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COMMON LIMIT RANGE PROPERTY IN NEUTROSOPHIC MENGER SPACES

M. JEYARAMAN*, A.N. MANGAYARKKARASI AND V. JEYANTHI

Abstract. The aim of this paper is to prove some common fixed-point theorems for weakly compatible mappings in Neutrosophic Menger Spaces satisfying common limit range property. Some examples are also given which demonstrate the validity of our results. As an application of our main result, we present a common fixed point theorems for four finite families of self-mappings in Menger spaces.

1. Introduction

There have been a number of generalizations of metric spaces. One of such generalization is a probabilistic metric space, briefly, PM-spaces, introduced in 1942 by Menger [13]. In the PM-space, we do not know exactly the distance between two points, but we know the probabilities of possible values of this distance. Many mathematicians proved several common fixed point theorems for contraction mappings in Menger spaces by using different notions viz. compatible mappings, weakly compatible mappings, property (E.A), common property (E.A). Modifying the idea of Kramosil and Michalek [9], George and Veeramani [5] introduced fuzzy metric spaces which are very similar to Menger spaces.

Recently, using the idea of intuitionistic fuzzy set, Atanassov as [1], which is a generalization of a fuzzy set, Zadeh [19], Park [14] introduced the notion of intuitionistic fuzzy metric spaces as a generalization of fuzzy metrics paces due to George and Veeramani [5]. Kutukcu et.al [10] introduced the notion of intuitionistic Menger spaces as a generalization of Menger spaces. In 2011, Sintunavarat and Kumam introduced the concept of common limit range (CLR) property. Very recently, Chauhan et. al [2] introduced the concept of joint common limit range (JCLR) property. The importance of these properties is that we don't require the closedness of subspaces for the existence of fixed points.

In 1998, Smarandache [15-17] characterized the new concept called neutrosophic logic and neutrosophic set and explored many results in it. In the idea of neutrosophic sets, there is T degree of membership, I degree of indeterminacy and F degree of non-membership. In 2019, Kirisci et al [8] defined neutrosophic metric space as a generalization of intuitionistic

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*Corresponding author.

fuzzy metric space and brings about fixed point theorems in complete neutrosophic metric space. Sowndrarajan, Jeyaraman and Florentin Smarandache [18] proved some fixed point results for contraction theorems in neutrosophic metric spaces. Jeyaraman and Florentin Smarandache et. al [7] prove common fixed point theorems for four sub compatible maps of type (J-1) in weak non-Archimedean neutrosophic metric space.

The purpose of this paper is to prove some common fixed-point theorems for weakly compatible mappings in neutrosophic menger spaces satisfying CLR property. Some examples are also given which demonstrate the validity of our results.

2. Preliminaries

Definition 2.1. [9] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm [CTN] if it satisfies the following conditions :

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0,1]$,
(iv) $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,1]$.

Definition 2.2. [9] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm [CTC] if it satisfies the following conditions :

(i) $\diamond$ is commutative and associative,
(ii) $\diamond$ is continuous,
(iii) $\varepsilon_1 \diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0,1]$,
(iv) $\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0,1]$.

Definition 2.3. [4]

(i) A distance distribution function is a function $F : \mathbb{R} \to \mathbb{R}^+$ which is left continuous on R, non-decreasing and $\inf_{t\in\mathbb{R}} F(t) = 0, \sup_{t\in\mathbb{R}} F(t) = 1$. We will denote by D the family of all distance distribution functions and by H a special element of D defined by
$$H(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ 1, & \text{if} \quad t > 0 \end{cases}.$$
If X is an non-empty set, $F : X \times X \to D$ is called a probabilistic distance on X and $F(x,y)$ is usually denoted by F_{xy}.

(ii) A non-distance distribution function is a function $L : \mathbb{R} \to \mathbb{R}^+$ which is right continuous on R, non-increasing and $\inf_{t\in\mathbb{R}} L(t) = 1, \sup_{t\in\mathbb{R}} L(t) = 0$. We will denote by E the family of all non-distance distribution functions and by G a special element of E defined by
$$G(t) = \begin{cases} 1, & \text{if} \quad t \leq 0 \\ 0, & \text{if} \quad t > 0 \end{cases}.$$
If X is a non-empty set, $L : X \times X \to E$ is called a probabilistic non distance on X and $L(x,y)$ is usually denoted by L_{xy}.

(iii) A non-distance distribution function is a function $P : \mathbb{R} \to \mathbb{R}^+$ which is right continuous on R, non-increasing and $\inf_{t\in\mathbb{R}} P(t) = 1, \sup_{t\in\mathbb{R}} P(t) = 0$. We will denote by U the family of all non-distance distribution functions and by R a special element of U defined by
$$R(t) == \begin{cases} 1, & \text{if} \quad t \leq 0 \\ 0, & \text{if} \quad t > 0 \end{cases}.$$

If X is a non-empty set, $P : X \times X \to U$ is called a probabilistic non distance on X and $P(x, y)$ is usually denoted by P_{xy}.

Definition 2.4. A 4-tuple (X, F, L, P) is said to be an neutrosophic probabilistic metric space if X is an arbitrary set, F is a probabilistic distance and L and P are probabilistic non-distance on X satisfying the following conditions; for all $x, y, z \in X$ and $t, s > 0$.

(i) $F_{xy}(t) + L_{xy}(t) + P_{xy}(t) \leq 3$,
(ii) $F_{xy}(0) = 0$,
(iii) $F_{xy}(t) = 1$ if and only if $x = y$,
(iv) $F_{xy}(t) = F_{yx}(t)$,
(v) If $F_{xy}(t) = 1$ and $F_{yz}(s) = 1$, then $F_{xz}(t + s) = 1$,
(vi) $L_{xy}(0) = 1$,
(vii) $L_{xy}(t) = 0$ if and only if $x = y$,
(viii) $L_{xy}(t) = L_{yx}(t)$,
(ix) If $L_{xy}(t) = 0$ and $L_{yz}(s) = 0$, then $L_{xz}(t + s) = 0$.
(x) $P_{xy}(0) = 1$,
(xi) $P_{xy}(t) = 0$ if and only if $x = y$,
(xii) $P_{xy}(t) = P_{yx}(t)$,
(xiii) If $P_{xy}(t) = 0$ and $P_{yz}(s) = 0$, then $P_{xz}(t + s) = 0$.

Definition 2.5. A 6-tuple $(X, F, L, P, *, \diamond)$ is said to be an Neutrosophic Menger Space (shortly NMS) if (X, F, L, P) is an neutrosophic probabilistic metric space and in addition, the following in equalities hold: for all $x, y, z \in X$ and $t, s > 0$,

(i) $F_{xy}(t) * F_{yz}(s) \geq F_{xz}(t + s)$,
(ii) $L_{xy}(t) \diamond L_{yz}(s) \leq L_{xz}(t + s)$,
(iii) $P_{xy}(t) \diamond P_{yz}(s) \leq P_{xz}(t + s)$.

Where $*$ is a continous t-norm and $\diamond$ is a continous t-conorm. The functions F_{xy} denote the degree of nearness, L_{xy} and P_{xy} the degree of non-nearness between x and y with respect to t respectively.

Remark. *In NMS $(X, F, L, P, *, \diamond) F_{xy}$ is non-decreasing and L_{xy} and P_{xy} non-increasing for all $x, y \in X$.*

Remark. *If the t - norm $*$ and the t - conorm $\diamond$ of an NMS $(X, F, L, P, *, \diamond)$ satisfy the conditions $\sup_{t\in(0,1)}(t*t) = 1, \inf_{t\in(0,1)}((1-t)\diamond(1-t)) = 0$ and $\inf_{t\in(0,1)}((1-t)\diamond(1-t)) = 0$, then $(X, F, L, P, *, \diamond)$ is a Hausdorff topological space in the (ε, λ) topology, i.e., the family of sets $\{U_x(\varepsilon, \lambda), \varepsilon > 0, \lambda \in (0, 1], x \in X\}$ is a basis of neighborhoods of point x for a Hausdorff topology $\tau(F, L)$, or (ε, λ) topology on X, where $\{U_x(\varepsilon, \lambda) = \{y \in X : F_{xy}(\varepsilon) > 1 - \lambda, L_{xy}(\varepsilon) < \lambda$ and $P_{xy}(\varepsilon) < \lambda\}$.*

Remark. *Every an neutrosophic metric space $(X, F, L, P, *, \diamond)$ is an NMS by considering $F : X \times X \to D, L : X \times X \to E$ and $P : X \times X \to U$ defined by $F_{xy}(t) = M(x, y, t), L_{xy}(t) = N(x, y, t)$ and $P_{xy}(t) = U(x, y, t)$ for all $x, y \in X$. Throughout this paper, $(X, F, L, P, *, \diamond)$ is an NMS with the following conditions:*

$$\lim_{t\to+\infty} F_{xy}(t) = 1, \lim_{t\to+\infty} L_{xy}(t) = 0 \quad \textit{and} \quad \lim_{t\to+\infty} P_{xy}(t) = 0, \tag{2.1}$$

for all $x, y \in X$ and $t > 0$.

Definition 2.6. (a) Let $(X, F, L, P, *, \diamond)$ be an NMS. A sequence $\{x_n\}_{n\in N}$ in X is said to be convergent to a point $x \in X$, if for each $t > 0$ and $\varepsilon \in (0, 1)$, there exists a positive integer $n_0 = n_0(t, \varepsilon)$ such that for all $n \geq n_0, F_{x_n}x(t) > 1 - \varepsilon, L_{x_n}x(t) < \varepsilon$ and $P_{x_n}x(t) < \varepsilon$.

(b) A sequence $\{x_n\}_{n\in N}$ in X is called a Cauchy sequence if for all $t > 0$ and $\varepsilon \in (0,1)$, there exists a positive integer $n_0 = n_0(t,\varepsilon)$ such that for all $n, m \geq n_0, F_{x_n x_m}(t) > 1-\varepsilon, L_{x_n x_m}(t) < \varepsilon$ and $L_{x_n x_m}(t) < \varepsilon$

(c) An NMS in which every Cauchy sequence is convergent is said to be complete.

Remark. *An induced NMS $(X, F, L, P, *, \diamond)$ is complete if (X, d) is complete.*

Definition 2.7. Let $(X, F, L, P, *, \diamond)$ be an NMS.
A sequence $\{x_n\}_{n\in N}$ in X is called a Cauchy sequence if and only if $\lim\limits_{n\to+\infty} F_{x_n x}(t) = 1$, $\lim\limits_{n\to+\infty} L_{x_n x}(t) = 0$ and $\lim\limits_{n\to+\infty} P_{x_n x}(t) = 0$, for all $t > 0$.

Lemma 2.1. *Let $(X, F, L, P, *, \diamond)$ be an NMS and $\{x_n\}, \{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$, respectively. Then $\lim_{n\to+\infty} \inf F_{x_n y_n}(t) \geq F_{xy}(t), \lim_{n\to+\infty} \sup L_{x_n y_n}(t) \leq L_{xy}(t)$ and $\lim\limits_{n\to+\infty} \sup P_{x_n y_n}(t) \leq P_{xy}(t)$, for all $t > 0$. If $t > 0$ is a continuous point of F_{xy}, L_{xy} and P_{xy}, then $\lim\limits_{n\to+\infty} F_{x_n y_n}(t) = F_{xy}(t)$, $\lim\limits_{n\to+\infty} L_{x_n y_n}(t) = L_{xy}(t)$ and $\lim\limits_{n\to+\infty} P_{x_n y_n}(t) = P_{xy}(t)$.*

Lemma 2.2. *Let $\{x_n\}_{n\in N}$ be a sequence in an NMS with the condition (2.1). If there exists a number $k \in (0,1)$ such that for $x, y \in X, t > 0$ and $n = 0, 1, 2, \ldots$. $F_{x_{n+2}x_{n+1}}(kt) \geq F_{x_{n+1}x_n}(t), L_{x_{n+2}x_{n+1}}(kt) \leq L_{x_{n+1}x_n}(t)$ and $P_{x_{n+2}x_{n+1}}(kt) \leq P_{x_{n+1}x_n}(t)$, then $\{x_n\}$ is a Cauchy sequence in X.*

Lemma 2.3. *Let $(X, F, L, P, *, \diamond)$ be an NMS. If there exists a number $k \in (0,1)$ such that for all $x, y \in X$ and $t > 0, F_{xy}(kt) \geq F_{xy}(t), L_{xy}(kt) \leq L_{xy}(t)$ and $P_{xy}(kt) \leq P_{xy}(t)$, then $x = y$.*

Definition 2.8. Two self - mappings A and S of an NMS are said to be compatible if $\lim\limits_{n\to+\infty} F_{ASx_n SAx_n}(t) = 1$, $\lim\limits_{n\to+\infty} L_{ASx_n SAx_n}(t) = 0$ and $\lim\limits_{n\to+\infty} P_{ASx_n SAx_n}(t) = 0$, for all $t > 0$. Whenever $\{x_n\} \subset X$ such that $\lim\limits_{n\to+\infty} Ax_n = \lim\limits_{n\to+\infty} Sx_n = z$, for some $z \in X$.

Definition 2.9. Two self - mappings A and S of an NMS are said to be non - compatible if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to+\infty} F_{Ax_n,z}(t) = \lim\limits_{n\to+\infty} F_{Sx_n,z}(t) = 1$, for some $z \in X$, but for some $t > 0$, either $\lim\limits_{n\to+\infty} F_{ASx_n,SAx_n}(t) \neq 1$ or $\lim\limits_{n\to+\infty} L_{ASx_n,SAx_n}(t) \neq 0$ or $\lim\limits_{n\to+\infty} P_{ASx_n,SAx_n}(t) \neq 0$ or one of the limits do not exist.

Definition 2.10. A pair of self - mappings A and S of an NMS $(X, F, L, P, *, \diamond)$ is said to be satisfy the property (E. A) if there exists a sequence $\{x_n\}$ in X such that for some $z \in X$,

$$\begin{aligned}
&\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = 1, \\
&\lim_{n\to+\infty} L_{Ax_n,z}(t) = \lim_{n\to+\infty} L_{Sx_n,z}(t) = 0 \quad \text{and} \\
&\lim_{n\to+\infty} P_{Ax_n,z}(t) = \lim_{n\to+\infty} P_{Sx_n,z}(t) = 0, \text{for all } t > 0. \qquad (2.2)
\end{aligned}$$

Remark. *It is easy to see that two non - compatible self - mappings of an NMS satisfy the property (E.A), but the converse is not true in general.*

Definition 2.11. Two pairs (A, S) and (B, T) of self - mappings of an NMS $(X, F, L, P, *, \diamond)$ are said to satisfy the common property (E. A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$

in X such that some $z \in X$ and for all $t > 0$.

$$\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1,$$
$$\lim_{n\to+\infty} L_{Ax_n,z}(t) = \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0 \quad \text{and}$$
$$\lim_{n\to+\infty} P_{Ax_n,z}(t) = \lim_{n\to+\infty} P_{Sx_n,z}(t) = \lim_{n\to+\infty} P_{By_n,z}(t) = \lim_{n\to+\infty} P_{Ty_n,z}(t) = 0.$$

If $B = A$ and $T = S$ in this definition we get the definition of the property (E.A).

Definition 2.12. A pair of self - mappings A and S of an NMS $(X, F, L, P, *, \diamond)$ is said to satisfy the common limit range property with respect to the mapping S (briefly CLR property), if there exists a sequence $\{x_n\}$ in X such that (2.2) holds, where $z \in S(X)$.

Now, we given an example of self - mappings A and S satisfying the CLR_s property.

Example 2.13. Let $(X, F, L, P, *, \diamond)$ be an NMS, where $X = [0, \infty)$, the t - norm $*$ is defined by $a * b = \min\{a, b\}$, the t -conorm $\diamond$ is defined by $a \diamond b = \max\{a, b\}$ and also defined $F_{xy}(t) = H(t - |x - y|), L_{xy}(t) = G(t - |x - y|)$ and $P_{xy}(t) = J(t - |x - y|)$ for all $x, y \in X$ and $t > 0$.
Define self - mappings A and S on X by: $Ax = x + 4, Sx = 5x$. Let a sequence $\{x_n = 1 + \frac{1}{n}\}_{n\in N}$ in X. Since $\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = 5$, then

$$\lim_{n\to+\infty} F_{Ax_n,5}(t) = \lim_{n\to+\infty} F_{Sx_n,5}(t) = 1,$$
$$\lim_{n\to+\infty} L_{Ax_n,5}(t) = \lim_{n\to+\infty} L_{Sx_n,5}(t) = 0 \quad \text{and}$$
$$\lim_{n\to+\infty} P_{Ax_n,5}(t) = \lim_{n\to+\infty} P_{Sx_n,5}(t) = 0.$$

for all $t > 0$, where $5 \in S(X)$. Therefore, the mappings A and S satisfy the CLR_s property. From this example, it is clear that a pair (A, S) satisfying the property (E.A) with the closedness of the subspace $S(X)$ always verifies the CLR property.

Definition 2.14. Two pairs (A, S) and (B, T) of self - mappings of an NMS $(X, F, L, P, *, \diamond)$ are said to satisfy the common limit range property with respect to mappings S and T (briefly, CLR_{ST} property), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$.

$$\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1,$$
$$\lim_{n\to+\infty} L_{Ax_n,z}(t) = \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0 \quad \text{and}$$
$$\lim_{n\to+\infty} P_{Ax_n,z}(t) = \lim_{n\to+\infty} P_{Sx_n,z}(t) = \lim_{n\to+\infty} P_{By_n,z}(t) = \lim_{n\to+\infty} P_{Ty_n,z}(t) = 0,$$

where $z \in S(X) \cap T(X)$.

Remark. *If $B = A$ and $T = S$ in this definition we get the definition of CLR_S property.*

Remark. *The CLR_{ST} property implies the common property (E.A), but the converse is not true in general,*

Remark. *If the pairs (A, S) and (B, T) satisfy the common property (E.A) and $S(X)$ and $T(X)$ are closed subsets of X, then the pairs satisfy also the CLR_{ST} property.*

Definition 2.15. Two pairs (A, S) and (B, T) of self - mappings of an NMS $(X, F, L, P, *, \diamond)$ are said to satisfy the joint common limit range property with respect to mappings S and T (briefly $JCLR_{ST}$ property), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$

$\lim_{n\to+\infty} F_{Ax_n,z}(t) = \lim_{n\to+\infty} F_{Sx_n,z}(t) = \lim_{n\to+\infty} F_{By_n,z}(t) = \lim_{n\to+\infty} F_{Ty_n,z}(t) = 1$,
$\lim_{n\to+\infty} L_{Ax_n,z}(t) = \lim_{n\to+\infty} L_{Sx_n,z}(t) = \lim_{n\to+\infty} L_{By_n,z}(t) = \lim_{n\to+\infty} L_{Ty_n,z}(t) = 0$ and
$\lim_{n\to+\infty} P_{Ax_n,z}(t) = \lim_{n\to+\infty} P_{Sx_n,z}(t) = \lim_{n\to+\infty} P_{By_n,z}(t) = \lim_{n\to+\infty} P_{Ty_n,z}(t) = 0$,
where $z = Su = Tu, u \in X$.

Remark. *If $B = A$ and $T = S$ in this definition we get the definition of CLR_S property.*

Definition 2.16. Two families of self - mappings $\{A_i\}$ and $\{S_j\}$ are said to be pairwise commuting if

(1) $A_i, A_j = A_j A_i, i, j \in \{1, 2, \ldots, m\}$,
(2) $S_k S_l = S_l S_k, k, l \in \{1, 2, \ldots, n\}$,
(3) $A_i S_k = S_k A_i, i \in \{1, 2, \ldots, m\}, k \in \{1, 2, \ldots, n\}$.

3. Main Results

Lemma 3.1. *Let A, B, S and T be self - mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the following conditions.*

(i) The pairs (A, S) satisfies the CLR_S Property or the pair (B, T) satisfies the CLR_T property,
(ii) $A(X) \subseteq T(X)$ or $B(X) \subseteq S(X)$,
(iii) $T(X)$ or $S(X)$ is a closed subset of X.
(iV) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges.

$$\begin{aligned}(1 + \alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) \\ > \alpha \min\left\{\ F_{Ax,Sx}(t)\, F_{By,Ty}(t),\ F_{Sx,By}(t)\, F_{Ax,Ty}(t)\ \right\} \\ + \min\left\{\begin{array}{c} F_{Sx,Ty}(t),\ \sup\limits_{t_1+t_2=\frac{2}{k}t} \min\{F_{Ax,Sx}(t_1), F_{Sx,By}(t_2)\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t} \min\{F_{By,Ty}(t_3), F_{Ax,Ty}(t_4)\} \end{array}\right\}\end{aligned} \tag{3.1.1}$$

$$\begin{aligned}(1 + \beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) \\ < \beta \max\left\{\ L_{Ax,Sx}(t)\, L_{By,Ty}(t),\ L_{Sx,By}(t)\, L_{Ax,Ty}(t)\ \right\} \\ + \max\left\{\begin{array}{c} L_{Sx,Ty}(t),\ \inf\limits_{t_1+t_2=\frac{2}{k}t} \max\{L_{Ax,Sx}(t_1), L_{Sx,By}(t_2)\}, \\ \inf\limits_{t_3+t_4=\frac{2}{k}t} \max\{L_{By,Ty}(t_3), L_{Ax,Ty}(t_4)\} \end{array}\right\}\end{aligned} \tag{3.1.2}$$

$$\begin{aligned}(1 + \gamma P_{Sx,Ty}(t))\, P_{Ax,By}(t) \\ < \gamma \max\left\{\ P_{Ax,Sx}(t)\, P_{By,Ty}(t),\ P_{Sx,By}(t)\, P_{Ax,Ty}(t)\ \right\} \\ + \max\left\{\begin{array}{c} P_{Sx,Ty}(t),\ \inf\limits_{t_1+t_2=\frac{2}{k}t} \max\{P_{Ax,Sx}(t_1), P_{Sx,By}(t_2)\}, \\ \inf\limits_{t_3+t_4=\frac{2}{k}t} \max\{P_{By,Ty}(t_3), P_{Ax,Ty}(t_4)\} \end{array}\right\}\end{aligned} \tag{3.1.3}$$

for all $x, y \in X, t > 0$ for some $\alpha, \beta, \gamma \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. Suppose that the pair (A, S) satisfies the CLR_S property and $T(X)$ is a closed subset of X. Then, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \text{ where } z \in S(X).$$

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$.
So

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z, \text{ where } z \in S(X) \cap T(X).$$

Thus, $Ax_n \to z, Sx_n \to z$ and $Ty_n \to z$. Now, we show that $By_n \to z$.
Let $\lim_{n\to\infty} F_{By_n,\mu}(t_0) = 1, \lim_{n\to\infty} L_{By_n,\mu}(t_0) = 0$ and $\lim_{n\to\infty} P_{By_n,\mu}(t_0) = 0$. We assert that $\mu = z$. Assume that $\mu \neq z$. We prove that there exists $t_0 > 0$ such that

$$F_{z,\mu}\Big(\frac{2}{k}t_0\Big) > F_{z,\mu}(t_0),\ L_{z,\mu}\Big(\frac{2}{k}t_0\Big) < L_{z,\mu}(t_0) \text{ and } P_{z,\mu}\Big(\frac{2}{k}t_0\Big) < P_{z,\mu}(t_0). \quad (3.1.4)$$

Suppose that contrary. Therefore, for all $t > 0$ we have

$$F_{z,\mu}\Big(\frac{2}{k}t\Big) > F_{z,\mu}(t),\ L_{z,\mu}\Big(\frac{2}{k}t\Big) < L_{z,\mu}(t) \text{ and } P_{z,\mu}\Big(\frac{2}{k}t\Big) < P_{z,\mu}(t). \quad (3.1.5)$$

Using repeatedly(3.1.5), we obtain

$$\begin{aligned}
&F_{z,\mu}(t) \geq F_{z,\mu}\Big(\frac{2}{k}t\Big) \geq \cdots \geq F_{z,\mu}\Big(\Big(\frac{2}{k}\Big)^n t\Big) \to 1,\\
&L_{z,\mu}(t) \leq L_{z,\mu}\Big(\frac{2}{k}t\Big) \leq \cdots \leq L_{z,\mu}\Big(\Big(\frac{2}{k}\Big)^n t\Big) \to 0 \text{ and}\\
&P_{z,\mu}(t) \leq P_{z,\mu}\Big(\frac{2}{k}t\Big) \leq \cdots \leq P_{z,\mu}\Big(\Big(\frac{2}{k}\Big)^n t\Big) \to 0,
\end{aligned}$$

as $n \to \infty$, is shows that $F_{z,\mu}(t) = 1,\ L_{z,\mu}(t) = 0$ and $P_{z,\mu}(t) = 0$, for all $t > 0$, which contradicts $\mu \neq z$ and hence (3.1.4) is proved.

Without loss of generality, we may assume that t_0 in (3.1.4) is a continuous point of $F_{z,\mu},\ L_{z,\mu}$ and $P_{z,\mu}$. Since every distance distribution function is left-continuous and every a non-distance distribution function is right continuous, (3.1.4) implies that there exists $\varepsilon > 0$ such that (3.1.4) holds for all $t \in (t_0 - \varepsilon, t_0)$. Since $F_{z,\mu}$ is non-decreasing, $L_{z,\mu}$ and $P_{z,\mu}$ are non- increasing,the set of all discontinuous points of $F_{z,\mu},\ L_{z,\mu}$ and $P_{z,\mu}$ is a countable set at most. Thus, when t_0 is a discontinuous point of $F_{z,\mu},\ L_{z,\mu}$ and $P_{z,\mu}$, we can choose a continuous point t_1 of $F_{z,\mu}$ and $F_{z,\mu}$ in $(t_0 - \varepsilon, t_0)$ to replace t_0. Using the inequality (3.1.1), (3.1.2) and (3.1.3) with $x = x_n,\ y = y_n$, we get for some

$t_0 > 0$

$$\begin{aligned}
&(1+\alpha F_{Sx_n,Ty_n}(t_0))F_{Ax_n,By_n}(t_0)\\
&\quad > \alpha \min\Big\{F_{Ax_n,Sx_n}(t_0)F_{By_n,Ty_n}(t_0), F_{Sx_n,By_n}(t_0)F_{Ax_n,Ty_n}(t_0)\Big\}\\
&\quad + \min\Big\{F_{Sx_n,Ty_n}(t_0), \min\{F_{Ax_n,Sx_n}(\varepsilon), F_{Sx_n,By_n}(\frac{2}{k}t_0-\varepsilon)\},\\
&\quad \min\{F_{By_n,Ty_n}(2t_0-\varepsilon), F_{Ax_n,Ty_n}(\varepsilon)\Big\}\\
&(1+\beta L_{Sx_n,Ty_n}(t_0))L_{Ax_n,By_n}(t_0)\\
&\quad < \beta \max\Big\{L_{Ax_n,Sx_n}(t_0)L_{By_n,Ty_n}(t_0), L_{Sx_n,By_n}(t_0)L_{Ax_n,Ty_n}(t_0)\Big\}\\
&\quad + \max\Big\{L_{Sx_n,Ty_n}(t_0), \max\{L_{Ax_n,Sx_n}(\varepsilon), L_{Sx_n,By_n}(\frac{2}{k}t_0-\varepsilon)\},\\
&\quad \max\{L_{By_n,Ty_n}(2t_0-\varepsilon), L_{Ax_n,Ty_n}(\varepsilon)\Big\} \text{ and}\\
&(1+\gamma P_{Sx_n,Ty_n}(t_0))P_{Ax_n,By_n}(t_0)\\
&\quad < \gamma \max\Big\{P_{Ax_n,Sx_n}(t_0)P_{By_n,Ty_n}(t_0), P_{Sx_n,By_n}(t_0)P_{Ax_n,Ty_n}(t_0)\Big\}\\
&\quad + \max\Big\{P_{Sx_n,Ty_n}(t_0), \max\{P_{Ax_n,Sx_n}(\varepsilon), P_{Sx_n,By_n}(\frac{2}{k}t_0-\varepsilon)\},\\
&\quad \max\{P_{By_n,Ty_n}(2t_0-\varepsilon), P_{Ax_n,Ty_n}(\varepsilon)\Big\}
\end{aligned}$$

for all $\varepsilon \in (0, \frac{2}{k}t_0)$. Letting $n \to \infty$, we have

$$\begin{aligned}
F_{z,\mu}(t_0) &\geq \min\left\{1, F_{\mu,z}\left(\frac{2}{k}t_0-\epsilon\right)\right\} = F_{\mu,z}\left(\frac{2}{k}t_0-\epsilon\right),\\
L_{z,\mu}(t_0) &\leq \max\left\{0, L_{\mu,z}\left(\frac{2}{k}t_0-\epsilon\right)\right\} = L_{\mu,z}\left(\frac{2}{k}t_0-\epsilon\right) \quad \text{and}\\
P_{z,\mu}(t_0) &\leq \max\left\{0, P_{\mu,z}\left(\frac{2}{k}t_0-\epsilon\right)\right\} = P_{\mu,z}\left(\frac{2}{k}t_0-\epsilon\right).
\end{aligned}$$

As $\epsilon \to 0$, we obtain

$$F_{z,\mu}(t_0) \geq F_{z,\mu}\left(\frac{2}{k}t_0\right), L_{z,\mu}(t_0) \leq L_{z,\mu}\left(\frac{2}{k}t_0\right) \quad \text{and} \quad P_{z,\mu}(t_0) \leq P_{z,\mu}\left(\frac{2}{k}t_0\right)$$

which contradicts (3.1.4) and so we have $z = \mu$. Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Theorem 3.2. *Let A, B, S and T be self - mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the inequality of lemma (3.1). If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. Since the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. Hence, there exist $u, v \in X$ such that $Su = Tv = z$. Now, we show that $Au = Su = z$. If $z \neq Au$, putting $x = u$ and $y = y_n$ in the inequality (3.1.1), (3.1.2) and (3.1.3), letting $n \to +\infty$, we have for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$.

$FA_{u,z}(t_0) \geq FA_{u,z}\left(\frac{2}{k}t_0 - \epsilon\right), LA_{u,z}(t_0) \leq LA_{u,z}\left(\frac{2}{k}t_0 - \epsilon\right)$
and $PA_{u,z}(t_0) \leq PA_{u,z}\left(\frac{2}{k}t_0 - \epsilon\right)$.
Letting $\epsilon \to 0$, we have
$FA_{u,z}(t_0) \geq FA_{u,z}\left(\frac{2}{k}t_0\right), LA_{u,z}(t_0) \leq LA_{u,z}\left(\frac{2}{k}t_0\right)$ and $PA_{u,z}(t_0) \leq PA_{u,z}\left(\frac{2}{k}t_0\right)$,
which contradicts (3.1.4) and so $Au = Su = z$.
Therefore, u is a coincidence point of the pair (A, S).
Now, we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.1.1), (3.1.2) and (3.1.3), we get for some $t_0 > 0$

$$(1 + \alpha F_{Su,Tv}(t_0))F_{Au,Bv}(t_0) > \alpha F_{z,Bv}(t_0) + F_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right),$$

$$(1 + \beta L_{Su,Tv}(t_0))L_{Au,Bv}(t_0) < \beta L_{z,Bv}(t_0) + L_{z,Bv}\left(\frac{2}{k}t_0 - \varepsilon\right) \text{ and}$$

$$(1 + \gamma P_{Su,Tv}(t_0))P_{Au,Bv}(t_0) < \gamma P_{z,Bv}(t_0) + P_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right), \text{ for all } \varepsilon \in \left(0, \frac{2}{k}t_0\right).$$

As $\varepsilon \to 0$ we have
$F_{z,Bv}(t_0) \geq F_{z,Bv}\left(\frac{2}{k}t_0\right), L_{z,Bv}(t_0) \leq L_{z,Bv}\left(\frac{2}{k}t_0\right)$ and $P_{z,Bv}(t_0) \leq P_{z,Bv}\left(\frac{2}{k}t_0\right)$
which contradicts (3.1.4) and so $Bv = Tv = z$.
Therefore, v is a coincidence point of the pair (B, T). Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.1.1), (3.1.2) and (3.1.3) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$.

$$(1 + \alpha F_{Sz,Tv}(t_0))F_{Az,Bv}(t_0) > \alpha \left(F_{Az,z}(t_0)\right)^2 + \min\left\{F_{Az,z}(t_0), F_{Az,z}\left(\frac{2}{k}t_0\right)\right\},$$

$$(1 + \beta L_{Sz,Tv}(t_0))L_{Az,Bv}(t_0) < \beta \left(L_{Az,z}(t)\right)^2 + \max\left\{L_{Az,z}(t_0), L_{Az,z}\left(\frac{2}{k}t_0\right)\right\} \quad \text{and}$$

$$(1 + \gamma P_{Sz,Tv}(t_0))P_{Az,Bv}(t_0) < \gamma \left(P_{Az,z}(t)\right)^2 + \max\left\{P_{Az,z}(t_0), P_{Az,z}\left(\frac{2}{k}t_0\right)\right\}.$$

Hence

$$F_{Az,z}(t_0) > \min\left\{F_{Az,z}(t_0), F_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = F_{Az,z}(t_0),$$

$$L_{Az,z}(t_0) < \max\left\{L_{Az,z}(t_0), L_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = L_{Az,z}(t_0) \quad \text{and}$$

$$P_{Az,z}(t_0) < \max\left\{P_{Az,z}(t_0), P_{Az,z}\left(\frac{2}{k}t_0\right)\right\} = P_{Az,z}(t_0)$$

which is impossible and so $Az = z = Sz$, which show that z is a common fixed point of A and S. Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarity, we can prove that z is a common fixed point of B and T. Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.1.1), (3.1.2) and (3.1.3). □

Remark. *If $B = A$ and $T = S$ in theorems (3.2), we obtain a common fixed point for a pair of self-mappings.*

Corollary 3.3. *Let $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n, \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self - mappings of an NMS $(X, F, L, P, *, \diamond)$, where $*$ is a continuous t - norm and $\diamond$ is a continuous t - conorm with $A = A_1A_2\dots A_m, B = B_1B_2\dots B_n, S = S_1S_2\dots S_p$*

and $T = T_1T_2 \dots T_q$ satisfying the inequality (3.1.1), (3.1.2) and (3.1.3) of lemma (3.1). Suppose that the pairs (A, S) and (B, T) verify the CLR_{ST} property.
Then $\{A_i\}_{i=1}^m, \{B_r\}_{r=1}^n \{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ a unique common fixed point in X provided that the pairs of families $(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p)$ and $(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q)$ commute pairwise.

Lemma 3.4. *Let A, B, S and T be self - mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the conditions $(i), (ii), (iii), (iv)$ of lemma (3.1) and*

$$\begin{aligned}
&(1+\alpha F_{Sx,Ty}(t))\, F_{Ax,By}(t) \\
&\quad > \alpha \min\left\{\ F_{Ax,Sx}(t)\ F_{By,Ty}(t),\ F_{Sx,By}(t)\ F_{Ax,Ty}(t)\ \right\} \\
&\quad + \min\left\{ \begin{array}{c} F_{Sx,Ty}(t),\ \sup\limits_{t_1+t_2=\frac{2}{k}t} \max\{F_{Ax,Sx}(t_1), F_{Sx,By}(t_2)\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t} \max\{F_{By,Ty}(t_3), F_{Ax,Ty}(t_4)\} \end{array} \right\} \qquad (3.4.1)
\end{aligned}$$

$$\begin{aligned}
&(1+\beta L_{Sx,Ty}(t))\, L_{Ax,By}(t) \\
&\quad < \beta \max\left\{\ L_{Ax,Sx}(t)\ L_{By,Ty}(t),\ L_{Sx,By}(t)\ L_{Ax,Ty}(t)\ \right\} \\
&\quad + \max\left\{ \begin{array}{c} L_{Sx,Ty}(t),\ \inf\limits_{t_1+t_2=\frac{2}{k}t} \min\{L_{Ax,Sx}(t_1), L_{Sx,By}(t_2)\}, \\ \inf\limits_{t_3+t_4=\frac{2}{k}t} \min\{L_{By,Ty}(t_3), L_{Ax,Ty}(t_4)\} \end{array} \right\} \qquad (3.4.2)
\end{aligned}$$

$$\begin{aligned}
&(1+\gamma P_{Sx,Ty}(t))\, P_{Ax,By}(t) \\
&\quad < \gamma \max\left\{\ P_{Ax,Sx}(t)\ P_{By,Ty}(t),\ P_{Sx,By}(t)\ P_{Ax,Ty}(t)\ \right\} \\
&\quad + \max\left\{ \begin{array}{c} P_{Sx,Ty}(t),\ \inf\limits_{t_1+t_2=\frac{2}{k}t} \min\{P_{Ax,Sx}(t_1), P_{Sx,By}(t_2)\}, \\ \inf\limits_{t_3+t_4=\frac{2}{k}t} \min\{P_{By,Ty}(t_3), P_{Ax,Ty}(t_4)\} \end{array} \right\} \qquad (3.4.3)
\end{aligned}$$

for all $x, y \in X, t > 0$, for some $\alpha, \beta \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. As in the proof of lemma (3.1), $Ax_n \to z, Sx_n \to z$, and $Ty_n \to z$.
Now, we show that $By_n \to z$. We assert that $\mu = z$. Assume that $\mu \neq z$.
Using the inequality (3.4.1), (3.4.2) and (3.4.3) with $x = x_n, y = y_n$, and letting $n \to +\infty$ we have for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. $F_{z,\mu}(t_0) = 1, L_{z,\mu}(t_0) = 0$ and $P_{z,\mu}(t_0) = 0$ and so we have $z = \mu$.
Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Theorem 3.5. *Let A, B, S and T be self mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the inequality (3.4.1), (3.4.2) and (3.4.3) of lemma (3.4). If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem (3.2), there exist $u, v \in X$ such that $Au = Su = Bv = Tv = z$. Therefore, u is a coincidence point of the pair $(A, \bar{S})$ and v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality

(3.4.1), (3.4.2) and (3.4.3) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$(1 + \alpha F_{Sz,Tv}(t_0))F_{Az,Bv}(t_0) > \alpha \left(F_{Az,z}(t_0)\right)^2 + \min\left\{F_{Az,z}(t_0), F_{Az,z}(t_0)\right\},$$

$$(1 + \beta L_{Sz,Tv}(t_0))L_{Az,Bv}(t_0) < \beta \left(L_{Az,z}(t_0)\right)^2 + \max\left\{L_{Az,z}(t_0), L_{Az,z}(t_0)\right\} \quad \text{and}$$

$$(1 + \gamma P_{Sz,Tv}(t_0))P_{Az,Bv}(t_0) < \gamma \left(P_{Az,z}(t_0)\right)^2 + \max\left\{P_{Az,z}(t_0), P_{Az,z}(t_0)\right\}.$$

Hence

$$F_{Az,z}(t_0) > F_{Az,z}(t_0), L_{Az,z}(t_0) < L_{Az,z}(t_0) \text{and} P_{Az,z}(t_0) < P_{Az,z}(t_0)$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T by putting $x = y = z$ in the inequality (3.4.1), (3.4.2) and (3.4.3). Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.4.1), (3.4.2) and (3.4.3). □

Let Φ be the set of all non-decreasing and continuous functions $\varphi : (0, 1] \to (0, 1]$ such that $\varphi(t) > t$ for all $t \in (0, 1]$, Ψ be the set of all non-increasing and continuous functions $\psi : (0, 1] \to (0, 1]$ such that $\psi(t) < t$ for all $t \in (0, 1]$ and Ω be the set of all non-increasing and continuous functions $\phi : (0, 1] \to (0, 1]$ such that $\phi(t) < t$ for all $t \in (0, 1]$.

Lemma 3.6. *Let A, B, S and T be self - mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the conditions $(i), (ii), (iii), (iv)$ of lemma (3.1) and*

$$\begin{aligned}&(1 + \alpha F_{Sx,Ty}(t))F_{Ax,By}(t)\\ &\geq \alpha \min\left\{F_{Ax,Sx}(t)F_{By,Ty}(t), F_{Sx,By}(t)F_{Ax,Ty}(t)\right\}\\ &+ \varphi\left(\min\left\{\begin{array}{c} F_{Sx,Ty}(t), \sup\limits_{t_1+t_2=\frac{2}{k}t} \min\left\{F_{Ax,Sx}(t_1), F_{Sx,By}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t} \min\left\{F_{By,Ty}(t_3), F_{Ax,Ty}(t_4)\right\}\end{array}\right\}\right)\end{aligned} \tag{3.6.1}$$

$$\begin{aligned}&(1 + \beta L_{Sx,Ty}(t))L_{Ax,By}(t)\\ &\leq \beta \max\left\{L_{Ax,Sx}(t)L_{By,Ty}(t), L_{Sx,By}(t)L_{Ax,Ty}(t)\right\}\\ &+ \psi\left(\max\left\{\begin{array}{c} L_{Sx,Ty}(t), \inf\limits_{t_1+t_2=\frac{2}{k}t} \max\left\{L_{Ax,Sx}(t_1), L_{Sx,By}(t_2)\right\}, \\ \inf\limits_{t_3+t_4=\frac{2}{k}t} \max\left\{L_{By,Ty}(t_3), L_{Ax,Ty}(t_4)\right\}\end{array}\right\}\right)\end{aligned} \tag{3.6.2}$$

$$\begin{aligned}&(1 + \gamma P_{Sx,Ty}(t))P_{Ax,By}(t)\\ &\leq \gamma \max\left\{P_{Ax,Sx}(t)P_{By,Ty}(t), P_{Sx,By}(t)P_{Ax,Ty}(t)\right\}\\ &+ \phi\left(\max\left\{\begin{array}{c} P_{Sx,Ty}(t), \inf\limits_{t_1+t_2=\frac{2}{k}t} \max\left\{P_{Ax,Sx}(t_1), P_{Sx,By}(t_2)\right\}, \\ \inf\limits_{t_3+t_4=\frac{2}{k}t} \max\left\{P_{By,Ty}(t_3), P_{Ax,Ty}(t_4)\right\}\end{array}\right\}\right)\end{aligned} \tag{3.6.3}$$

for all $x, y \in X, t > 0$ for some $\alpha, \beta, \gamma \geq 0$ and $1 \leq k < 2$, where $\varphi \in \Phi, \psi \in \Psi$ and $\Omega \in \phi$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof. It follows as in the proof of lemma (3.1) □

Theorem 3.7. *Let A, B, S and T be self mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the inequality (3.6.1), (3.6.2) and (3.6.3) of lemma (3.6). If the pairs (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have coincidence points. Moreover,* *if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. As in the proof of theorem (3.2) $z = Au = Su = Bv = Tv$. Since the pair (A, S) is weakly compatible and $Au = Su$, we obtain $Az = Sz$.

Now, we assert that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.6.1), (3.6.2) and (3.6.3) with $x = z$ and $y = v$, we obtain for some $t_0 > 0$

$$(1+\alpha F_{Sz,Tv}(t_0))F_{Az,Bv}(t_0) \geq \alpha\left(F_{Az,z}(t_0)\right)^2 + \varphi\left(\min\left\{F_{Az,z}(t_0), \min\left\{F_{Az,z}(\epsilon), F_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right)$$

$$(1+\beta L_{Sz,Tv}(t_0))L_{Az,Bv}(t_0) \leq \beta\left(L_{Az,z}(t_0)\right)^2 + \psi\left(\max\left\{L_{Az,z}(t_0), \max\left\{L_{Az,z}(\epsilon), L_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right)$$

$$(1+\gamma P_{Sz,Tv}(t_0))P_{Az,Bv}(t_0) \leq \gamma\left(P_{Az,z}(t_0)\right)^2 + \phi\left(\max\left\{P_{Az,z}(t_0), \max\left\{P_{Az,z}(\epsilon), P_{Az,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}\right\}\right)$$

for all $\varepsilon \in \left(0, \frac{2}{k}t_0\right)$. Letting $\varepsilon \to 0$, we get

$$F_{Az,z}(t_0) \geq \varphi\left(\min F_{Az,z}(t_0)\right) > F_{Az,z}(t_0),$$
$$L_{Az,z}(t_0) \leq \psi\left(\max L_{Az,z}(t_0)\right) < L_{Az,z}(t_0),$$
$$P_{Az,z}(t_0) \leq \phi\left(\max P_{Az,z}(t_0)\right) < P_{Az,z}(t_0),$$

which is impossible and so $Az = z = Sz$, which shows that z is a common fixed point of A and S.

Since the pair (B, T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T by putting $x = y = z$ in the inequality (3.6.1), (3.6.2) and (3.6.3). Therefore, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequality (3.6.1), (3.6.2) and (3.6.3). □

Remark. *In theorems (3.2), (3.5) and (3.7) by a similar manner, we can prove that A, B, S and T have a unique common fixed point in X if we assume that the pairs (A, S) and (B, T) verify $JCLR_{ST}$ property or CLR_{AB} property instead of CLR_{ST} property.*

Theorem 3.8. *Let A, B, S and T be self - mappings of an NMS $(X, F, L, P, *, \diamond)$ satisfying the conditions of lemma (3.1) or lemma (3.4)or lemma (3.6). If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. In view of lemma (3.1), lemma (3.4), lemma (3.6), the pairs (A, S) and (B, T) verify the CLR_{ST} property, therefore there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. The rest of the proof follows as in the proof of theorems (3.2), (3.5) and (3.7). □

Example 3.1. Let $(X, F, L, P, *, \diamond)$ be an NMS, where $X = [3, 11)$, $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$ with $F_{xy}(t) = H(t - |x - y|), L_{xy}(t) = G(t - |x - y|)$ and $P_{xy}(t) = R(t - |x - y|)$ for all $x, y \in X$ and $t > 0$. Define the self – mappings A, B, S

and T by

$$Ax = \begin{cases} 3 & if \quad x \in \{3\} \cup (5, 11) \\ 10 & if \quad x \in (3, 5] \end{cases}, \quad Bx = \begin{cases} 3 & if \quad x \in \{3\} \cup (5, 11) \\ 9 & if \quad x \in (3, 5] \end{cases}$$

$$Sx = \begin{cases} 3 & if \quad x = 3 \\ 6 & if \quad x \in (3, 5) \\ \frac{x+1}{2} & if \quad x \in [5, 11) \end{cases}, \quad Tx = \begin{cases} 3 & if \quad x = 3 \\ 9 & if \quad x \in (3, 5) \\ x - 2 & if \quad x \in [5, 11) \end{cases}$$

Therefore, $A(X) = \{3, 4\} \subset [3, 9] = T(X)$ and $B(X) = \{3, 5\} \subset [3, 6] = S(X)$. Thus, all the conditions of theorem (3.5) are satisfied and 3 is a unique common fixed point of the pairs (A, S) and (B, T). Also, it is noted that theorem (3.2) cannot be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X.

4. Conclusion

In this paper, we have used the concept of CLR_{ST} property to prove certain popular fixed point theorems for weakly compatible mappings in neutrosophic menger spaces. Some illustrative examples are also given to show the usability of the presented results.

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Associate Professor, P.G. and Research Department of Mathematics,
Raja Doraisingam Govt. Arts College, Sivagangai,
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
Email address: jeya.math@gmail.com, ORCID: https://orcid.org/0000-0002-0364-1845

Part Time Research Scholar, Government Arts College for Women, Sivagangai,
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
Department of Mathematics, Nachiappa Swamigal Arts & Science College,
Karaikudi, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
Email address: murugappan.mangai@gmail.com

Dept. of Mathematics, Government Arts College for Women, Sivagangai,
Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
Email address: jeykaliappa@gmail.com