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ON RELATIVE $(2, 2)$-Γ-HYPERIDEALS OF 2-DUO ORDERED Γ-SEMIHYPERGROUPS

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ABSTRACT. In this paper, first we obtain the necessary and sufficient condition for an ordered Γ-semihypergroup H to be a relative completely regular 2-duo ordered Γ-semihypergroup for any relative $(2, 2)$-Γ-hyperideal as well as for any relative $(2, 2)$-quasi-Γ-hyperideal of H. Then, we find the necessary and sufficient condition that $Q = (Q^2]_S$ for every relative $(2, 2)$-Γ-hyperideal Q of H to be a relative quasi-Γ-prime with $S \subseteq H$. Finally, we prove the necessary and sufficient condition for relative $(2, 2)$-Γ-hyperideal to be a relative quasi-Γ-prime for a relative completely regular and relative $(2, 2)$-Γ-hyperideal of H making a chain with inclusive relation.

1. INTRODUCTION

The notion of bi-ideal was introduced by Good and Hughes [20] and Lajos [21] generalized this notion of bi-ideal in the form of (m, n)-ideal. The concept of quasi-deal was introduced by Steinfeld [18], interestingly enough, in two algebraic structures, viz., in rings and semigroups.

The notion of Γ-semihypergroup as a generalization of semigroup, semihypergroup and Γ-semigroup was introduced by Davvaz et al. [8, 9, 22]. The notion of ordered Γ-semihypergroup was introduced and studied by Kondo and Lekkoksung [14] as an extension of the notion of ordered semihypergroups. It was further investigated by many authors [13].

The theory of relative ideal in semigroup(resp. left, right relative ideals) was given by Wallace [7]. Khan et al. [16], [17] generalized this concept in ordered semigroups. Thereafter, Basar et al [2], [3], [4], [5], [6] studied these ideals in different algebraic structures.

The concept of hyperstructures was given by Marty [11]. The theory was extensively studied from theoretical point of view as well as for its applications to many subjects in pure

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and applied mathematics. For useful review of various algebraic hyperstructures and their applications in different fields, one can refer [10], [12].

The theory of ordered semihypergroups is one that of generalizations of the concept of semihypergroups, introduced by Heidari and Davvaz [10], in a sense that every semihypergroup can be considered as an ordered semihypergroup. Kehayopulu [14] gave the notion of duo ordered semigroups. In this paper, we study the necessary and sufficient condition for an ordered Γ-semihypergroup H to be a relative completely regular 2-duo ordered Γ-semihypergroup for any relative $(2,2)$-Γ-hyperideal as well as for any relative $(2,2)$-quasi-Γ-hyperideal of H. We also obtain the necessary and sufficient condition $Q = (Q^2]_S$ for every relative $(2,2)$-Γ-hyperideal of H to be a relative quasi-Γ-prime with $S \subseteq H$. Finally, we show the necessary and sufficient condition for relative $(2,2)$-Γ-hyperideal to be a relative quasi-Γ-prime for a relative completely regular and relative $(2,2)$-Γ-hyperideal of H forming a chain with inclusive relation studied in [19] and [23].

2. Review of Basic Definitions and Fundamentals

In this section, we make the recollection of the essentially required concepts for the completion of materials which will be used throughout this paper and discussed in [5].
A hyperstructure H is a nonvoid set equipped with an hyperoperation " $\circ$ " on H defined as follows:

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (x,y) \to (x \circ y)$$

and an operation " $*$ " on $\mathcal{P}^*(H)$ defined as follows:

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (X,Y) \to X * Y$$

such that

$$X * Y = \bigcup_{(x,y)\in X\times Y} (x \circ y)$$

for any $X, Y \in \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the nonempty subsets of H. A hyperoperation " $\circ$ " on H gives rise to an operation " $*$ " on $\mathcal{P}^*(H)$. Conversely, an operation " $*$ " on $\mathcal{P}^*(H)$ gives rise to a hyperoperation " $\circ$ " on H, defined as follows: $x \circ y = \{x\} * \{y\}$. Therefore, a hypersemigroup $(H, \circ, *)$ can be identified by $(H, \circ)$ because of the interdependency of the operation " $*$" and the hyperoperation " $\circ$". Clearly, we have $X \subseteq Y \Rightarrow X * D \subseteq Y * D, D * X \subseteq D * Y$ for any $X, Y, D \in \mathcal{P}^*(H)$ and $H * H \subseteq H$. For a subset X of an hypersemigroup H, we define by $(X]$ the subset of H as follows:

$$(X] = \{s \in H :\mid s \leq x \text{ for some x} \in \text{X}\}.$$

If " $\leq$ " is an order relation on a hypersemigroup H, we define the order relation " $\preceq$ " on $\mathcal{P}^*(H)$ as follows:

$$\preceq := \{(X,Y) \mid \forall x \in X \; \exists \, y \in Y \text{ such that x} \leq \text{y}\}.$$

Therefore, for $X, Y \in \mathcal{P}^*(H)$, we denote $X \preceq Y$ if for every $x \in X$, there exists $y \in Y$ such that $x \leq y$. This is indeed, a reflexive and transitive relation on $\mathcal{P}^*(H)$.
A hyperstructure $(H, \circ)$ is called a semihypergroup if for all $x, y, z \in H$, $(x \circ y) \circ z = x \circ (y \circ z)$, i. e.,

$$\bigcup_{m\in x\circ y} m \circ z = \bigcup_{n\in y\circ z} x \circ n.$$

A nonempty subset A of a semihypergroup $(H, \circ)$ is called a subsemihypergroup of H if $A * A \subseteq A$. A semihypergroup $(H, \circ)$ equipped with a partial order " $\leq$ " on H that is compatible with semihypergroup operation " $\preceq$ " such that for all $x, y, z \in H$,

$$x \leq y \Rightarrow z \circ x \preceq z \circ y \text{ and } \mathrm{x} \circ \mathrm{z} \preceq \mathrm{y} \circ \mathrm{z},$$

is called an ordered Γ-semihypergroup. Throughout this paper, H will denote an ordered Γ-semihypergroup unless otherwise stated.

Definition 2.1. [1] Suppose that $(H, \circ, \leq)$ is an ordered Γ-semihypergroup and $S \subseteq H$. Then, a nonempty subset I of H is called a right (resp., left) relative Γ-hyperideal of H if

(i) $I \circ \Gamma \circ S \subseteq I$(resp., $S \circ \Gamma \circ I \subseteq I$); and
(ii) if $x \in I$ and $S \ni y \leq x$, then $y \in I$, i. e., if $(I]_S = I$.

A subset of H which is both a right and left relative Γ-hyperideal of H is called a relative Γ-hyperideal of H. We see that $I \circ \Gamma \circ S \subseteq I(resp.,\ S \circ \Gamma \circ I \subseteq I)$ if and only if $x \circ s \subseteq I(resp.,\ s \circ \Gamma \circ x \subseteq I)$ for every $x \in I$, and every $s \in S$. Clearly, every right(resp., left) relative Γ-hyperideal of an ordered Γ-semihypergroup H is a sub-Γ-semihypergroup of H.

Definition 2.2. [1] Suppose that $(H, \circ, \leq)$ is an ordered Γ-semihypergroup, and let $S \subseteq H$. A nonempty subset Q of H is called a relative quasi Γ-hyperideal of H if

(i) $(S \circ \Gamma \circ Q]_S \cap (Q \circ \Gamma \circ S]_S \subseteq Q$; and
(ii) $p \in Q, S \ni q \leq p \Rightarrow q \in Q$, i. e., $(Q]_S = Q$.

Definition 2.3. [1] Suppose that $(H, \circ, \leq)$ is an ordered Γ-semihypergroup, $S \subseteq H$ and m, n are nonnegative integers. A nonempty subset Q of H is called a relative (m, n)-quasi Γ-hyperideal of H if

(i) $(S^m \circ \Gamma \circ Q]_S \cap (Q \circ \Gamma \circ S^n]_S \subseteq Q$; and
(ii) $p \in Q, S \ni q \leq p \Rightarrow q \in Q$, i. e., $(Q]_S = Q$.

Definition 2.4. [1] Let $(H, \circ, \leq)$ be an ordered Γ-semihypergroup and let S be any nonempty subset of H. Then, a sub-Γ-semihypergroup B of H is said to be a relative bi-Γ-hyperideal of H if

(i) $B \circ \Gamma \circ S \circ \Gamma \circ B \subseteq B$; and
ii) for all $t \in B$, $S \ni g \leq t \Rightarrow g \in B$.

Definition 2.5. [1] An ordered Γ-semihypergroup H is called relative regular (resp. relative left regular, relative right regular) if for every $s \in S \subseteq H$, $s \in (s \circ \Gamma \circ S \circ \Gamma \circ s]_S$(resp. $s \in (S \circ \Gamma \circ s^2]_S, s \in (s^2 \circ \Gamma \circ S]_S$).

Definition 2.6. [1] An ordered Γ-semihypergroup H is called completely relative regular if it is both relative right regular and relative left regular.

Ardekani and Davvaz [15] defined the following notion for ordered semihypergroup. We define it in ordered Γ-semihypergroup in terms of relative ordered Γ-hyperideals as follows:

Definition 2.7. [1] An ordered Γ-semihypergroup H is called right (resp. left) relative duo if the right (resp. left) relative Γ-hyperideals of H are two-sided. Also, H is called relative duo if it is both right relative duo as well as left relative duo.

Definition 2.8. [1] Suppose that H is an ordered Γ-semihypergroup and let n be a positive integer. Then H is said to be an n-duo ordered Γ-semihypergroup if it satisfies the following conditions:

(1): every relative $(n,0)$-Γ-hyperideal of H is a relative $(0,n)$-Γ-hyperideal of H; and

(2): every relative $(0,n)$-Γ-hyperideal of H is a relative $(n,0)$-Γ-hyperideal of H.

Definition 2.9. Suppose that H is an ordered Γ-semihypergroup and Q is a relative $(2,2)$-Γ-hyperideal of H. Then, Q is called quasi-Γ-prime if for any relative $(2,2)$-Γ-hyperideal L and M of H, we have $L \circ \Gamma \circ M \subseteq Q \Rightarrow L \subseteq Q$ or $M \subseteq Q$.

Also, Q is called relative quasi-Γ-semiprime if for any $(2,2)$-Γ-hyperideal L of H, $L \subseteq Q$. Suppose that H is an ordered Γ-semihypergroup and A is any non-empty subset of H. Then the relative (m,n)-Γ-hyperideal $[A]_{m,n}$ is called the relative (m,n)-Γ-hyperideal of H generated by A. Similarly, $[A]_{m,0}$ and $[A]_{0,n}$ are called the relative $(m,0)$ -Γ-hyperideal and the relative $(0,n)$-Γ-hyperideal of H generated by A, respectively. Thus we have the following:

$$[A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ \Gamma \circ H \circ \Gamma \circ A^n].$$

Furthermore, if $A = \{a\}$, we denote $[\{a\}]_{m,n}$ by $[a]_{m,n}$. It is to be noted that if H is a 2-duo ordered Γ-semihypergroup, then $[a]_{0,2} = [a]_{2,0}$ for all $a \in H$.

3. Relative $(2,2)$-Γ-hyperideals of 2-duo ordered Γ-semihypergroups

The following result gives us the necessary and sufficient condition for an ordered Γ-semihypergroup H to be a relative completely regular 2-duo ordered Γ-semihypergroup in terms of a relative $(2,2)$-Γ-hyperideal of H.

Theorem 3.1. *Suppose that H is an ordered Γ-semihypergroup and $S \subseteq H$. Then H is a relative completely regular 2-duo ordered Γ-semihypergroup if and only if*

$$((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S = T = ((T^2 \cup S \circ \Gamma \circ T^2)^2]_S$$

for any relative $(2,2)$-Γ-hyperideal T of H and $S \subseteq H$.

Proof. Suppose that H is a relative completely regular 2-duo ordered Γ-semihypergroup. Furthermore, suppose that T is a relative $(2,2)$-Γ-hyperideal of H. We now show the following:

$$\begin{aligned}((T^2 \cup T^2 \circ \Gamma \circ T \circ S)^2]_S &= (T^2 \circ \Gamma \circ T^2 \cup T^2 \circ \Gamma \circ T^2 \circ \\ &\quad \Gamma \circ S \cup T^2 \circ \Gamma \circ S \circ \Gamma \circ T^2 \cup T^2 \circ \Gamma \circ S \circ T^2 \circ \Gamma \circ S]_S \\ &\subseteq (T \cup T \circ \Gamma \circ S]_S \\ &\subseteq (T \cup (T^2 \circ \Gamma \circ S \circ T^2]_S \circ \Gamma \circ S]_S \\ &\subseteq (T \cup (T \circ \Gamma \circ (T^2 \circ \Gamma \circ T^2]_S \circ \Gamma \circ S \circ \Gamma \circ T^2]_S] \circ \Gamma \circ S]_S \\ &\subseteq (T \cup T \circ \Gamma \circ T^2 \circ \Gamma \circ S \circ \Gamma \circ T^2 \circ \Gamma \circ T^2 \circ \Gamma \circ S]_S.\end{aligned}$$

As $(S \circ \Gamma \circ T^2]_S$ is a relative $(0,2)$-Γ-hyperideal of S and S is a 2-duo ordered Γ-semihypergroup, then $(S \circ \Gamma \circ T^2]_S$ is a relative $(2,0)$-Γ-hyperideal of S. That is, $(S \circ \Gamma \circ T^2]^2 \circ \Gamma \circ S \subseteq (S \circ \Gamma \circ \Gamma \circ T^2]_S$. Therefore, we have the following:

$$\begin{aligned}(T \cup T \circ T^2 \circ \Gamma \circ S \circ \Gamma \circ T^2 \circ \Gamma \circ S \circ T^2 \circ \Gamma \circ S]_S &\subseteq (T \cup T^3 \circ \Gamma \circ (S \circ \Gamma \circ T^2]_S \circ \Gamma \circ (S \circ \Gamma \circ T^2] \circ \Gamma \circ S]_S \\ &\subseteq (T \cup T^3 \circ \Gamma \circ (S \circ \Gamma \circ T^2]_S]_S \\ &\subseteq (T \cup T^2 \circ \Gamma \circ S \circ \Gamma \circ T^2]_S \\ &\subseteq (T]_S = T.\end{aligned}$$

Thus, $((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S \subseteq T$. We now have the following: $T = (T^2]_S$. Then, we do have the following:

$$\begin{aligned} T &= (T \circ \Gamma \circ T]_S \\ &= ((T^2]_S \circ \Gamma \circ (T^2]_S]_S \\ &= (T^4]_S \\ &\subseteq ((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S. \end{aligned}$$

Therefore, $((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S = T$. The assertion $T = ((T^2 \cup S \circ \Gamma \circ T^2)^2]_S$ can be proved in a similar fashion. Hence, $((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S = T = ((T^2 \cup S \circ \Gamma \circ T^2)^2]_S$.
$\Leftarrow$ Let T be a $(0,2)$-Γ-hyperideal of H. Then, T is a relative $(2,2)$-Γ-hyperideal of H, and

$$T^2 \circ \Gamma \circ S \circ \Gamma \circ T^2 \subseteq T^2 \circ \Gamma \circ T \subseteq T^2 \subseteq T,$$

and $(T]_S = T$. By the given hypothesis, we receive $((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S = T$. In a similar fashion, let I be a relative $(2,0)$-Γ-hyperideal of H. Therefore, I is a relative $(2,2)$-Γ-hyperideal of H, and therefore

$$((I^2 \cup S \circ \Gamma \circ I^2)^2]_S = I.$$

Then, we find that H is a relative completely regular 2-duo ordered Γ-semihypergroup. $\square$

In the following result, we again derive the necessary and sufficient criterion for an ordered Γ-semihypergroup H to be a relative completely regular 2-duo ordered Γ-semihypergroup but now that in the setting of a relative $(2,2)$-quasi-Γ-hyperideal of H.

Theorem 3.2. *Let H be an ordered Γ-semihypergroup. Then, H is a relative completely regular 2-duo ordered Γ-semihypergroup if and only if $((Q^2 \cup Q^2 \circ \Gamma \circ S)^2]_S = Q = ((Q^2 \cup S \circ \Gamma \circ Q^2)^2]_S$ for any $(2,2)$-quasi-Γ-hyperideal Q of H for $S \subseteq H$.*

Proof. Suppose that H is a relative completely regular 2-duo ordered Γ-semihypergroup. Furthermore, suppose that Q is a relative $(2,2)$-quasi- Γ-hyperideal of H and $S \subseteq H$. Then, we receive the following:

$$\begin{aligned} ((Q^2 \cup Q^2 \circ \Gamma \circ S)^2]_S &= (Q^2 \circ \Gamma \circ Q^2 \cup Q^2 \circ \Gamma \circ Q^2 \circ \Gamma \circ S \cup Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2 \cup Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2 \circ \Gamma \circ S]_S \\ &\subseteq (S \circ \Gamma \circ Q^2 \circ \Gamma \circ S \cup Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2]_S \\ &\subseteq (S \circ \Gamma \circ Q \circ \Gamma \circ (Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2]_S \circ \Gamma \circ S \cup S \circ \Gamma \circ Q^2]_S \\ &\subseteq (S \circ \Gamma \circ Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2 \circ \Gamma \circ S \cup S \circ \Gamma \circ Q^2]_S. \end{aligned}$$

As $(S \circ \Gamma \circ Q^2]_S$ is a relative $(0,2)$-Γ-hyperideal of H, and H is a 2-duo ordered Γ-semihypergroup, then $(S \circ \Gamma \circ Q^2]_S$ is a relative $(2,0)$-Γ-hyperideal of H. Therefore, we process the following:

$$\begin{aligned} (S \circ \Gamma \circ Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2 \circ \Gamma \circ S \cup S \circ \Gamma \circ Q^2]_S &\subseteq ((S \circ \Gamma \circ Q^2]_S \circ \Gamma \circ (S \circ \Gamma \circ Q^2]_S \circ \Gamma \circ S \circ \Gamma \circ Q^2 \\ &\subseteq ((S \circ \Gamma \circ Q^2]_S \cup S \circ \Gamma \circ Q^2]_S \\ &\subseteq (S \circ \Gamma \circ Q^2]_S \cup S \circ \Gamma \circ Q^2]_S \\ &\subseteq (S \circ \Gamma \circ Q^2]_S. \end{aligned}$$

Thus, $((Q^2 \cup Q^2 \circ \Gamma \circ S)^2]_S \subseteq (Q^2 \circ \Gamma \circ S]_S \circ \Gamma \circ (S \circ \Gamma \circ Q^2]_S \subseteq Q$. By the hypothesis, we obtain the following:

$$Q \subseteq (Q^2 \circ \Gamma \circ S \circ \Gamma \circ Q^2]_S \subseteq ((Q^2 \cup Q^2 \circ \Gamma \circ S)^2]_S.$$

The condition that $Q = ((Q^2 \cup S \circ \Gamma \circ Q^2)^2]_S$ can be proved in a similar fashion. Hence, $((Q^2 \cup Q^2 \circ \Gamma \circ S)^2]_S = Q = ((Q^2 \cup S \circ \Gamma \circ Q^2)^2]_S$.
$\Leftarrow$ Suppose that T is a relative $(0,2)$-Γ-hyperideal of H. Then, T is a relative $(2,2)$-quasi-Γ-hyperideal of H as $(T^2 \circ \Gamma \circ S]_S \cap (S \circ \Gamma \circ T^2]_S \subseteq (S \circ \Gamma \circ T^2]_S \subseteq T$ and $(T]_S = T$. By the hypothesis, we find that

$$((T^2 \cup T^2 \circ \Gamma \circ S)^2]_S = T.$$

In a similar fashion, let I be a relative $(2,0)$-Γ-hyperideal of H. Therefore, I is a relative $(2,2)$-quasi-Γ-hyperideal of H. So, $((I^2 \cup S \circ \Gamma \circ I^2)^2]_S = I$. Hence we receive that H is a relative completely 2-duo ordered Γ-semihypergroup. $\square$

The following result provides us the necessary and sufficient condition for an ordered Γ-semihypergroup H to be a relative completely regular ordered Γ-semihypergroup in the form of a relative $(2,2)$-Γ-hyperideal of H.

Lemma 3.3. *Suppose that H is an ordered Γ-semihypergroup. Then H is a relative completely regular if and only $I = (I^2]_S$ for any relative $(2,2)$-Γ-hyperideal I of H.*

Proof. Suppose that H is a relative regular completely ordered Γ-semihypergroup. Furthermore, suppose that I is a relative $(2,2)$-Γ-hyperideal of H. We then receive the following:

$$\begin{aligned}
I &\subseteq (I^2 \circ \Gamma \circ S \circ \Gamma \circ I^2]_S \\
&\subseteq (I^2 \circ \Gamma \circ S \circ \Gamma \circ (I^2 \circ \Gamma \circ S \circ \Gamma \circ I^2]_S \circ \Gamma \circ (I^2 \circ \Gamma \circ S \circ \Gamma \circ I^2]_S]_S \\
&\subseteq (I^2 \circ \Gamma \circ (S \circ \Gamma \circ I^2 \circ \Gamma \circ S) \circ \Gamma \circ I^2 \circ \Gamma \circ I^2 \circ \Gamma \circ S \circ \Gamma \circ I^2]_S \\
&\subseteq ((I^2 \circ \Gamma \circ S \circ \Gamma \circ I^2) \circ \Gamma \circ (I^2 \circ \Gamma \circ S \circ \Gamma \circ I^2]_S \\
&\subseteq (I^2]_S \\
&\subseteq (I^2]_S \\
&\subseteq (I]_S = I.
\end{aligned}$$

Therefore, $I = (I^2]_S$.
$\Leftarrow$ Suppose that $I = (I^2]_S$ for all $(2,2)$-Γ-hyperideal I of H. Furthermore, suppose that $s \in S$. Then, we receive the following:

$$\begin{aligned}
s \in [s]_{2,2} &= (([s]_{2,2})^2]_S \\
&= ((s \cup s^2 \cup s^3 \cup s^4 \cup s^2 \circ \Gamma \circ S \circ \Gamma \circ s^2]^2]_S \\
&\subseteq (s^2 \cup s^3 \cup s^4 \cup s^2 \circ \Gamma \circ S \circ \Gamma \circ s^2]_S \\
&= (s^2]_S \cup (s^3]_S \cup (s^4]_S \cup (s^2 \circ \Gamma \circ S \circ \Gamma \circ s^2]_S.
\end{aligned}$$

Therefore, $s \leq s^2$ or $s \leq s^3$ or $s \leq s^4$ or $s \in (s^2 \circ \Gamma \circ S \circ \Gamma \circ s^2]_S$. Hence in either case, S is a relative completely regular ordered Γ-semihypergroup $\square$

In the following result, we deduce an expression of a relative $(2,2)$-Γ-hyperideal for all $(2,2)$-Γ-hyperideal of an ordered Γ-semihypergroup H given that H is a relative completely regular.

Lemma 3.4. *Suppose that H is an ordered Γ-semihypergroup. If H is a relative completely regular , then $(P \circ \Gamma \circ Q]_S$ is a relative $(2,2)$-Γ-hyperideal of H for all $(2,2)$-Γ-hyperideal P, Q of H.*

Proof. We have the following:

$$\begin{aligned}(P\circ\Gamma\circ Q]_S^2\circ\Gamma\circ S\circ\Gamma\circ(P\circ\Gamma\circ Q]_S^2 &= (P\circ\Gamma\circ Q]_S\circ\Gamma\circ(P\circ\Gamma\circ Q]_S\circ\Gamma\circ S]_S\circ\\ &\quad\Gamma\circ(P\circ\Gamma\circ Q]_S\circ\Gamma\circ(P\circ\Gamma\circ Q]_S\circ\Gamma\circ S]_S\\ &\subseteq (P\circ\Gamma\circ Q\circ\Gamma\circ S\circ\Gamma\circ P\circ Q]_S\\ &\subseteq (P\circ\Gamma\circ S\circ\Gamma\circ P\circ\Gamma\circ Q]_S\\ &\subseteq ((P^2\circ\Gamma\circ S\circ\Gamma\circ P^2]_S\circ\Gamma\circ S\circ(P^2\circ\Gamma\circ S\circ\Gamma\circ P^2]_S\circ\Gamma\circ]_S\\ &\subseteq (P^2\circ\Gamma\circ(S\circ\Gamma\circ P^2\circ\Gamma\circ S\circ\Gamma\circ P^2\circ\Gamma\circ S)\circ\Gamma\circ P^2\circ\Gamma\circ Q]_S\\ &\subseteq (P^2\circ\Gamma\circ S\circ\Gamma\circ P^2\circ\Gamma\circ Q]_S\\ &\subseteq (P\circ\Gamma\circ Q]_S.\end{aligned}$$

Therefore, we receive the following:

$$(P\circ\Gamma\circ Q]_S^2\circ\Gamma\circ S\circ\Gamma\circ(P\circ\Gamma\circ Q]_S^2\subseteq(P\circ\Gamma\circ Q]_S$$

and $((P\circ\Gamma\circ Q]_S]_S=(P\circ\Gamma\circ Q]_S$. Hence, $(P\circ\Gamma\circ Q]_S$ is a relative $(2,2)$-Γ-hyperideal of H. □

In the following result, we derive the necessary and sufficient condition for a relative $(2,2)$-Γ-hyperideal of an ordered Γ-semihypergroup H to become a relative quasi-Γ-prime.

Lemma 3.5. *Suppose that H is an ordered Γ-semihypergroup. Then, $Q=(Q^2]_S$ for every relative $(2,2)$-Γ-hyperideal Q of H if and only if a relative $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-prime.*

Proof. Suppose that Q is a relative $(2,2)$-Γ-hyperideal of H such that $I=(I^2]_S$. Furthermore, suppose that Q is a relative $(2,2)$-Γ-hyperideal of H such that $I^2\subseteq Q$. Then, we receive the following: $I=(I^2]_S\subseteq Q=Q$. Therefore, Q is a relative quasi-Γ-semiprime $(2,2)$-Γ-hyperideal of H.
$\Leftarrow$ Suppose that every relative $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-semiprime. Furthermore, suppose that I a relative $(2,2)$-Γ-hyperideal of H. Then, $(I^2]_S\subseteq I$. Moreover, we show that $I\subseteq(I^2]_S$. For the following:

$$\begin{aligned}(I^2]_S^2\circ\Gamma\circ S\circ\Gamma\circ(I^2]_S^2 &= (I^2]_S\circ\Gamma\circ(I^2]_S\circ\Gamma\circ(S]_S\circ\Gamma\circ(I^2]_S\circ\Gamma\circ(I^2]_S\\ &\subseteq (I^2\circ\Gamma\circ S\circ\Gamma\circ I^2\circ\Gamma\circ I]_S\\ &\subseteq (I\circ\Gamma\circ I]_S\\ &\subseteq (I^2]_S,\end{aligned}$$

and $((I^2]_S]_S=(I^2]_S$, this shows that $(I^2]_S$ is a relative $(2,2)$-Γ-hyperideal of H. By the hypothesis, we obtain $(I^2]_S$ is a relative quasi-Γ-semiprime. As $I^2\subseteq(I^2]_S$, and $I\subseteq(I^2]_S$. Hence $I=(I^2]_S$. □

In the following result, we show the necessary and sufficient condition for $(2,2)$-Γ-hyperideal to be a relative quasi-Γ-semiprime if a relative completely regular and relative $(2,2)$-Γ-hyperideal of H forms a chain with inclusion in case of 2-duo ordered Γ-semihypergroup.

Theorem 3.6. *Suppose that H is a 2-duo ordered Γ-semihypergroup. Then every relative $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-semiprime if and ony if H is a relative completely regular and relative $(2,2)$-Γ-hyperideal of H forms a chain with inclusion.*

Proof. Suppose that every relative $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-prime. We know that every relative quasi-Γ-semiprime $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-semiprime $(2,2)$-Γ-hyperideal of H, therefore, we observe that every quasi-Γ-semiprime $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-semiprime. We now have $I = (I^2]_S$ for any relative $(2,2)$-Γ-hyperideal I of H. We know that every quasi-Γ-prime $(2,2)$-Γ-hyperideal of H is a relative quasi-Γ-semiprime $(2,2)$-Γ-hyperideal of H, we observe that for every quasi-Γ-prime $(2,2)$-Γ-hyperideal of H. We thus obtain that H is a relative completely regular. Furthermore, let P and Q be relative $(2,2)$-Γ-hyperideal of H. We observe that $(P \circ \Gamma \circ Q]_S$ is a relative $(2,2)$-Γ-hyperideal of H. By the given hypothesis, we see that $(P \circ \Gamma \circ Q]_S$ is a relative quasi-Γ-prime. It then follows the below two cases:

Case I: Let $P \subseteq (P \circ \Gamma \circ Q]_S$. We now receive the following:

$$\begin{aligned} Q &\subseteq (P \circ \Gamma \circ Q]_S \\ &\subseteq ((P^2 \circ \Gamma \circ S \circ \Gamma P^2]_S \circ \Gamma \circ]_S \\ &\subseteq ((P^2 \circ \Gamma \circ S \circ \Gamma \circ P^2]_S \circ \Gamma \circ P \circ \Gamma \circ S \circ \Gamma \circ P^2]_S \\ &\subseteq (P^2 \circ \Gamma \circ S \circ \Gamma \circ P^2 \circ \Gamma \circ S \circ \Gamma \circ P^2]_S. \end{aligned}$$

As $(S \circ \Gamma \circ P^2]_S$ is a relative $(0,2)$-Γ-hyperideal of H which is a 2-duo ordered Γ-semihypergroup, then $(S \circ \Gamma \circ P^2]_S$ is a relative $(2,0)$-Γ-hyperideal of H. Therefore, we receive the following:

$$\begin{aligned} (P^2 \circ \Gamma \circ S \circ \Gamma \circ P^2 \circ \Gamma \circ S \circ \Gamma \circ P^2 \circ \Gamma \circ S]_S &\subseteq (P^2 \circ \Gamma \circ (S \circ \Gamma \circ P]_S \circ \Gamma \circ (S \circ \Gamma \circ P^2]_S \circ \Gamma \circ S]_S \\ &\subseteq (P^2 \circ \Gamma \circ (S \circ \Gamma \circ P^2]_S]_S \\ &\subseteq (P^2 \circ \Gamma \circ S \circ \Gamma \circ P^2]_S \\ &\subseteq (P]_S \\ &\subseteq (P]_S = P. \end{aligned}$$

Thus, $Q \subseteq P$. From both the cases, we have that a relative $(2,2)$-Γ-hyperideal of H form a chain by inclusion.

$\Leftarrow$ Suppose that H is a relative regular completely ordered Γ-semihypergroup and a relative $(2,2)$-Γ-hyperideals of H form a chain by inclusion. Let P, Q and K be relative $(2,2)$-Γ-hyperideals of H such that $P \circ \Gamma \circ Q \subseteq K$. We now receive $P = (P^2]_S$ and $Q = (Q^2]_S$. If $P \subseteq Q$, then $P = (P^2]_S \subseteq (P \circ \Gamma \circ A]_S \subseteq (K]_S = K$. In a similar fashion, if $Q \subseteq P$, we get $Q = (Q^2]_S \subseteq (P \circ \Gamma \circ Q]_S \subseteq (K]_S = K$. Hence, K is a relative quasi-Γ-prime. □

4. Conclusions and/or Discussions

As an insight and impact of this paper, for our unique research contributions, we have shown the necessary and sufficient condition for an ordered Γ-semihypergroup to be a relative completely regular 2-duo ordered Γ-semihypergroup with a relative $(2,2)$-Γ-hyperideal and any relative $(2,2)$-quasi-Γ-hyperideal. Then, we have obtained the necessary and sufficient condition $Q = (Q^2]_S$ for every relative $(2,2)$-Γ-hyperideal Q to be a relative quasi-Γ-prime. Also, we have proved the necessary and sufficient condition for relative $(2,2)$-Γ-hyperideal to be a relative quasi-Γ-prime of a relative completely regular and relative $(2,2)$-Γ-hyperideal of H forming a chain with the desired inclusive condition. One can explore this much sought after area of research studied in the paper to the context of po-ternary Γ-semihypergroups and other possible algebraic structures in terms of relative Γ-hyperideals as some of potential directions for future work.

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