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ORDERED Γ-SEMIHYPERGROUP OF THE ASSOCIATED Γ-SEMIHYPERGROUP WITH ALL RELATIVE BI-Γ-HYPERIDEALS

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ABSTRACT. In this paper, the main goal is to study an ordered Γ-semihypergroup H in the context of the characterizations of the associated Γ-semihypergroup $\mathcal{B}(H)$ of all bi-Γ-hyperideals of H. We show that an ordered Γ-semihypergroup H is a Clifford ordered Γ-semihypergroup if and only if $\mathcal{B}(H)$ is a semilattice. We also show that a Γ-semihypergroup $\mathcal{B}(H)$ is a normal band if and only if the ordered Γ-semihypergroup H is simultaneously regular and intra regular. Furthermore, for each subclass S with many bands, we prove that for an ordered Γ-semihypergroup H, the conditional inclusion $\mathcal{B}(H) \in S$ holds true.

1. INTRODUCTION

The notion of bi-ideal was introduced by Good and Hughes [27] and Lajos [29] generalized this notion of bi-ideal in the form of (m, n)-ideal. The concept of quasi-deal was introduced by Steinfeld [25], [26], interestingly enough, in algebraic system, that is, in rings and semigroups as well. A semigroup S (without order) by the set of all bi-ideals of semigroup was also characterized and studied by Lajos [30].

Nambooripad [18] proved that a regular semigroup S is locally testable if and only if for every $f \in E(S)$, fSf is a semilattice. Zalcstein [32] proved that a locally testable semigroup is a band if and only if it is a normal band. Hansda [16] defined a Clifford (left Clifford) ordered semigroup. Furthermore, Hansda [17] studied minimal bi-ideals in regular and completely regular ordered semigroups. Kehayopulu et al [22] studied bi-ideals in ordered semigroups and ordered groups. Bhuniya et al [2] studied completely regular and Clifford ordered semigroups. Kehayopulu et al [20] introduced bi-ideal for ordered semigroup. A locally testable finite semigroup was defined by Nambooripad in [18]. Moreover, Mallick et al [28] studied the semigroup of bi-ideals of an ordered semigroup. Kehayopulu

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[21] defined Green's relations on an ordered semigroup. A left group like ordered semigroup was defined in [2]. Right group like ordered semigroup was defined in dually.

The concept of hyperstructures was given by Marty [13]. The theory was extensively studied from theoretical point of view as well as for its applications to many subjects in pure and applied mathematics. For useful study of references of various algebraic hyperstructures and their applications in different fields, one can refer [12], [14].

The notion of Γ-semihypergroup as a generalization of semigroup, semihypergroup and Γ-semigroup was introduced by Davvaz et al. [10, 11, 31]. In 2010, Anvariyeh et al. [31] introduced the notion of Γ-semihypergroups for generalization of semihypergroup and studied the Γ-hyperideals of Γ-semihypergroups. The notion of ordered Γ-semihypergroup was further studied by Kondo and Lekkoksung [19] as an extension of the notion of ordered semihypergroups. It was also investigated by some authors [15]. The theory of ordered semihypergroups is one that of generalizations of the concept of semihypergroups. It was introduced by Heidari and Davvaz [12] for that every semihypergroup can be considered as an ordered semihypergroup.

The theory of ideals play significant role in different algebraic structures. In [1] Almasarwah, A. G. Ahmad and G. Muhiuddin studied Doubt N-ideals theory in BCK-algebras based on N-structures. The theory of relative ideal in semigroup(resp. left, right relative ideals) was given by Wallace [8], [9]. Khan et al. [23], [24] generalized this concept in ordered semigroups. Thereafter, Basar et al [3], [4], [5], [6], [7] studied these ideals in different algebraic structures.

In this paper, motivated by the previous work on ordered semigroup in [28] for the context of ordered semigroup, we make an attempt in the present paper to study relative Γ-hyperideals of ordered Γ-semihypergroups in some detail.

In this paper, the main motivation and the importance of considering the present study over the existing studies is to improve over previous results. We show that an ordered Γ-semihypergroup H in the context of the characterizations of the associated Γ-semihypergroup $B(S)$ of all bi-Γ-hyperideals of H. We prove that an ordered Γ-semihypergroup H is a Clifford ordered Γ-semihypergroup if and only if $B(S)$ is a semilattice. We show that Γ-semihypergroup $B(S)$ is a normal band if and only if the ordered Γ-semigroup H is both regular and intra regular. For each subclass S of many bands, we prove that an ordered Γ-semihypergroup H holds such that $B(H) \in S$.

2. Preliminaries on Basic Definitions and Fundamental Results

In this section, we recollect basic concepts from the references in the paper and also define notions that are subsequently necessary in the study of this paper for further course of action itself.

Definition 2.1. [31] Let H be a non-empty set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The pair $(H, \circ)$ is called a hypergroupoid.

Definition 2.2. [31] A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$, which meas that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

Moreover, if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a hypergroup.

Definition 2.3. [19] An algebraic hyperstructure $(S, \Gamma, \leq)$ is called an ordered Γ-semihypergroup if (S, Γ) is a Γ-semihypergroup and $(S, \leq)$ is a partially ordered set such that for any $x, y, z \in S, x \leq y$ and $\gamma \in \Gamma$ implies $z\gamma x \leq z\gamma y$ and $x\gamma z \leq y\gamma z$. Here, $A \leq B$ means that for any $a \in A$, there exists $b \in B$ such that $a \leq b$, for all non-empty subsets A and B of S.

A non-empty subset A of an ordered Γ-semihypergroup $(S, \Gamma, \leq)$ is called a sub Γ-semihypergroup of S if $A\Gamma A \subseteq A$.

Example 2.4. [19] Let $(S, \circ, \leq)$ be an ordered semihypergroup and Γ a non-empty set. We define $x\gamma y = x \circ y$ for every $x, y \in S$ and $\gamma \in \Gamma$. Then $(S, \Gamma, \leq)$ is an ordered Γ-semihypergroup.

Definition 2.5. A sub-Γ-semihypergroup(non-empty subset) B of an ordered Γ-semihypergroup H is called a relative (generalized) bi-Γ-hyperideal of H if $B\Gamma S\Gamma B \subseteq B$ and $(B]_B = B$ for $S \subseteq H$.

The set of all relative (generalized) bi-Γ-hyperideal of H is denoted, in this paper, by $\mathcal{B}$. The principal left relative Γ-hyperideal, right relative Γ-hyperideal and relative bi-Γ-hyperideal generated by $s \in S \subseteq H$ and represented by $L(s), R(s), I(s)$ and $\mathcal{B}(s)$, respectively. They are defined by $L(s) = (s \cup S\Gamma s]_S, R(s) = (s \cup s\Gamma S]_S$, $I(s) = (s \cup S\Gamma s \cup s\Gamma S \cup S\Gamma s\Gamma S]_S$, $\mathcal{B}(s) = (s \cup s^2 \cup s\Gamma S\Gamma s]_S$ for $S \subseteq H$.

Definition 2.6. An ordered Γ-semihypergroup H is said to be relative regular if for every $s \in S, s \in (s\Gamma S\Gamma s]_S$ and is relative intra-regular if for every $s \in S, s \in (S\Gamma s^2\Gamma S]_S$ for $S \subseteq H$.

Definition 2.7. A relative band H is a Γ-semihypergroup $(H, \cdot)$ with the property $a^2 = a$ for every $a \in S \subseteq H$. A relative band $(H, \cdot)$ is called relative rectangular if for every $a, b \in S \subseteq H$, $a\gamma_1 b\gamma_2 a = a$. A left(right) zero relative band is a relative band $(H, \cdot)$ with the property $a\gamma b = a(b\gamma a = a)$ for every $a, b \in S \subseteq H$. A relative band $(H, \cdot)$ is said to be left (right) relative normal band if for every $a, b, c \in S \subseteq H$, $a\gamma_1 b\gamma_2 c = a\gamma_3 c\gamma_4\gamma_5 b\gamma_5 (a\gamma_6 b\gamma_7 c = b\gamma_8 a\gamma_9 c)$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9 \in \Gamma$ and H is said to be relative normal if $a\gamma_1 b\gamma_2 c\gamma_3 a = a\gamma_4 c\gamma_5 b\gamma_6 a$. A commutative band is called a semilattice.

Definition 2.8. A Γ-semihypergroup in which every finitely generated sub-Γ-semihypergroup is finite called locally finite. A locally finite Γ-semihypergroup H is called locally testable if for every idempotent f of H, $f\Gamma S\Gamma f$ is a semilattice.

3. Main Results

Proposition 3.1. *Suppose that H is an ordered Γ-semihypergroup. Then H is relative regular if and only if the Γ-semihypergroup $\mathcal{B}(S)$ of all bi-Γ-hyperideals is relative regular for $S \subseteq H$.*

Proof. Let $\mathcal{B}(S)$ be a relative regular Γ-semihypergroup, where $S \subseteq H$. Let $s \in S$. Then $B(s) \in B(S)$. As $B(S)$ is relative regular, there is $C \in B(S)$ such that $B(s) = B(s) * \Gamma * C * \Gamma * B(s) = (B(s)\Gamma C \Gamma B(s)]_S$. Since $s \in \mathcal{B}(s)$, there are $b \in B(s), x \in C$ and $c \in B(s)$ such that $s \leq b\alpha x\beta c$ for $\alpha, \beta \in \Gamma$. Moreover, for $b, c \in B(s)$ there are $s_1, s_2 \in S$ such that $b \leq s$ or $b \leq s\alpha s_1\beta a$ and $c \leq s$ or $c \leq s\alpha s_2\beta s$. Thus, in either ways $s \leq b\alpha x\beta c$ for $\alpha, \beta \in \Gamma$ implies that $s \in (s\Gamma S\Gamma s]_S$. Hence, H is a relative regular ordered Γ-semihypergroup. □

Theorem 3.2. *Suppose that H is an ordered Γ-semihypergroup. Then $R(S)\Gamma(L(S))$ is a band and $B(S) = R(S)\Gamma L(S)$ for $S \subseteq H$*

Proof. Suppose that $R \in \mathcal{R}(S)$ and $r \in R$. As H is a relative regular, there exist $y \in S$ in order that $r \leq r\alpha y\beta r$ for $\alpha, \beta \in \Gamma$. Furthermore, $r\alpha y \subseteq R$ provides us that $r \in (R\Gamma R]_S = R * \Gamma * R = R^2$, and therefore, we have $R \subseteq R^2$. Thus, we receive $R^2 = R$. Hence R is a relative band. In a similar fashion, one can prove that $\mathcal{L}(S)$ is a relatice band. Now consider $R \in \mathcal{R}(S)$, and $L \in \mathcal{L}(S)$. Suppose that $B = R * \Gamma * L$. Then, $B = (R\Gamma L]_S$ and B is a sub-Γ-semihypergroup of H. Next, we have $B\Gamma S\Gamma B = (RL]_S\Gamma S\Gamma(R\Gamma L]_S \subseteq (R\Gamma L\Gamma S\Gamma R\Gamma L]_S \subseteq (R\Gamma L]_S = B$. This implies that $B \in \mathcal{B}(S)$ and therefore, $\mathcal{R}(S)\Gamma\mathcal{L}(S) \subseteq \mathcal{B}(S)$. Moreover, consider $D \in \mathcal{B}(S)$. Now, $D \in \mathcal{B}(S) \subseteq \mathcal{R}(S)\Gamma\mathcal{L}(S)$. Hence, $\mathcal{B}(S) = \mathcal{R}(S)\Gamma\mathcal{L}(S)$. □

Theorem 3.3. *An ordered Γ-semihypergroup H is both regular and intra-regular if and only if $\mathcal{B}(S)$ is a relative band for $S \subseteq H$.*

Proof. Suppose that H is regular as well as intra-regular ordered Γ-semihypergroup and $S \subseteq H$. Let $B \in \mathcal{B}(S)$ and $a \in B$. Therefore, $a \leq a\alpha s\beta a \leq a\alpha s\beta a\gamma s\delta a$ for some $s \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. As H is relative intra-regular, there exist $s_1, s_2 \in S$ such that $a \leq s_1\alpha a^2\beta s_2$ for $\alpha, \beta \in \Gamma$ which shows that $a \leq a\gamma_1 s\gamma_2 s_1\gamma_3 a^2\gamma_4 s_2\gamma_5 s\gamma_6 a \leq (a\gamma_1 s\gamma_2 s_1\gamma_3 a)\gamma_4(a\gamma_5 s_2\gamma_6 s\gamma_7 a)$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7 \in \Gamma$. As $a\alpha s\beta s_1\gamma a \in B\Gamma S\Gamma B \subseteq B$, we have $a\gamma_1 s\gamma_2 s_1\gamma_3 a^2\gamma_4 s_2\gamma_5 s\gamma_6 a \in B^2$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \Gamma$ in order that $a \in (B\Gamma B]_S = B * \Gamma * B = B^2$. Furthermore, $B^2 \subseteq B$. Hence $B^2 = B$.

Conversely, suppose that $\mathcal{B}(S)$ is a relative band. Let $s \in S$. Then $B(s) \in \mathcal{B}(S)$ and therefore, $s \in B(s) = B(s)^2 = B(s) * \Gamma * B(s) = (B(s)\Gamma B(s)]_S$. Therefore, $s \leq b\gamma c$ for some $b, c \in B(s)$ and $\gamma \in \Gamma$. This shows that $b \leq s$ or $b \leq s\alpha s^1\beta s$ for some $s^1 \in S^1$ for $\alpha, \beta \in \Gamma$. Also, $c \leq s$ or $c \leq s\alpha t\beta s$ for some $t \in S^1$ for $\alpha, \beta \in \Gamma$. It then follows that $s \leq b\alpha c$ shows that either $s \leq s^2$ for $\alpha \in \Gamma$ or $s \in (s\Gamma S\Gamma s^2\Gamma S\Gamma s]_S$ which shows that s is both relative intra-regular and relative regular. Hence, H is both relative regular and relative intra-regular. □

Lemma 3.4. *Suppose that H is a regular as well as intra-regular ordered Γ-semihypergroup and $S \subseteq H$. Then*

- **(1):** *for every $B, C, D \in B(S)$, $((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S = (B\Gamma C\Gamma B]_S \cap (B\Gamma D\Gamma B]_S$;*
- **(2):** *$\mathcal{B}(S)$ is relative locally testable Γ-semihypergroup.*

Proof. (1) Suppose that $S \subseteq H$. We have the following:

$$((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S \subseteq ((B\Gamma C\Gamma B]\Gamma(B]_S]_S \subseteq ((B\Gamma C\Gamma B]_S]_S \subseteq (B\Gamma C\Gamma B]_S.$$

In a similar fashion, we receive $((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S \subseteq (B\Gamma D\Gamma B]_S$. Therefore, we have the following:

$$((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S \subseteq (B\Gamma C\Gamma B]_S \cap (B\Gamma D\Gamma B]_S.$$

Let $v \in (B\Gamma C\Gamma B]_S \cap (B\Gamma D\Gamma B]_S$. Then there exist $b \in B, c \in C, d \in D$ so that $v \leq b\alpha c\beta b$ and $v \leq b\alpha d\beta b$ for $\alpha, \beta \in \Gamma$. As H is both relative regular and relative intra-regular, there are $x, t, s \in S$ such that $v \leq v\alpha x\beta v, b \leq b\alpha t\beta b$ and $b \leq s_1\alpha b^2\beta s_2$ for $\alpha, \beta \in \Gamma$. This shows that

$$\begin{aligned} v \leq b\gamma_1 c\gamma_2 b\gamma_3 x\gamma_4 b\gamma_5 d\gamma_6 b &\leq b\gamma_1 c\gamma_2 b\gamma_3 t\gamma_4 b\gamma_5 x\gamma_6 b\gamma_7 d\gamma_8 b \\ &\leq b\gamma_1 c\gamma_2 b\gamma_3 t\gamma_4 s_1\gamma_5 b^2\gamma_6 s_2\gamma_7 x\gamma_8 b\gamma_9 d\gamma_{10} b \\ &\leq (b\gamma_1 c\gamma_2 b\gamma_3 t\gamma_4 s_1\gamma_5 b)\gamma_6(b\gamma_7 s_2\gamma_8 x\gamma_9 b\gamma_{10} d\gamma_{11} b). \end{aligned}$$

for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11} \in \Gamma$. Therefore, $v \in ((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S$. Thus, $(B\Gamma C\Gamma B]_S \cap (B\Gamma D\Gamma B]_S \subseteq ((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S$. Hence, $((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S = (B\Gamma C\Gamma B]_S \cap (B\Gamma D\Gamma B]_S$.

(2) Suppose that $S \subseteq H$. Consider $B \in \mathcal{B}(S)$. Then $B\Gamma\mathcal{B}(S)\Gamma B$ is a sub-Γ-semihypergroup of $\mathcal{B}(S)$ and so is a relative band. For every $C, D \in \mathcal{B}(S)$, we have the following:

$$\begin{aligned} (B\Gamma C\Gamma B]_S * (B\Gamma D\Gamma B]_S &= ((B\Gamma C\Gamma B]_S\Gamma(B\Gamma D\Gamma B]_S]_S \\ &= (B\Gamma C\Gamma B]_S \cap (B\Gamma D\Gamma B]_S \\ &= (B\Gamma D\Gamma B]_S \cap (B\Gamma C\Gamma B]_S \\ &= ((B\Gamma D\Gamma B]_S\Gamma(B\Gamma C\Gamma B]_S]_S \\ &= (B\Gamma D\Gamma B]_S * \Gamma * (B\Gamma C\Gamma B]_S \end{aligned}$$

shows that $B\Gamma\mathcal{B}(S)B$ is a semilattice. Thus, $\mathcal{B}(S)$ is a relative locally testable. □

Corollary 3.5. *Suppose that H is an ordered Γ-semihypergroup and $S \subseteq H$. If H is both relative regular and relative intra-regular then $\mathcal{B}(S)$ is a relative band if and only if $\mathcal{B}(S)$ is a relative normal band.*

Theorem 3.6. *Suppose that H is an ordered Γ-semihypergroup. Then $B(S)$ is relatve rectangular band if and only if H is relative regular and relative simple.*

Proof. Suppose that $\mathcal{B}(S)$ is a relative rectangular band and $S \subseteq H$. Let $x, y \in S$. Then $B(x), B(y) \in \mathcal{B}(S)$. As $\mathcal{B}(S)$ is a relative rectangular band, we find $B(x) = B(x) * \Gamma * B(y) * \Gamma * B(x)$. Also, $B(y) = B(y) * \Gamma * B(x) * \Gamma * B(y)$. Moreover, by Theorem 3.3, S is a relative regular. As $x \in B(x) = B(x) * \Gamma * B(y) * \Gamma * B(x) = (B(x) * \Gamma * B(y) * \Gamma * B(x)]_S$, there exist $w, z \in B(x), v \in B(y)$ such that $x \leq z\alpha v\beta w$ for $\alpha, \beta \in \Gamma$. As $w, z \in B(x), z \leq x\alpha s_1\beta x$ and $w \leq x\alpha s_2\beta x$ for some $s_1, s_2 \in S$ and $\alpha, \beta \in \Gamma$. Furthermore, we observe that for $v \in B(y)$ there exists $s_3 \in S$ such that $v \leq y\alpha s_3\beta y$ for $\alpha, \beta \in \Gamma$. Therefore, $x \leq (x\gamma_1 s_1\gamma_2 x\gamma_3 y\gamma_4 s_3)\gamma_5 y\gamma_6(x\gamma_7 s_2\gamma_8 x)$. That is, $x \leq a\alpha y\beta b$ for some $a, b \in S$ and $\alpha, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8 \in \Gamma$. Hence, H is relative simple.

Conversely, suppose that H is a relative regular and simple ordered Γ-semihypergroup and $S \subseteq H$. For $a \in S$, with the given condition, we receive $a \in (S\Gamma a^2\Gamma S\Gamma]_S$ so that H is relative intra-regular. Therefore, by Theorem 3.3, $\mathcal{B}(S)$ is a relative band. Again, let $A, B \in \mathcal{B}(S)$. We prove that $A = A * \Gamma * B * \Gamma * A$. For this purpose, suppose that $a \in A$ and $b \in B$. Since $a, a\alpha b\beta a \subseteq S$ and $a\mathcal{J}b$, therefore $a \leq y_1\gamma_1 a\gamma_2 b\gamma_3 a\gamma_4 y_2$ for some $y_1, y_2 \in S$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$. The relative regularity of H gives rise to $a \leq a\alpha x\beta a \leq a\gamma_1 x\gamma_2 a\gamma_3 x\gamma_4 a$ for some $x \in S$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \alpha, \beta \in \Gamma$. Then $a \in (a\gamma_1 x\gamma_2 y_1\gamma_3 a)\gamma_4 b\gamma_5(a\gamma_6 y_2\gamma_7 x\gamma_8 a)$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8 \in \Gamma$. Therefore, $a \in ((A\Gamma S\Gamma A)\Gamma B\Gamma(A\Gamma S\Gamma A)]_S \subseteq (A\Gamma B\Gamma A]_S = A * \Gamma * B * \Gamma * A$. It then shows that $A \subseteq A * \Gamma * B * \Gamma * A$. Again $A * \Gamma * B * \Gamma * A \subseteq (A\Gamma S\Gamma A]_S = A$. Thus, $A = A * \Gamma * B * \Gamma * A$. Hence $\mathcal{B}(S)$ is a relative rectangular band. □

Theorem 3.7. *Suppose that H is an ordered Γ-semihypergroup and $S \subseteq H$. Then $\mathcal{B}(S)$ is a left (right) relative zero band if and only if H is a left (right) group like ordered Γ-semihypergroup.*

Proof. Suppose that $\mathcal{B}(S)$ is a left relative zero band. Then by Proposition 3.1, H is a relative regular. Let $x, y \in S$. Then, $B(x), B(y) \in \mathcal{B}(S)$. As $\mathcal{B}(S)$ is a left relative zero band, $B(x) = B(x) * \Gamma * B(y)$, so $x \in (B(x)\Gamma B(y)]_S$. Then there exist $z \in B(x)$ and $w \in B(y)$ such that $x \leq z\alpha w$ for $\alpha \in \Gamma$. Moreover, $w \leq y\gamma_1 s\gamma_2 y$ for some $s \in S$ and $\gamma_1, \gamma_2 \in \Gamma$. Therefore, $x \leq (z\gamma_1 y\gamma_2 s)\gamma_3 y$ for $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. Hence H is a left group like ordered Γ-semihypergroup.

Conversely, suppose that H is a left group like ordered Γ-semihypergroup and $S \subseteq H$. Let $B, C \in B(S)$. Let $u \in B * \Gamma * C$, then there exist $b \in B$ and $c \in C$ such that $u \leq b\alpha c$ for $\alpha \in \Gamma$. Thus, H is a left group like ordered Γ-semihypergroup, we have $c \leq t\alpha b$ for some $t \in S$ and $\alpha \in \Gamma$. Then for $c \leq t\alpha b$ equipped with $u \leq b\gamma_1 c \leq b\gamma_2 t\gamma_3 b$ for $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ gives $u \in B$. Thus, $B * \Gamma * C \subseteq B$. Now for any $d \in B, d \leq d\alpha t\beta d$ for some $t \in S$ and $\alpha, \beta \in \Gamma$. Since $d, d\gamma c \in S, d \leq t_1\alpha d\beta c$ for some $t_1 \in S$ and $\alpha, \beta, \gamma \in \Gamma$. In fact, $d \leq d\gamma_1 t\gamma_2 t_1\gamma_3 d\gamma_4 c$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$. For clearly, we have $d \in B\Gamma S\Gamma B \subseteq B$ in order that $d \in (B\Gamma C]_S = B * \Gamma * C$. Therefore, $B = B * \Gamma * C$. Hence, B is a left zero relative band.

□

Theorem 3.8. *Suppose that H is an ordered Γ-semihypergroup. Then $\mathcal{B}(S)$ is both left zero and right zero relative band if and only if H is a group like ordered Γ-semihypergroup for $S \subseteq H$.*

Proof. The proof is similar to the proof of the Theorem 3.7 and hence omitted. □

Theorem 3.9. *Suppose that H is an ordered Γ-semihypergroup. Then the following assertions are equivalent for $S \subseteq H$:*

- **(1):** *H is a Clifford ordered Γ-semihypergroup;*
- **(1):** *$B_1 * \Gamma * B_2 = B_1 \cap B_2$ for all $B_1, B_2 \in \mathcal{B}(S)$;*
- **(1):** *$(\mathcal{B}(S), *)$ is a semilattice.*

Proof. $(1) \Rightarrow (2)$ Suppose that H is a Clifford ordered Γ-semihypergroup and $S \subseteq H$. Furthermore, suppose that $B_1, B_2 \in \mathcal{B}(S)$ and $u \in B_1 * \Gamma * B_2$. Then $u \leq b_1\alpha b_2$ for $b_1 \in B_1$, $b_2 \in B_2$ and $\alpha \in \Gamma$. As H is relative regular ordered Γ-semihypergroup, there exists $x \in S$ such that $u \leq u\gamma_1 x\gamma_2 u \leq b_1\gamma_3 b_2\gamma_4 x\gamma_5 b_1\gamma_6 b_2$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in \Gamma$. As H is Clifford ordered Γ-semihypergroup, there exists $x_1 \in S$ in order that $b_1\gamma_1 b_2 \leq b_2\gamma_1 x_1\gamma_2 b_1$ so that $u \leq b_1\gamma_1 b_2\gamma_2 x\gamma_3 b_2\gamma_4 x_1\gamma_5 b_1$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \Gamma$. This implies $u \in B1$. Similarly $u \in B2$. Hence, $B_1 * \Gamma * B_2 * \Gamma * B_1 \cap B_2$. Furthermore, let $b \in B_1 \cap B_2$. Since H is relative regular, there exists $y \in S$ such that $b \leq b\gamma_1 y\gamma_2 b \leq b\gamma_3 y\gamma_4 b\gamma_5 y\gamma_6 b$. As H is Clifford ordered Γ-semihypergroup, $y\gamma_1 b \leq b\gamma_2 z\gamma_3 y$ for some $z \in S$. Therefore, $b \leq b\gamma_1 b\gamma_2 z\gamma_3 y_2\gamma_4 b$. As $b \in B_2$ and B_2 is a relative bi-Γ-hyperideal of H, it gives that $bzy_2\ b \in B_2\Gamma S\Gamma B_2 \subseteq B_2$. Moreover, $b \in B_1$ so that $b \in (B_1\Gamma B_2]_S = B_1 * \Gamma * B_2$. Hence, $B_1 * \Gamma * B_2 = B_1 \subseteq B_2$.
$(3) \Rightarrow (1)$ This is clear.

(2) $\Rightarrow$ (3) Assume that $(B(S), *)$ is a semilattice. Then H is a regular ordered Γ-semihypergroup by Theorem 3.3. Consider $a, b \in S$. Then $a\gamma b \in B(a) * \Gamma * B(b) = B(b) * \Gamma * B(a)$ implies that $a\gamma_1 b \leq v\gamma_2 u$ for some $u \in B(a)$ and $v \in B(b)$. As H is relative regular, there exist $s, t \in S$ such that $u \leq a\gamma_1 s\gamma_2 a$ and $v \leq b\gamma_3 t\gamma_4 b$. Thus, $a\gamma_1 b \leq b\gamma_2 t\gamma_3 b\gamma_4 a\gamma_5 s\gamma_6 a = b\gamma_7 z\gamma_8 a$ where $z = t\gamma_1 b\gamma_2 a\gamma_3 s \in S$ for $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\gamma_6, \gamma_7, \gamma_8, \gamma \in \Gamma$. Hence, H is a Clifford ordered Γ-semihypergroup. □

Theorem 3.10. *Suppose that H is an ordered Γ-semihypergroup. Then $\mathcal{B}(S)$ is a left normal relative band if and only if H is a left Clifford ordered Γ-semihypergroup for $S \subseteq H$.*

Proof. Suppose that H is a left Clifford ordered Γ-semihypergroup. Let A, B and $C \in B(S)$ and $x \in A*\Gamma*B*\Gamma*C$. Then $x \in (A\Gamma B\Gamma C]_S$ so $x \leq abc$ for some $a \in A, b \in B$ and $c \in C$. Since H is relative regular, there is $s \in S$ such that $x \leq x\gamma_1 s\gamma_2 x$ so that $x \leq a\gamma_1 b\gamma_2 c\gamma_3 s\gamma_4 a\gamma_5 b\gamma_6 c$. As H is a left Clifford ordered Γ-semihypergroup, it follows that $b\gamma_1 c \leq s_1\gamma_2 b$ for some $s_1 \in S$, so $x \leq a\gamma_1 b\gamma_2 c\gamma_3(s\gamma_4 a\gamma_5 s_1)\gamma_6 b \leq a\gamma_7 b\gamma_8 s_2\gamma_9 c\gamma_{10} b$ for $s_2 \in S$. As H is relative regular there is $t \in S$ such that $a \leq a\gamma_1 t\gamma_2 a$ implies $x \leq a\gamma_3 t\gamma_4 a\gamma_4 b\gamma_5 s_2 c\gamma_6 b$. Moreover, there exist $s_3, s_4 \in S, x \leq a\gamma_1 t\gamma_2 s_3\gamma_3 a\gamma_4 s_2\gamma_5 c\gamma_6 b \leq a\gamma_7 t\gamma_8 s_3\gamma_9 s_4\gamma_{10} a\gamma_{11} c\gamma_{12} b$ implies that $x \in A*\Gamma*C*\Gamma*B$. Therefore, $A*\Gamma*B*\Gamma*C \subseteq A*\Gamma*C*\Gamma*B$. In a similar fashion, it can be shown that $A*\Gamma*C*\Gamma*B \subseteq A*\Gamma*B*\Gamma*C$. Therefore, $A*\Gamma*B*\Gamma*C = A*\Gamma*C*\Gamma*B$. Hence, B(S) is a relative left normal band.

Conversely, suppose that $B(S)$ is a relative left normal band. Then H is a relative regular, by Theorem 3.3. Suppose that $a, b \in S$. Then there exists $x \in S$ such that $a\gamma_1 b \leq a\gamma_2 b\gamma_3 x\gamma_4 a\gamma_5 b$ which shows that $a\gamma_1 b \subseteq (B(a\gamma_2 b\gamma_3 x)\Gamma B(a)\Gamma B(b)]_S = (B(a\gamma_4 b\gamma_5 x)\Gamma B(b)\Gamma B(a)]_S$, since $B(S)$ is a relative left normal band. Then $a\gamma_1 b \leq u\gamma_2 v\gamma_3 w$ for $c, \gamma_2, \gamma_3 \in \Gamma$, where $u \in B(a\gamma_1 b\gamma_2 x), v \in B(b), w \in B(a)$. Again, $w \leq a\gamma_1 s\gamma_2 a$ for some $s \in S$. Now $a\gamma_1 b \leq u\gamma_2 v\gamma_3 w \leq (u\alpha v\beta a\gamma s)\delta a \leq s_1 a$, where $s_1 = u\alpha v\beta a\gamma s \in S$ for $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence, H is left Clifford ordered Γ-semihypergroup. □

4. Conclusions and/or Discussions

In this paper, we have studied an ordered Γ-semihypergroup H in the settings of the characterizations of the associated Γ-semihypergroup $\mathcal{B}(H)$ of all bi-Γ-hyperideals of H. We have proved that an ordered Γ-semihypergroup H is a Clifford ordered Γ-semihypergroup if and only if $\mathcal{B}(H)$ is a semilattice. We have also proved that a Γ-semihypergroup $B(H)$ is a normal band if and only if the ordered Γ-semihypergroup H is simultaneously relative regular and relative intra regular. Furthermore, for each subclass S with many bands, we have shown that for an ordered Γ-semihypergroup H, the conditional inclusion $\mathcal{B}(H) \in S$ holds true. These results are refinement and improvement over previous results, and these can be studied in various other possibly researchable areas of different algebraic structures possessing potential for future work direction.

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References

[1] A. Al-masarwah, A. G. Ahmad and G. Muhiuddin, Doubt N-ideals theory in BCK-algebras based on N-structures, Annals of Communications in Mathematics, 3 (1) (2020), 54-62.

[2] A. K. Bhuniya and K. Hansda, On completely regular and Clifford ordered semigroups, Afrika Matematika (2020), https://doi.org/10.1007/s13370-020-00778-1

[3] A. Basar, Shaista and B. Satyanarayana, A note on completely regular 2-duo ordered Γ-semihypergroups by relative ordered quasi-Γ-hyperideals and relative ordered Γ-hyperideals, Alochana Journal, 13(1)(2024), 69-77.

[4] A. Basar, S. Kumari and B. Satyanarayana, On Some Properties of Relative Hyperideals In Relative Regular Ordered Krasner Hyperrings, GIS Science Journal, 10(1)(2023), 306-328.

[5] A. Basar, A. Ahmad, B Satyanarayana, P. K. Sharma and M. Sanjeel, On relative ordered Gamma-hyperideals and relative prime segments in chain ordered Gamma-semihypergroups, Journal of Xi'an University of Architecture & Technlogy, 14(8)(2022), 300-309.

[6] A. Basar, A. Ahmad, P. K. Sharma, and S. Bhavanari, On relative ordered Hoehnke Γ-hyperideal and Γ-hyperideals in chain ordered Γ-semihypegroups, GIS Science Journal, 9(7)(2022), 2793-2803.

[7] A. Basar, Shaista, B. Satyanarayana and A. Ahmad, On Relative Quasi-ideals and Relative Bi-ideals in AG-Rings, International Conference On "Exploring Excellence in Mathematical Sciences", 4-6 November, 2023, organized by Nalanda Open University, Nalanda in Collaboration with Bihar Mathematical Society(Submitted).

[8] A. D. Wallace, Relative Ideals in Semigroups I, Colloq. Math., (1962), 55-61.

[9] A. D. Wallace, Relative Ideals in Semigroups II, Acta Mathematica Hungarica, 14(1-2)(1963), 137-148.

[10] D. Heidari and B. Davvaz, Γ-Hypergroups and Γ-Semihypergroups Associated to Binary Relations., Iran. J. Sci. Technol. Trans. A Sci., 35(2) (2011), 69–80.

[11] D. Heidari, S. Ostadhadi Dehkordi, and B. Davvaz, Γ-semihypergroups and their Properties. Politehn. Univ., Bucharest Sci. Bull. Ser. A Appl. Math. Phys.,72(1) (2010), 195–208.

[12] D. Heidari and B. Davvaz, On ordered hyperstructures, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 73 (2) (2011), 85-96.

[13] F. Marty, Surune generalization de la notion de groupe, 8eme 'Congres Math. Scandinaves, Stockholm, (1934), 45-49.

[14] B. Davvaz, A brief survey on algebraic hyperstructures: Theory and applications, Journal of Algebraic Hyperstructures and Logical Algebras, 1(3)(2020), 15-29.

[15] B. Davvaz and S. Omidi, C-Γ-hyperideal theory in ordered Γ-semihypergroups, Journal of Mathematical and Fundamental Sciences, 49(2)(20179, 181-192.

[16] K. Hansda, Bi-ideals in Clifford ordered semigroup, Discuss. Math. Gen. Algebra Appl.,33 (2013), 73–84. https://doi.org/10.7151/dmgaa.1195

[17] K. Hansda, Minimal bi-ideals in regular and completely regular ordered semigroups, Quasigroup and Related Systems, 27(1), 63–72.

[18] K. S. S. Nambooripad, Pseudo-semilattices and biordered sets I, Simon Stevin, 55 (1981), 103– 110.

[19] M. Kondo and N. Lekkoksung, On Intra-Regular Ordered Γ-Semihypergroups, Int. J. Math. Anal. (Ruse) 7 , 25-28(2013), 1379–1386.

[20] N. Kehayopulu and M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, Inform. Sci., 171 (2005), 13–28. https://doi.org/10.1016/j.ins.2004.03.015

[21] N. Kehayopulu, Note on Green's relation in ordered semigroup, Math. Japonica, 36 (1991), 211–214.

[22] N. Kehayopulu, J. S. Ponizovskii and M. Tsingelis, Bi-ideals in ordered semigroups and ordered groups, J. Math. Sci., 112 (4) (2002), 4353–4354. https://doi.org/10.1023/A:1020347003781

[23] N. M. Khan and M. F. Ali, Relative bi-ideals and relative quasi ideals in ordered semigroups, Hacet. J. Math. Stat., 20(10) (2020), 1–12.

[24] M. F. Ali, N. M. Khan and Relative Ideals in Ordered Semigroups, Indian Journal of Pure and Applied Mathematics, 51 (2020), 1497–1514.

[25] O. Steinfeld, On ideal-quotients and prime ideals, Acta. Math. Acad. Sci. Hung., 4(1953), 289-298.

[26] O. Steinfeld, Quasi ideals in Rings and Semigroups, vol. 10 of Disquisitiones Mathematicae Hungaricae, Akademiai Kiado, Budapest, Hungary, 1978.

[27] R. A. Good and D. R. Hughes, Associated groups for a semigroup, Bull. Amer. Math. Soc., (58)(1952), 624-625.

[28] S. Mallick and K. Hansda, On the semigroup of bi-ideals of an ordered semigroup, Kragujevac Journal of Mathematics, 47(3) (2023), 339–345.

[29] S. Lajos, On (m, n)-ideals of semigroups, Abstracts of Second Hungar. Math. Congress I, (1960), 42-44.

[30] S. Lajos, On the bi-ideals in semigroups, Proc. Japan Acad, 45(8) (1969), 710–712. https://doi.org/10.3792/pja/1195520625

[31] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, On Γ-Hyperideals in Γ-Semihypergroups, Carpathian J. Math., 26(1)(2010), 11–23.

[32] Y. Zalcstein, Locally testable semigroups, Semigroup Forum, 5(1973), 216–227. https://doi.org/10.1007/BF02572893

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