ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 7, Number 1 (2024), 42-46
DOI: 10.62072/acm.2024.070104
ISSN: 2582-0818

DISTRIBUTIONAL BOUNDARY VALUES OF ANALYTIC FUNCTIONS

ALEXANDER G. RAMM

ABSTRACT. Let D be a connected bounded domain in $\mathbb{R}^2$, S be its boundary which is closed, connected and smooth. Let $\Phi(z) = \frac{1}{2\pi i}\int_S \frac{\phi(s)ds}{s-z}$, $\phi \in X$, $z = x+iy$, X is a Banach space of linear bounded functionals on H^μ, a Banach space of distributions, and H^μ is the Banach space of Hoelder-continuous functions on S with the usual norm. As X one can use also the space Hoelder continuous of bounded linear functionals on the Sobolev space H^ℓ on S. Distributional boundary values of $\Phi(z)$ on S are studied in detail. The function $\Phi(t)$, $t \in S$, is defined in a new way. Necessary and sufficient conditions are given for $\phi \in X$ to be a boundary value of an analytic in D function. The Cauchy formula is generalized to the case when the boundary values of an analytic function in D are tempered distributions. The Sokhotsky-Plemelj formulas are derived for $\phi \in X$.

1. INTRODUCTION

Let D be a connected bounded domain on the complex plane, S be its boundary, which is closed and $C^{1,a}$-smooth, $0 < a \le 1$. Consider an analytic function in D, defined as

$$\Phi(z) = \frac{1}{2\pi i}\int_S \frac{\phi(s)}{s-z}ds, \quad \frac{1}{2\pi i} := c. \tag{1.1}$$

We assume that $\phi \in X$, where X is a Banach space of distributions, which are linear bounded functionals on H^μ, $\mu \in (0,1)$, and H^μ is the Banach space of the Hoelder-continuous functions ψ with the usual norm $\|\psi\|_{H^\mu} = \sup_{z\in S}|\psi(z)| + \sup_{z,z'\in S}\frac{|\psi(z)-\psi(z')|}{|z-z'|}$. The norm in X is defined as $\|\phi\|_X = \sup_{\|\psi\|_{H^\mu}=1}(\phi,\psi)$, where (ϕ,ψ) is the value of the functional $\phi \in X$ on the element $\psi \in H^\mu$. One may consider by the same methods other distributional spaces, for example, linear bounded functionals over Sobolev spaces on S, or a more general distributional space on S. We also consider the case when the boundary values of an analytic function in D are tempered distributions.

The basic new assumption is $\phi \in X$. Let us denote $c\int_S \frac{\phi(s)}{s-z}ds = \left(\phi, \frac{c}{s-z}\right)$, the value of the functional $\phi \in X$ on the element $\frac{c}{s-z} \in H^\mu$, $z \notin S$.

In the literature, it was assumed that ϕ is Hoelder-continuous or $\phi \in L^p(S)$, $p > 1$, see [2], [6], [9]. This paper is an extension of the results in [9], where ϕ belonged to $L^1(S)$.

We prove an analog of the Sokhotsky-Plemelj formulas for $\phi \in X$. They are important in applications to singular integral operators and boundary value problems, [2], [7], [9].

2010 *Mathematics Subject Classification.* 30E25, 30B30.
Key words and phrases. analytic functions; distributional boundary values of analytic functions.
Received: December 27, 2023. Accepted: March 15, 2024. Published: March 31, 2024.

By D^+ we denote D, by $D^- = D'$, where $D' = \mathbb{R}^2 \setminus \bar{D}$, the $\bar{D}$ is the closure of D. The function $\Phi(z)$ is analytic in D and in D', $\Phi(\infty) = 0$. Boundary value of $\Phi(z)$ on S has been studied by many authors, [1]–[9]. In [2] and [7] it is assumed that $\phi \in H^\mu(S)$. In [6] it is assumed that $f \in L^p(S)$, $p > 1$. The case $p = 1$ is discussed in [9], where for the first time a new definition of the boundary values of $\Phi(z)$ is given and the Sokhotsky-Plemelj formulas are proved for $\phi \in L^1(S)$. These formulas are valid almost everywhere but not everywhere if $\phi \in L^1(S)$. In this work, $\phi \in X$ and the Sokhotsky-Plemelj formulas hold in the distributional sense, as bounded linear functionals on H^μ. In [2], pp.144-146, there is a discussion of works on Riemann's problem with non-smooth coefficients, see also [10]. Our results will be useful in this direction and also in a theory of singular integral equations with distributional coefficients.

Denote the limiting values of $\Phi(z)$ when $z \to t \in S$ along the non-tangential to S directions, by $\Phi^+(t)$ when $z \in D$, $z \to t$, and by $\Phi^-(t)$ and $\psi^-(t)$ when $z \in D'$, $z \to t$. By $\Phi(t)$ we denote the values of $\Phi(z)$ when $z = t \in S$. Let us assume that $z \to t$ along the unit normal N_t to S, where the normal is directed into D. The reason for this choice of this normal will be clear later.

We are interested in the following questions:

1. In what sense the boundary values of $\Phi(z)$ can be understood?

2. How to characterize elements $\phi \in X$ which are the boundary values of analytic functions in D?

3. How to prove an analog of the Sokhotsky-Plemelj formulas when $\phi \in X$?

We answer this question in Theorem 1.

2. BASIC RESULTS

First, let us answer question 1. Below we assume everywhere (unless otherwise stated) that $\phi \in X$.

The boundary values of $\Phi(z)$ we define as follows:

$$\Big(\Phi^\pm(z), \psi(t)\Big) := \lim_{\delta \to 0} \Big(\Phi(t \pm \delta N_t), \psi(t)\Big), \quad \forall \psi \in H^\mu, \tag{2.1}$$

where $\Big(h, \psi\Big)$ is the value of $h \in X$ on the element $\psi \in H^\mu$ and N_t is the unit normal to S at the point $t \in S$; N_t is directed into D.

The reason for the choice of the normal N_t pointing into D is clear: since $\delta > 0$ the points $t \pm \delta N_t \in D^\pm$.

In more detail: at a point $t \in S$ we choose the x-axis to be tangential to S at the point t; the x-axis is directed so that its positive direction forms angle $\pi/2$ with the positive direction of y-axis, that is, with N_t; the y-axis is directed along N_t, and $\delta > 0$ to be a number that will tend to 0. With this choice, the point $t + N_t\delta$ belongs to D.

Let us prove the existence of the limit (2.1). We rewrite (2.1) as

$$\lim_{\delta \to 0} \Big(c \int_S \frac{\phi(s)ds}{s - t \mp \delta N_t}, \psi(t)\Big) = \Big(\phi, c \int_S \frac{\psi(t)dt}{s - t} \pm \frac{1}{2}\psi(s)\Big), \quad \forall \psi \in H^\mu, \tag{2.2}$$

where the functional ϕ acts on a function of s. The integral $c \int_S \frac{\phi(s)ds}{s-t\mp\delta N_t}$ is understood as $c\Big(\phi, \frac{1}{s-t\mp\delta N_t}\Big)$, where the functional ϕ acts on the element $\frac{1}{s-t\mp\delta N_t} \in H^\mu$ and $t \pm \delta N_t$ is a parameter, so the value $\Big(\phi, \frac{1}{s-t\mp\delta N_t}\Big)$ does make sense although the functional ϕ is not defined pointwise.

We have

$$\lim_{\delta\to 0} c\Big(\frac{1}{s-t\mp\delta N_t},\psi\Big) = c\Big(\frac{1}{s-t},\psi\Big) \pm \frac{\psi(s)}{2}, \tag{2.3}$$

where the expression $\Big(\frac{1}{s-t},\psi\Big)$ we write as $\int_S \frac{\psi(t)dt}{s-t}$. This integral makes sense classically.

Thus, the formula (2.2) is proved.From (2.1) and (2.2) it follows that

$$\Big(\Phi^+ - \Phi^-,\psi\Big) = \Big(\phi,\psi\Big), \quad \forall\psi\in H^\mu. \tag{2.4}$$

Similarly, we derive

$$\Big(\Phi^+ + \Phi^-,\psi\Big) = -\Big(\phi,B\psi\Big), \quad \forall\psi\in H^\mu, \tag{2.5}$$

where

$$B\psi := \frac{1}{i\pi}\int_S \frac{\psi(t)dt}{t-s}. \tag{2.6}$$

Formulas (2.4)–(2.5) are the Sokhotsky-Plemelj formulas for $\phi\in X$.

Thus, we have proved the version of the Sokhotsky-Plemelj formulas valid for $\phi\in X$:

$$\Big(\Phi^+ - \Phi^-,\psi\Big) = \Big(\phi,\psi\Big), \quad \Big(\Phi^+ + \Phi^-,\psi\Big) = -\Big(\phi,B\psi\Big), \quad \forall\psi\in H^\mu. \tag{2.7}$$

Indeed,

$$(\Phi^+ + \Phi^-,\psi) = (\phi,\frac{2}{2i\pi}\int_S\frac{\psi(t)dt}{s-t}) = -(\phi,B\psi). \tag{2.8}$$

This is formula (2.5).

We have

$$\Big(B\phi,\psi\Big) := -\Big(\phi,B\psi\Big), \quad \forall\psi\in H^\mu. \tag{2.9}$$

Therefore, we have answered questions 1 and 3.

Let us answer question 2. If $z\in D$, $z\notin S$, then the expression $c\int_S\frac{\phi ds}{s-z}$ makes sense since the generalized function $\phi\in X$ acts on a function $c\frac{1}{s-z}$ which is an element of H^μ and z is a parameter. The boundary value Φ^+ makes sense as the value $-\Big(\phi,B\psi\Big) + \frac{1}{2}\Big(\phi,\psi\Big)$. The functional ϕ is the boundary value Φ^+ on S if and only if $\Phi(z)$ is analytic in D and $\Phi^+=\phi$, that is, $-\Big(\phi,B\psi\Big)+\frac{1}{2}\Big(\phi,\psi\Big) = \Big(\phi,\psi\Big)$. So, one has:

$$\Big(\phi,B\psi\Big) + \frac{1}{2}\Big(\phi,\psi\Big) = 0, \quad \forall\psi\in H^\mu. \tag{2.10}$$

Under these conditions $\Phi^-(z)=0$ in D^-.

Let us formulate the results.

Theorem 1. *Let $\phi \in X$. The boundary values of analytic function $\Phi(z)$ are understood as in* (2.1)*. These boundary values do exist and satisfy the generalized Sokhotsky-Plemelj conditions* (2.7)*. The element $\phi \in X$ is a boundary value of an analytic function* (1.1) *in D^+ if and only if* (2.10) *holds. Under this condition $\Phi(z) = 0$ in D^-.*

Remark 1. *One can use as X the space $H^{-\ell}$ dual to H^ℓ, where $H^\ell = H^\ell(S)$ is the Sobolev space, $\ell \geq 0$. This is of interest because the elements of X for large ℓ have high singularities. Thus, the methods, developed in this paper, allow one to study analytic functions whose boundary values are highly singular, for example, they can be tempered distributions or distributions in $\mathcal{D}$.*

3. Boundary values that are tempered distributions

If $z \notin S$ then a tempered distribution ϕ is well defined on $(s-z)^{-1}$. If $z = t + N_t\delta$, then

$$\lim_{\delta\to 0} \frac{1}{s - t - N_t\delta} = \frac{1}{s-t-i0} = \frac{1}{s-t} + i\pi\delta(s-t), \quad s, t \in S, \tag{3.1}$$

where $\delta(s-t)$ is the delta-function. Let $\psi(t) \in C^\infty(S)$ be any test function. Then (3.1) is understood as

$$\lim_{\delta\to 0}\left(\frac{1}{s-t-N_t\delta}, \psi(t)\right) = \int_S \frac{\psi(t)dt}{s-t} + i\pi \int_S \delta(s-t)\psi(t)dt = -\int_S \frac{\psi(t)dt}{t-s} + i\pi\psi(s). \tag{3.2}$$

If $\psi \in C^\infty(S)$ and $S \in C^\infty$, as we have assumed in this Section, then $\int_S \frac{\psi(t)dt}{t-s} \in C^\infty(S)$. Therefore, the value

$$((\phi, \frac{1}{s-t}), \psi(t)) = -(\phi, \int_S \frac{\psi(t)dt}{t-s})$$

is well defined. By (ϕ, η) we denote the value of the tempered distribution ϕ on a test function η. As in Section 2, one proves the Sokhotsky-Plemelj formulas (2.4)–(2.5). A necessary and sufficient condition for a distribution ϕ to be a boundary value of an analytic in D function (1.1) is equation (2.10).

If this equation holds, then the Cauchy formula holds

$$\Phi(z) = \frac{1}{2\pi i} \int_S \frac{\phi(s)ds}{s-z}, \quad z \in D. \tag{3.3}$$

The integral in (3.3) is understood as the value of the distribution ϕ on the test function $\frac{1}{s-z}$, $z \in D$, $z \notin S$.

4. CONCLUSION

Let D be a connected bounded domain in $\mathbb{R}^2$, S be its boundary, which is closed, connected and smooth. Let $\Phi = \Phi(z),\ z = x + iy$, be an analytic function in D. Its boundary values are defined by formula (2.1). The limits in this formula do exist.

A necessary and sufficient condition for a generalized function $\phi \in X$ to be a boundary value of an analytic in D function $\Phi(z)$, defined in (1.1), is equation (2.10). The space X can be, for example, distributional space $H^{-\ell}$, the space of tempered distributions, or the space $\mathcal{D}$.

An analog (2.7) of the Sokhotsky-Plemelj formulas under the assumption $\phi \in X$ is proved. This analog holds also when ϕ is a tempered distribution on S.

REFERENCES

[1] A. Calderon, Cauchy integral on Lipschitz curves and related operators, Proc. Natl. Acad. Sci. USA, 74, N4, (1977), 1324-1327.
[2] F. Gahov, *Boundary value problems*, Nauka, Moscow, 1977. (in Russian)
[3] I. Gradshtein, I. Ryzhik, *Tables of integrals, series and products*, Gos. Izdat. Fiz.-Math. Lit., Moscow, 1962. (In Russian)
[4] I. Gel'fand, G. Shilov, *Generalized functions, Vol. 1*, Gos. Izdat. Fiz.-Math. Lit., Moscow, 1959. (In Russian)
[5] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1983.
[6] S. Mikhlin, S. Prössdorf, *Singular integral operators*, Springer-Verlag, New York, 1986.
[7] N. Muskhelishvili, *Singular integral equations*, Nauka, Moscow, 1968. (In Russian)
[8] I. Privalov, *Boundary values of univalent analytic functions*, Gostekhizdat, Moscow, 1950. (In Russian)
[9] A. G. Ramm, Boundary values of analytic functions, Far East Journal of Appl. Math., 116, N3, (2023), 215-227. http://dx.doi.org/10.17654/0972096023011
[10] A. G. Ramm, New definition of singular integral operator, Annals Communic. Math., 6. N4, (2023), 220-224.

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KS 66506, USA.
Email address: ramm@ksu.edu