitrary elements. This means $\varphi^2(x) = x$ and $\varphi^2(y) = y$. Then

$$\varphi^2(x \cdot y) = \varphi(\varphi(x \cdot y)) = \varphi(\varphi(x) \cdot \varphi(y)) = \varphi^2(x) \cdot \varphi^2(y) = x \cdot y$$

according to (4). Hence, $x \cdot y \in H(A)$. □

The following proposition gives some of the basic properties of the set $H(A)$.

Proposition 3.9. *Let A be a JU-algebra. Then holds $H(A) \subseteq L(A)$.*

Proof. Let $a \in A$ be an arbitrary element such that $a \in H(A)$. This means $\varphi^2(a) = a$. Let us take $x \in A$ such that $x \leqslant a$. From here, applying (5) twice, we get $a = \varphi^2(a) = \varphi^2(x) \leqslant x$ with respect to (6). Thus $a = x$, which proved that a is an atom in A. So, $a \in L(A)$. □

We conclude this subsection with the following statement:

Theorem 3.10. *Subset $L(A)$ of a JU-algebra A is an anti-chain.*

Proof. Let us take the elements $a, b \in L(A)$ of the JU-algebra A such that $a \neq b$ and assume that they are comparable, that is, $a \leqslant b$ or $b \leqslant a$ holds. If it were $b \leqslant a$ we would have $b = 1$ or $b = a$ because a is an atom in A. Since both options are impossible, we conclude that elements a and b are not comparable. □

3.2. **Extension of (weak) KU-algebra to JU-algebra.** Referring to the procedures of extension (weak) BCC-algebra to weak BCC-algebra by adding one element, presented in theorems 2.4. and 2.5 in the article [5], one can prove the validity of analogous theorems for the extension of (weak) KU-algebra to JU-algebra.

Theorem 3.11. *Any KU-algebra can be extended to a JU-algebra.*

Proof. Let $(A, \cdot, 1)$ be a JU-algebra and let $a \notin A$. Then $(A \cup \{a\}, *, 1)$ with the operation $*$, defined by

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \text{ and } y \in A, \\ a & \text{for } x \in A \text{ and } y = a, \\ a & \text{for } x = a \text{ and } y \in A, \\ 1 & \text{for } x = a \text{ and } y = a, \end{cases}$$

is a JU-algebra containing $(A, \cdot, 1)$ since $a * 1 = a \neq 1$. The proof can be demonstrated by direct verification if one, two or all three variables are replaced by the letter a in the KU axiom system. For illustration, we show those procedures.

If $x = a$, then we have $(y \cdot z) \cdot ((z \cdot a) \cdot (y \cdot a)) = (y \cdot z) \cdot (a \cdot a) = (y \cdot z) \cdot 1 = 1$.
If $y = a$, then we have

$$(a \cdot z) \cdot ((z \cdot x) \cdot (a \cdot x)) = a \cdot ((z \cdot x) \cdot a) = (z \cdot x) \cdot (a \cdot a) = (z \cdot x) \cdot 1 = 1.$$

If $z = a$, we have $(y \cdot a) \cdot ((a \cdot x) \cdot (y \cdot x)) = a \cdot (a \cdot (y \cdot x)) = a \cdot a = 1$.
If $x = a$ and $y = a$, then we have $(a \cdot z) \cdot ((z \cdot a) \cdot (a \cdot a)) = a \cdot (a \cdot 1) = a \cdot a = 1$.
If $x = a$ and $z = a$, then we have $(y \cdot a) \cdot ((a \cdot a) \cdot (y \cdot a)) = a \cdot (1 \cdot a) = a \cdot a = 1$.
If $y = a$ and $z = a$, then $(a \cdot a)((a \cdot x) \cdot (a \cdot x)) = 1 \cdot (a \cdot a) = 1 \cdot 1 = 1$. □

Example 3.7. Let A be as in the example 2.2. Let us put $B = A \cup \{a\}$ and define the operation $*$ on B in the following way

$*$	1	2	3	4	a
1	1	2	3	4	a
2	1	1	1	2	a
3	1	3	1	2	a
4	1	1	1	1	a
a	a	a	a	a	1

Then $(B, *, 1)$ is a JU-algebra bat not a KU-algebra because $a \cdot 1 = a \neq 1$. Order relation on this algebra is

$$\leqslant = \{(1,1),(2,2),(3,3),(4,4),(a,a),(1,2),(1,3),(1,4),(2,4),(3,2),(3,4)\}.$$

The subsets $\{1\}$, $\{1,2,3,4\}$ and $\{1,a\}$ are the only nontrivial JU-ideals in A. The subset $K = \{1,2\}$ is not a JU-ideal in A because, for example, we have $2 \in K$ and $2 \cdot 4 = 2 \in K$ but $4 \notin K$. Also, the subset $L = \{1,2,3\}$ is not a JU-ideal in A because, for example, we have $3 \in L$ and $3 \cdot 4 = 2 \in L$ but $4 \notin L$. However, both of these last subsets are subalgebras in A.

As a consequence of the previous theorem, we have: that the element a is an atom in the extension $A \cup \{a\}$ designed in this way.

Corollary 3.12. *The element a is an atom in the extension $A \cup \{a\}$.*

Proof. According to the way the extension $A \cup \{a\}$ was created, the element a is an isolated element in $A \cup \{a\}$. Therefore, it is an atom in $A \cup \{a\}$ according to Theorem 3.2. □

The following theorem shows the design of a new JU-algebra $A \cup \{a\}$ from a given JU-algebra A by adding one element $a \notin A$. The specificity of this creation is $L(A \cup \{a\}) = L(A) \cup \{a\}$.

Theorem 3.13. *Any JU-algebra A can be extended to a JU-algebra $A \cup \{a\}$ containing one element more such that $L(A \cup \{a\}) = L(A) \cup \{a\}$.*

Proof. Let $(A, \cdot, 1)$ be a JU-algebra and let $a \notin A$. It is not difficult to see that $A \cup \{a\}$ with the operation $*$, defined by

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ \varphi(x) & \text{for } x \in A \setminus \{1\} \ \wedge \ y = a, \\ y & \text{for } x = a \ \wedge \ y \in A, \\ a & \text{for } x = 1 \ \wedge \ y = a, \\ 1 & \text{for } x = a \ \wedge \ y = a, \end{cases}$$

is a JU-algebra containing $(A, \cdot, 1)$. The proof can be demonstrated by direct verification if one, two or all three variables are replaced by the letter a in the JU axiom system. For illustration, we show some of those procedures.

If $x = a$, then we have

$$(y \cdot z) \cdot ((z \cdot a) \cdot (y \cdot a)) = (y \cdot z) \cdot (\varphi(z) \cdot \varphi(y)) = 1$$

according to (2)

If $y = a$, then $(a \cdot z) \cdot ((z \cdot a) \cdot (a \cdot x)) = z \cdot ((z \cdot x) \cdot x) = (z \cdot x) \cdot (z \cdot x) = 1$.
If $z = a$, then we have
$(y \cdot a) \cdot ((a \cdot x) \cdot (y \cdot x)) = \varphi(y) \cdot (x \cdot (y \cdot x)) = \varphi(y) \cdot (y \cdot (x \cdot x))$

$$= \varphi(y) \cdot \varphi(y) = 1.$$

If $x = a$ and $y = a$, then we have

$$(a \cdot z) \cdot ((z \cdot a) \cdot (a \cdot a)) = z \cdot (\varphi(z) \cdot 1) = z \cdot \varphi^2(z) = 1$$

by (6).

If $x = a$ and $z = a$, then we have

$$(a \cdot y) \cdot ((a \cdot a) \cdot (y \cdot a)) = \varphi(y) \cdot (1 \cdot \varphi(y)) = \varphi(y) \cdot \varphi(y) = 1.$$

If $y = a$ and $= a$, then we have $(a \cdot a) \cdot ((a \cdot x) \cdot (a \cdot x)) = 1 \cdot 1 = 1$.

It is easy to conclude that the element a is an atom in $A \cup \{a\}$ since it is an isolated element in $A \cup \{a\}$. □

Example 3.8. Let A be as in the example 2.6(2). Let us put $B = A \cup \{c\}$ and define the operation $*$ on B in the following way

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	b	b	1	b
c	1	a	b	1

Then $(B, *, 1)$ is a JU-algebra. Order relation on this algebra is

$$\leqslant = \{(1,1), (a,a), (b,b), (c,c), (1,a), (1,c), (c,a)\}.$$

The subsets $\{1\}$, $\{1, c\}$, $\{1, a, b\}$, $\{1, a, c\}$, $\{1, b, c\}$ are nontrivial JU-ideals in A. The subset $K = \{1, a\}$ is not a JU-ideal in A because, for example, we have $a \in K$ and $a \cdot c = 1 \in K$ but $c \notin K$. The subset $L = \{1, b\}$ is not a JU-ideal in A because, for example, we have $b \in L$ and $b \cdot a = b \in L$ but $a \notin L$.

4. Final comments

Properties of atoms in BCK/BCI/BCC/weak BCC algebras are described in more detail than was possible with properties of atoms in JU-algebras. In this paper, the criteria (Theorem 3.2 and Theorem 3.5) for recognizing atoms in JU-algebras are presented. In addition to the previous one, a procedure for the extension of KU-algebra to JU-algebra was designed (Theorem 3.11). Also, the possibility is proved (Theorem 3.13) that a JU-algebra A can be extended to a JU-algebra $A \cup \{a\}$ so that $L(A \cup \{a\}) = L(A) \cup \{a\}$.

Comparing the obtained results on the properties of atoms in JU-algebras with the properties of atoms in other types of logical algebras, it can be concluded that there is plenty of room for further and deeper research of these phenomena in JU-algebras.

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