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QUASISTATIC EVOLUTION OF DAMAGE IN THERMO-ELECTRO-ELASTIC-VISCOPLASTIC MATERIALS WITH NORMAL COMPLIANCE

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ABSTRACT. We have studied contact problem in a quasistatic process with boundary conditions, for which we couple a thermal effect and damage and electrical effect. We assume that the contact is with normal compliance in a form with a gap function and is associated to a slip rate-dependent version of Coulomb's law of dry friction. We give the variational formulation, then the existence and the uniqueness of the weak solution. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, and fixed points.

1. INTRODUCTION

Numerous contacts occur between deformable bodies or between deformable and rigid bodies in various industrial and daily life scenarios. Consequently, significant endeavors have been dedicated to modeling and analyzing these contact processes, resulting in a substantial body of engineering literature on the subject [5, 21, 14]. Due to the intricate nature of contact phenomena, they give rise to novel and compelling mathematical models. The mathematical scrutiny of contact issues relies on fundamental physical principles and demands expertise in partial differential equations, nonlinear analysis, and numerical methods.

A piezoelectric material is one that produces an electrical voltage when subjected to mechanical stress, and conversely, exhibits mechanical deformations when an electric field is applied. Such materials are commonly employed in various industries, serving as switches in radiotronics, electroacoustics, or measuring equipment. Comprehensive models for elastic materials with piezoelectric effects are detailed in [4, 24, 25, 26].

The study of problems involving the electro-mechanical properties of materials has been extensive. Furthermore, a natural extension of these coupled electro-elastic models involves incorporating temperature as an additional state variable to address thermal effects alongside piezoelectric effects.

To comprehend the impact of temperature on the behavior of real materials such as metals, magmas, and polymers, researchers in mathematics, physics, and engineering have

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delved into the study of thermo-elastic and thermo-viscoplastic constitutive laws. For detailed examples and insights, refer to [20, 22, 23, 7, 1].

Damage, as another internal state variable, plays a crucial role in engineering by directly impacting the structural integrity of machines. Due to its significance, an increasing number of engineering publications have explored damage models. Mathematical investigations into models considering the influence of internal material damage on the contact process have been undertaken. For a mathematical analysis of one-dimensional problems, see [12].

The earliest mechanical damage models based on thermodynamic considerations can be traced back to [8]. More recent and general models, as presented in [9, 10, 11, 13, 19], derive from the principle of virtual power. In these works, material damage is characterized by a damage function denoted by α, confined to values between zero and one. When $\alpha = 1$, the material is undamaged; when $\alpha = 0$, the material is entirely damaged, and for $0 < \alpha < 1$, partial damage occurs, resulting in reduced system capacity.

Contact problems involving the coupling between thermal and mechanical fields are considered in [6, 15, 16, 17, 18].

This paper is dedicated to exploring a novel mathematical model that characterizes quasi-static contact conditions involving normal compliance and adhering to the Coulomb law of dry friction between a thermo-piezoelectric body and an electrically conductive foundation.

Our examination of the contact phenomenon unfolds through various stages, encompassing mathematical modeling and variational analysis, which includes establishing the existence and uniqueness of the solution. In Section 2, the contact model is introduced, and insights are provided on the associated boundary conditions. Moving forward, Section 3 outlines the assumptions regarding the data and formulates the variational description. The subsequent Section 5 is dedicated to proving the existence and uniqueness of the solution.

2. PROBLEM STATEMENT

Consider a material body situated within a bounded domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$, featuring a well-defined boundary surface Γ. This surface is divided into three measurable parts, namely Γ_1, Γ_2 and Γ_3, with the condition $meas\Gamma_1 > 0$. The unit outward normal vector on Γ is denoted as ν. The body is fixed within a structure on Γ_1. Surface tractions of density f_2 and voluminal forces of density f_0, both varying slowly with respect to time, act on Γ_2 and Ω respectively. Additionally, the body is subject to electric charges with volume density q_0 and surface electric charges. To describe the latter, we introduce another partition of the boundary into three parts: Γ_a, Γ_b and Γ_3 with $meas(\Gamma_a) > 0$. Frictional contact with a conductive foundation occurs on Γ_3, where the electric potential vanishes on Γ_a, and a prescribed surface electric charge density q_2 is applied on Γ_b. At each time instant, Γ_3 is further divided into two parts: one where the body and foundation are in contact, and the other where they are separated. The boundary of the contact part represents a free boundary, determined by the solution of the problem. For generality, we assume the existence of a gap, denoted as $g = g(x)$, between Γ_3 and the foundation in the reference configuration. This gap is measured along the outer normal ν. It is crucial to note that the assumption $meas(\Gamma_1) > 0$ is essential in quasistatic problems. Without this assumption, the problem becomes noncoercive mathematically, and many estimates and results outlined below may not hold. This aligns with the physical scenario where, if $\Gamma_1 = 0$, the body is not restrained and can move freely in space as a rigid body.

The classical formulation of the mechanical problem involving an electro-elastic-viscoplastic body with damage and thermal effects can be articulated as follows.

Problem P**.** Find a displacement field $\boldsymbol{u} : \Omega\times(0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega\times(0,T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$, an electric displacement field $\boldsymbol{D} : \Omega\times(0,T) \to \mathbb{R}^d$, and a damage field $\alpha : \Omega\times(0,T) \to \mathbb{R}$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) =&\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) - \mathcal{E}^* E(\varphi(t))\\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(s)) + \mathcal{E}^* E(\varphi(t)), \varepsilon(\boldsymbol{u}(s)), \theta(s), \alpha(s))ds \\ &\qquad \text{in } \Omega,\ \text{a.e } .t \in (0,T),\end{aligned} \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi) \qquad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{2.3}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \qquad \text{in } \Omega \times (0,T), \tag{2.4}$$

$$\begin{aligned}\dot{\theta} - k_0\Delta\theta = \Theta\left(\boldsymbol{\sigma}(t) - \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^* E(\varphi(t)), \varepsilon\left(\boldsymbol{u}\right), \theta, \alpha\right) + \rho \\ \text{in } \Omega \times (0,T),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\dot{\alpha} - k_1\Delta\alpha + \partial\varphi_K(\alpha) \ni S\left(\boldsymbol{\sigma}(t) - \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^* E(\varphi(t)), \varepsilon(\boldsymbol{u}), \theta, \alpha\right) \\ \text{in } \Omega \times (0,T),\end{aligned} \tag{2.6}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0,T), \tag{2.7}$$

$$\boldsymbol{\sigma\nu} = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{2.8}$$

$$-\sigma_\nu = p\left(u_\nu - g\right) \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.9}$$

$$\left.\begin{aligned}&\|\boldsymbol{\sigma}_\tau\| \le \mu\left(\|\dot{\boldsymbol{u}}_\tau\|\right)|\sigma_\nu| \\ &-\boldsymbol{\sigma}_\tau = \mu\left(\|\dot{\boldsymbol{u}}_\tau\|\right)|\sigma_\nu| \tfrac{\dot{\boldsymbol{u}}_\tau}{\|\dot{\boldsymbol{u}}_\tau\|} \quad \text{if} \quad \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0}\end{aligned}\right\} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.10}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{2.11}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{2.12}$$

$$\boldsymbol{D}.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{2.13}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = \psi\left(u_\nu - g\right)\phi_l\left(\varphi - \varphi_0\right) \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.14}$$

$$k_0\frac{\partial\theta}{\partial\nu} + k_2\theta = 0 \qquad \text{on } \Gamma \times (0,T), \tag{2.15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. \tag{2.16}$$

Now we will provide some comments on the equalities, inclusion and boundary conditions (2.1)-(2.16) . Equations (2.1)-(2.2) represents the law of thermo-electro-elastic-viscoplastic behaviour with damage were $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and elasticity operators, respectively, and $\mathcal{G}$ is the viscoplastic operator, where θ represents the absolute temperature and α is the damage field. $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathcal{E}^*$ is its transposition, and (2.3)-(2.4) represent the equilibrium equations for the stress and electric displacement fields. (2.5) represents the conservation of energy where k_0 is a strictly positive constant, Θ is a nonlinear constitutive function that represents the heat generated by the internal forces, ρ is a data, which represents the source of heat density. The evolution of the damage field is governed by the inclusion (2.6) with k_1 is a strictly positive constant, $\partial\varphi_K$ represents the sub-differential of the indicator function φ_K of the set K of admissible damage functions defined by $K=\{\alpha\in H^1(\Omega)|\, 0\leq\alpha\leq 1 \text{ a.e. in} W\}$ and S is a given constituent function; which represents the source of the damage in the system. Equalities (2.7) and (2.8) are the conditions at the displacement-traction limits, respectively. Equation (2.9) represents the normal compliance condition on the contact surface Γ_3 where p is a given positive function, which will be described below. (2.10) represents a version of the Coulomb law of dry friction, (2.11) is a homogeneous Neumann boundary condition for the damage field on Γ

(2.12) and (2.13) represent the electric boundary conditions described on Γ_a and Γ_b. The electrical condition on the potential contact surface Γ_3 presented by Equality (2.14) , where ϕ_l is the potential of the electric foundation. The function ϕ_l is given by

$$\phi_l(s)=\begin{cases}-l & \text{if } s<-l,\\ s & \text{if } -l\leq s\leq l,\\ l & \text{if } s>l.\end{cases} \tag{2.17}$$

here l is a large positive constant, it may be arbitrarily large, higher than any possible peak voltage in the system. The function ψ is given bellow. For more details see [20]. (2.15) represent a Fourier boundary condition for the temperature, (2.16) are the initial date.

3. Variational formulation and preliminaries

Here are some notations and conventions that we will use throughout this paper. We denote by $\mathbb{S}^d(d=2,3)$ the space of symmetric tensors of order two on $\mathbb{R}^d$; $(,)$ and $\|.\|$ represent respectively the scalar product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$. Thus, we have

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\quad \forall\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d \quad\text{and}\quad \boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}\quad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d,$$
$$\|\boldsymbol{u}\|=(\boldsymbol{u}\cdot\boldsymbol{u})^{\frac{1}{2}},\quad\forall\boldsymbol{u}\in\mathbb{R}^d\quad\text{and}\quad\|\boldsymbol{\sigma}\|=(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{\frac{1}{2}},\quad\forall\boldsymbol{\sigma}\in\mathbb{S}^d.$$

Here and below, the indices i,j and k run from 1 to d and the summation convention over repeated indices is used. We denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on the boundary given by

$$v_\nu=\boldsymbol{v}.\boldsymbol{\nu},\quad \boldsymbol{v}_\tau=\boldsymbol{v}-u_\nu\boldsymbol{\nu}.$$

We denote by $\boldsymbol{\sigma}=\boldsymbol{\sigma}(\boldsymbol{x};t)$ the field of stresses, by $\boldsymbol{u}=\boldsymbol{u}(x;t)$ the field of displacements and by $\varepsilon(\boldsymbol{u})$ the field of infinitesimal deformations. To simplify the notations, we do not explicitly indicate the dependence of the functions on $x\in\overline{\Omega}$ and $t\in[0,T]$. For a stress field $\boldsymbol{\sigma}$ we denote by $\boldsymbol{\sigma}_\nu$ and $\boldsymbol{\sigma}_\tau$ the normal and tangential components at the boundary given by

$$\sigma_\nu=\sigma\nu\cdot\boldsymbol{\nu},\quad \boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}.$$

Now we introduce a Hilbert space

$$H^1(\Omega)=\{u\in L^2(\Omega)\quad|\quad \partial_iu\in L^2(\Omega), i=1,....d\}.$$

we introduce the spaces

$$H=L^2(\Omega;\mathbb{R}^d),\quad H_1=\{\boldsymbol{u}\in H\mid\varepsilon(\boldsymbol{u})\in\mathcal{H}\}=H^1(\Omega;\mathbb{R}^d),$$
$$\mathcal{H}=\big\{\boldsymbol{\sigma}=(\sigma_{ij})\mid\sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\big\}=L^2(\Omega;\mathbb{S}^d),\quad \mathcal{H}_1=\{\boldsymbol{\sigma}\in\mathcal{H}\mid \operatorname{Div}\boldsymbol{\sigma}\in H\}.$$

The spaces H, H_1; $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the scalar products given by

$$(\boldsymbol{u},\boldsymbol{v})_H=\int_\Omega u_iv_i dx\quad\forall\boldsymbol{u},\boldsymbol{v}\in H,\quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}=\int_\Omega\sigma_{ij}\tau_{ij}dx\quad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H},$$

$$(\mathbf{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u},\boldsymbol{v}\in H_1,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_H, \quad \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathcal{H}_1.$$

where $\varepsilon : H^1\left(\Omega;\mathbb{R}^d\right) \to L^2\left(\Omega;\mathbb{S}^d\right)$ and $\operatorname{Div} : \mathcal{H}_1 \to L^2\left(\Omega;\mathbb{R}^d\right)$ are respectively the operators of deformation and divergence, defined by

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{u}_{i,j}+\boldsymbol{u}_{j,i}\right), \quad \operatorname{Div}(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

The associated norms in H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$ respectively. Since the boundary Γ is Lipschitz, the exterior normal vector ν at the boundary is defined a.e. For any vector field $\boldsymbol{v}\in H_1$ we use the notation $\boldsymbol{v}$ to denote the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ. Recall that the trace map $\gamma : H_1 \to H_\Gamma$ is linear and continuous, but not surjective. The image of H_1 by this map is denoted by H_Γ, this subspace injects continuously into $L^2(\Gamma)^d$. Let H'_Γ be the dual of H_Γ, and $(.;.)$ the duality product between H'_Γ and H_Γ For all $\boldsymbol{\sigma}\in\mathcal{H}_1$, there exists an element $\boldsymbol{\sigma\nu}\in H'_\Gamma$ such that

$$(\boldsymbol{\sigma}\nu,\gamma v) = (\boldsymbol{\sigma},\varepsilon(v))_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},v)_H \quad \forall v\in H_1.$$

Also, if $\boldsymbol{\sigma}$ is regular enough (e.g. C^1), we have the formula

$$(\sigma\nu,\gamma v) = \int_\Gamma \sigma\nu.v da, \quad \forall v\in H_1.$$

So, for fairly regular $\boldsymbol{\sigma}$ we have the following formula (Green's Formula):

$$(\boldsymbol{\sigma},\varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div}\boldsymbol{\sigma},\boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\boldsymbol{v}da, \quad \forall\boldsymbol{v}\in H_1.$$

We define the closed subspaces of $L^2(\Omega$ and H_1

$$\mathcal{V} = \left\{v\in L^2(\Omega) \mid \varepsilon_{ij}(v)\in L^2(\Omega)\right\} = H^1(\Omega), \quad V = \left\{\boldsymbol{v}\in H_1 \mid v=0 \text{ sur } \Gamma_1\right\}. \quad (3.1)$$

Since $meas(\Gamma_1)>0$, Korn's inequality holds on V, then, there exists a constant $C_K>0$ depending only on Ω and Γ_1 such that

$$\|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}} \geq C_K\|\boldsymbol{v}\|_{H^1(\Omega)^d}, \quad \forall\boldsymbol{v}\in V.$$

A proof of Korn's inequality may be found in ([27] ,p.79). we consider the inner product and the associated norm given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}},\ \|\boldsymbol{v}\|_V = \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}},\ \boldsymbol{u},\boldsymbol{v}\in V. \quad (3.2)$$

It follows that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent norms on V and therefore $(V,\|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem and (3.2) there exists a constant $c_0>0$ depending only on Ω,Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\|\boldsymbol{v}\|_V, \quad \forall\boldsymbol{v}\in V. \quad (3.3)$$

In what follows, we define the Sobolev spaces associated with the electrical unknowns (field of the electrical displacement $\mathbf{D}$ and the electrical potential φ) of the electro-mechanical problem which will be introduced in this paper. Let the spaces

$$\mathcal{W} = \left\{\mathbf{D}=(D_i) \mid D_i\in L^2(\Omega), \operatorname{div}\mathbf{D}\in L^2(\Omega)\right\}, \quad (3.4)$$

$$W = \left\{\xi\in H^1(\Omega), \xi=0 \text{ on } \Gamma_a\right\}. \quad (3.5)$$

where $\operatorname{div} \mathbf{D} = (D_{i,i})$. These spaces $\mathcal{W}$ and W are real Hilbert spaces endowed with the scalar products given by

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = (\mathbf{D}, \mathbf{E})_H + (\operatorname{div} \mathbf{D}, \operatorname{div} \mathbf{E})_{L^2(\Omega)}, \tag{3.6}$$

$$(\varphi, \xi)_W = \int_\Omega \nabla\varphi . \nabla\xi dx, \tag{3.7}$$

and the associated norms $\|.\|_{\mathcal{W}}$ and $\|.\|_W$, respectively.

$$\|\mathbf{D}\|^2_{\mathcal{W}} = \|\mathbf{D}\|^2_{L^2(\Omega)^d} + \|\operatorname{div} \mathbf{D}\|^2_{L^2(\Omega)}, \|\phi\|_W = \|\nabla\phi\|_{L^2(\Omega)^d}$$

Since $meas(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality is satisfied, thus,

$$\|\nabla\zeta\|_H \geq c_F \|\zeta\|_{H^1(\Omega)}, \quad \forall\zeta \in W, \tag{3.8}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\zeta = (\zeta_{.,i})$. It follows from (3.8) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace theorem, there exist a constant $\tilde{c}_0$ such that

$$\|\phi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0 \|\phi\|_W, \quad \forall\phi \in W. \tag{3.9}$$

Moreover,recall that when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds

$$(\boldsymbol{D}, \nabla\zeta)_H + (\operatorname{div} \boldsymbol{D}, \zeta)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D} \cdot \boldsymbol{\nu} \zeta da, \quad \forall\zeta \in H^1(\Omega). \tag{3.10}$$

For any real Hilbert space X, we use the classical notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$, where $1 \leq p \leq \infty$ and $k \geq 1$. For $T > 0$ we denote by $C(0, T; X)$ and $C^1(0, T; X)$ the space of continuous and continuously differentiable functions from $[0, T]$ to X, respectively, with the norms

$$\|\boldsymbol{u}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{u}(t)\|_X.$$

$$\|\boldsymbol{u}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|\boldsymbol{u}(t)\|_X + \max_{t\in[0,T]} \|\dot{\boldsymbol{u}}(t)\|_X.$$

In order to obtain existence and uniqueness results in study of variational inequality that we have in contact mechanics. Given the operators $A : X \to X$ and $B : X \to X$, the functions $j : X \times X \to \mathbb{R}$ and $f : [0, T] \to X$. We consider the following problems

Problem I Find $u : [0, T] \to X$ such that

$$\begin{aligned} &(A\dot{u}(t), v - \dot{u}(t))_X + (Bu(t), v - \dot{u}(t))_X + j(u(t), v) \\ &\qquad -j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_X \quad \forall v \in X, t \in [0, T], \end{aligned} \tag{3.11}$$

$$u(0) = u_0. \tag{3.12}$$

Theorem 3.1. *Let X be a Hilbert space. Assume that the operator $A : X \to X$ satisfies*

$$\begin{cases} (a) \text{ There exists } m_A > 0 \text{ such that} \\ (Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|^2_X, \quad \forall u_1, u_2 \in X. \\ (b) \text{ There exists } L > 0 \text{ such that} \\ \|Au_1 - Au_2\|_X \leq L \|u_1 - u_2\|_X, \forall u_1, u_2 \in X. \end{cases} \tag{3.13}$$

The functional $j : X \times X \to \mathbb{R}$ satisfied

$$\begin{cases} \text{(a) For all } u \in X, j(u,.) \text{ is convex and l.s.c.} \\ \text{(b) There exists } \alpha > 0 \text{ such that} \\ j(u_1, v_2) - j(u_1, v_1) + j(u_2, v_1) - j(u_2, v_2) \\ \leq \alpha \|u_1 - u_2\|_X \|v_1 - v_2\|_X, \quad \forall u_1, u_2, v_1, v_2 \in X. \end{cases} \tag{3.14}$$

The operator $B : X \to X$ satisfies

$$\exists L_B > 0 \text{ such that } \|Bu - Bv\|_X \leq L_B \|u - v\|_X, \forall u, v \in X. \tag{3.15}$$

Finally, we assume that

$$f \in C([0,T], X). \tag{3.16}$$

$$u_0 \in X. \tag{3.17}$$

Then

*1) There is a unique solution $u \in C^1([0,T];X)$ to problem **I**.*

*2) If u_1 and u_2 are two solutions of the problem **I** corresponding to the data $f_1, f_2 \in C([0,T];X)$, then there exists $c > 0$ such that*

$$\|\dot{u}_1(t) - \dot{u}_2(t)\|_X \leq c\left(\|f_1(t) - f_2(t)\|_X + \|u_1(t) - u_2(t)\|_X\right), \quad \forall t \in [0,T]. \tag{3.18}$$

The proof of Theorem 3.1 can be found in reference [20].

Now we consider X and H two real Hilbert spaces such that the inclusion map of $(V, \|.\|_X)$ in $(H, \|.\|_H)$ is continuous and dense. Identifying the dual of H with itself, i.e. we can write the Gelfand triplet $X \subset H \subset X'$. The notations $\|.\|_X$, $\|.\|_{X'}$ and $(.,.)_{X' \times X}$ represent the norms on X, X' and the product of duality between X' and X, respectively

Theorem 3.2. *Let $X \subset H \subset X'$ be a Gelfand triple. Assume that $A : X \to X'$ is a hemicontinuous and monotone operator that satisfies*

$$(Av, v)_{X' \times X} \geq w\|v\|_X^2 + \varsigma \quad \forall v \in X, \tag{3.19}$$

$$\|Av\|_{X'} \leq C\left(\|v\|_X + 1\right) \quad \forall v \in X. \tag{3.20}$$

for some constants $w > 0, C > 0$ and $\varsigma \in \mathbb{R}$. Then, given $u_0 \in H$ and $f \in L^2(0,T;X')$, there exists a unique function u that satisfies

$$\begin{aligned} &u \in L^2(0,T;X) \cap C([0,T];H), \dot{u} \in L^2(0,T;X'), \\ &\dot{u}(t) + Au(t) = f(t) \quad \text{a.e. } t \in (0,T), \\ &u(0) = u_0. \end{aligned}$$

The previous abstract result can be found in [2, 3].

In the study of the problem P, we consider the following assumptions

The viscosity operator $\mathcal{A}:\Omega\times\mathbb{S}^d\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}(a)\text{ There exists } L_{\mathcal{A}}>0 \quad\text{such that}\\ \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|,\\ \quad\text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e } \boldsymbol{x}\in\Omega.\\ (b)\text{ There exists } m_{\mathcal{A}}>0 \quad\text{such that}\\ (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2,\\ \quad\text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e } \boldsymbol{x}\in\Omega.\\ (c)\text{ The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\\ \quad\text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ (d)\text{ The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\mathbf{0})\in\mathcal{H}.\end{cases}\tag{3.21}$$

The elastic function $\mathcal{B}:\Omega\times\mathbb{S}^d\times\mathbb{R}\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}(a)\text{ There exists } L_{\mathcal{B}}>0 \quad\text{such that}\\ \qquad\|\mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}_1)-\mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{B}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|,\quad\forall\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\forall\text{ a.e } \mathbf{x}\in\Omega.\\ (b)\text{The mapping } \mathbf{x}\mapsto\mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega,\\ \qquad\forall\boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ (c)\text{ The mapping } \mathbf{x}\mapsto\mathcal{B}(\mathbf{x},0)\in\mathcal{H}.\end{cases}\tag{3.22}$$

The plasticity operator $\mathcal{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{S}^d$ satisfies

$$\begin{cases}(a)\text{There exists a constant } L_{\mathcal{G}}>0 \quad\text{such that}\\ \|\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\theta_1,\alpha_1)-\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\theta_2,\alpha_2)\|\\ \qquad\le L_g\left(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|+\|\theta_1-\theta_2\|+\|\alpha_1-\alpha_2\|\right),\\ \forall\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\quad\forall\theta_1,\theta_2\in\mathbb{R},\quad\forall\alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e. } \mathbf{x}\in\Omega.\\ (b)\text{ The mapping } \mathbf{x}\mapsto\mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\theta,\alpha) \text{ is Lebesgue measurable on } \Omega,\\ \forall\boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d,\forall\theta,\alpha\in\mathbb{R}.\\ (c)\text{ The mapping } \mathbf{x}\mapsto\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},0,0)\in\mathcal{H}.\end{cases}\tag{3.23}$$

The function $\Theta:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies

$$\begin{cases}(a)\text{ There exists a constant } L_{\Theta}>0 \quad\text{such that}\\ \|\Theta(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\theta_1,\alpha_1)-\Theta(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\theta_2,\alpha_2)\|\le L_{\Theta}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ +\|\theta_1-\theta_2\|+\|\alpha_1-\alpha_2\|),\\ \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\in\mathbb{S}^d,\ \text{and } \theta_1,\theta_2,\alpha_1,\alpha_2\in\mathbb{R},\ \text{a.e } \boldsymbol{x}\in\Omega.\\ (b)\text{ For any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d,\ \text{and } \theta,\alpha\in\mathbb{R},\boldsymbol{x}\mapsto\Theta(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\theta,\alpha) \text{ is Lebesgue}\\ \quad\text{measurable on } \Omega.\\ (c)\text{ The mapping } \boldsymbol{x}\mapsto\Theta(\boldsymbol{x},\mathbf{0},\mathbf{0},0,0)\in L^2(\Omega).\end{cases}\tag{3.24}$$

The function of the damage source $S : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} (a)\text{There exists a constant } L_S > 0 \quad \text{such that} \\ \|S(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \theta_1, \alpha_1) - S(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \theta_2, \alpha_2)\| \\ \qquad\qquad \le L_S(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \forall \theta_1, \theta_2 \in \mathbb{R}, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto S(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \theta, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \forall \theta, \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \mathbf{x} \mapsto S(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0, 0) \in L^2(\Omega). \end{cases} \tag{3.25}$$

The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\psi > 0 \text{ such that} \\ \|\psi(., u_1) - \psi(., u_2)\| \le L_\psi \|u_1 - u_2\|, \text{ for all } u_1, u_2 \in \mathbb{R}. \\ \text{(b) There exists } M_\psi > 0 \text{ such that } \|\psi(\boldsymbol{x}, u)\| \le M_\psi, \\ \qquad \text{for all } u \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c)} \boldsymbol{x} \mapsto \psi(\boldsymbol{x},) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}, \\ \text{(d)} \boldsymbol{x} \mapsto \psi(\boldsymbol{x}, u) = 0, \quad \text{for all } u \le 0. \end{cases} \tag{3.26}$$

Electric permittivity operator $\boldsymbol{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } \boldsymbol{B}(\boldsymbol{x}, E) = (b_{ij}(\boldsymbol{x})E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } b_{ij} = b_{ji} \in L^\infty(\Omega), 1 \le i, j \le d. \\ \text{(c) There exists a constant } m_{\boldsymbol{B}} > 0 \text{ such that} \\ \quad \boldsymbol{B}E.E \ge m_{\boldsymbol{B}}\|E\|^2, \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{cases} \tag{3.27}$$

The piezoelectric operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{E} = (e_{ijk}), e_{ijk} \in L^\infty(\Omega), 1 \le i, j, k \le d. \\ (b) \quad \mathcal{E}(\mathrm{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}^*\boldsymbol{\tau}, \text{ for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases} \tag{3.28}$$

The normal compliance function $p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) There exists } L > 0 \text{ such that} \\ \quad \|p(\boldsymbol{x}, u_1) - p(\boldsymbol{x}, u_2)\| \le L\|u_1 - u_2\| \\ \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(d) For any } u \in \mathbb{R}, \boldsymbol{x} \mapsto p(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3 \\ \text{(e) } \boldsymbol{x} \mapsto p(\boldsymbol{x}, u) = 0, \text{ for all } u \le 0. \end{cases} \tag{3.29}$$

The coefficient of friction satisfies

$$\begin{cases} (a)\ \mu : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \\ (b) \text{ There exists } L_\mu > 0 \text{ such that} \\ \|\mu(\boldsymbol{x}, r_1) - \mu(\boldsymbol{x}, r_2)\| \le L_\mu\|r_1 - r_2\|, \quad \forall r_1, r_2 \in \mathbb{R}_+, \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \\ (c) \text{ The mapping } \boldsymbol{x} \longmapsto \mu(\boldsymbol{x}, r) \text{ is Lebesgue measurable on } \Gamma_3, \quad \forall r \in \mathbb{R}_+, \\ (d) \text{ The mapping } \boldsymbol{x} \longmapsto \mu(\boldsymbol{x}, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.30}$$

We assume that the body forces, surface tractions, volume heat source and initial date, satisfy

$$f_0 \in C(0,T,L^2(\Omega)), \quad f_2 \in C(0,T,L^2(\Gamma_2)^d), \quad \rho \in L^2(0,T,L^2(\Omega)). \tag{3.31}$$

$$\mathbf{u}_0 \in V, \quad \theta_0 \in \mathcal{V}, \alpha_0 \in K, q_0 \in C(0,T,L^2(\Omega)), q_2 \in C(0,T,L^2(\Gamma_b)), \tag{3.32}$$

$$k_i > 0; \quad i = 0,1,2. \tag{3.33}$$

We define the bilinear form $a_i : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}, (i = 0,1)$.

$$a_0(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi dx + k_2 \int_\Gamma \zeta\xi d\gamma. \tag{3.34}$$

$$a_1(\zeta,\xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi dx. \tag{3.35}$$

Next. We define four mappings $j : V \times V \to \mathbb{R},\ h : V \times W \to W,\ f : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p(u_\nu - g)v_\nu d\Gamma + \int_{\Gamma_3} \mu\left(\|\dot{\boldsymbol{u}}_\tau\|\right) p(u_\nu - g)\left\|\boldsymbol{v}_\tau\right\| d\Gamma \quad \forall \boldsymbol{u},\boldsymbol{v} \in V, \tag{3.36}$$

$$(h(\mathbf{u},\varphi),\zeta)_W = \int_{\Gamma_3} \psi\left(u_\nu - g\right)\phi_l\left(\varphi - \varphi_0\right)\zeta da, \tag{3.37}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} da, \tag{3.38}$$

$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta dx - \int_{\Gamma_b} q_2(t)\zeta da, \tag{3.39}$$

for all $\boldsymbol{u},\boldsymbol{v} \in V$, $\varphi,\zeta \in W$ and $t \in [0,T]$. Note that

$$\boldsymbol{f} \in C\left(0,T;V\right), \quad q \in C(0,T;W). \tag{3.40}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)-(2.16).

problem PV**.** Find a displacement field $\boldsymbol{u} : (0,T) \to V$, a stress field $\boldsymbol{\sigma} : (0,T) \to \mathcal{H}$, an electric potential $\varphi : (0,T) \to W$, a damage field $\alpha : (0,T) \to H^1(\Omega)$, and a temperature

$\theta : (0,T) \to H^1(\Omega)$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) =& \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\varepsilon(\boldsymbol{u}(t)) - \mathcal{E}^* E(\varphi(t)) \\ &+ \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(s)) + \mathcal{E}^* E(\varphi(t)), \varepsilon(\boldsymbol{u}(s)), \theta(s), \alpha(s)) ds \\ & \text{in } \Omega, \text{ a.e } .t \in (0,T),\end{aligned} \tag{3.41}$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\mathbf{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \tag{3.42}$$

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\zeta)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}(t)), \nabla\zeta)_H + (h(\boldsymbol{u}(t), \varphi(t)), \zeta)_W = (q(t), \zeta)_W, \tag{3.43}$$

$$\begin{aligned}&(\dot{\theta}(t), v)_{L^2(\Omega)} + a_0(\theta(t), v) \\ &= (\Theta(\boldsymbol{\sigma}(t) - \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^* E(\varphi(t)), \varepsilon(\boldsymbol{u}(t)), \theta(t), \alpha(t)), v)_{L^2(\Omega)} \\ &\quad + (\rho(t), v)_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega), \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.44}$$

$$\begin{aligned}&\alpha(t) \in K, (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\geq \left(S\left(\boldsymbol{\sigma}(t) - \mathcal{A}(\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^* E(\varphi(t)), \theta(t), \alpha(t)\right), \xi - \alpha(t)\right)_{L^2(\Omega)}, \\ &\qquad \forall \xi \in K, t \in (0,T),\end{aligned} \tag{3.45}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0. \tag{3.46}$$

Our main existence and uniqueness result for Problem PV is in the following section.

4. EXISTENCE AND UNIQUENESS

Theorem 4.1. *Assume that (3.21)-(3.33) hold, Then there exists a unique solution* $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \alpha, \boldsymbol{D})$ *to problem* PV. *Moreover, the solution has the regularity*

$$\boldsymbol{u} \in C^1(0,T;V), \tag{4.1}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}), \quad \operatorname{Div} \boldsymbol{\sigma} \in C(0,T;H), \tag{4.2}$$

$$\varphi \in C(0,T;W), \tag{4.3}$$

$$\theta \in L^2\left(0,T;\mathcal{V}\right) \cap C\left(0,T;L^2(\Omega)\right) \cap W^{1,2}\left(0,T;\mathcal{V}'\right), \tag{4.4}$$

$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{4.5}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{4.6}$$

A set of functions $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \alpha, \boldsymbol{D}$) satisfying (3.41)-(3.46) is called a weak solution of the thermo-elctro-elastic-vescoplastic contact problem P. We conclude that, under the conditions specified in the previous theorem, the problem (2.1)-(2.16) has a unique weak solution. We now deal with the proof of Theorem 4.1, which is based on classical results of evolution equations and inequalities with monotonic operators and fixed point arguments. To this end, we assume that (3.21)-(3.33) hold, and C will represent a strictly positive constant which may depend on the data of the problem but it is independent of time t, and its value may change from place to place. For simplicity, we remove in the following the explicit dependence of the various functions on $x \in \Omega \cup \Gamma$.

Let $\eta \in C(0,T;\mathcal{H})$, we consider the following variational problem.

Problem $\mathcal{P}_\eta^1$. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$, such that

$$\begin{gathered}(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}_\eta(t)), \varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\boldsymbol{u}_\eta(t)), \varepsilon(\boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)))_{\mathcal{H}} + (\eta(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t))_{\mathcal{H}} \\ +j\left(\boldsymbol{u}_\eta(t), \boldsymbol{v}\right) - j\left(\boldsymbol{u}_\eta(t), \dot{\boldsymbol{u}}_\eta(t)\right) \geq \left(f, \boldsymbol{v} - \dot{\boldsymbol{u}}_\eta(t)\right)_V \\ \text{a.e. } t \in (0,T), \quad \text{ for all } \boldsymbol{v} \in V,\end{gathered} \tag{4.7}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.8}$$

We have the following result for $\mathcal{P}_\eta^1$

Lemma 4.2. *There is a unique solution $\boldsymbol{u}_\eta \in C^1([0,T];V)$ to the problem (4.7)-(4.8). Moreover, if $\boldsymbol{u}_{\eta_1}$, and $\boldsymbol{u}_{\eta_2}$ Are two solutions of the problem (4.7)-(4.8) corresponding to the data $\eta_1, \eta_2 \in C([0,T];\mathcal{H})$, then there exists $c > 0$ such that*

$$\|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)\|_V \leq c\left(\|\eta_1(t) - \eta(t)\|_{\mathcal{H}} + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V\right), \quad \forall t \in [0,T]. \tag{4.9}$$

Proof. We use the Riesz-Fréchet representation theorem to define the operators $A : V \to V$, $B : V \to V$ and the function $\mathbf{f} : [0,T] \to V$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \tag{4.10}$$

$$(B\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{B}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \tag{4.11}$$

$$(\mathbf{f}_\eta(t), \boldsymbol{v})_V = (f(t), \boldsymbol{v})_V - (\eta(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}. \tag{4.12}$$

For all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $t \in [0,T]$. It follows from the assumptions (3.21)

$$\begin{aligned}(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V &= \left(\mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_1\right)\right) - \mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_2\right)\right), \varepsilon\left(\boldsymbol{u}_1\right) - \varepsilon\left(\boldsymbol{u}_2\right)\right)_{\mathcal{H}} \\ &\geq m_{\mathcal{A}} \left\|\varepsilon\left(\boldsymbol{u}_1\right) - \varepsilon\left(\boldsymbol{u}_2\right)\right\|_{\mathcal{H}}^2 \geq C\left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V^2.\end{aligned}$$

Also for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$ and $\boldsymbol{v} \in V$ we have

$$\begin{aligned}\|(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{v})_V\| &= \left\|\left(\mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_1\right)\right) - \mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_2\right)\right), \varepsilon(\boldsymbol{v})\right)\right\|_{\mathcal{H}}, \\ &\leq \left\|\mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_1\right)\right) - \mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_2\right)\right)\right\|_{\mathcal{H}} \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \\ &\leq L_{\mathcal{A}} \left\|\varepsilon\left(\boldsymbol{u}_1\right) - \varepsilon\left(\boldsymbol{u}_2\right)\right\|_{\mathcal{H}} \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \\ &\leq L_{\mathcal{A}} \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V \|\boldsymbol{v}\|_V.\end{aligned}$$

So

$$\|A\boldsymbol{u}_1 - A\boldsymbol{u}_2\|_V \leq C\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V.$$

Similarly, using (3.22) we obtain

$$\|B\boldsymbol{u}_1 - B\boldsymbol{u}_2\|_V \leq C\left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V.$$

By means of (3.29)-(3.30) and (3.3) we obtain

$$\begin{aligned}&j\left(\boldsymbol{u}_1, \boldsymbol{v}_2\right) - j\left(\boldsymbol{u}_1, \boldsymbol{v}_1\right) + j\left(\boldsymbol{u}_2, \boldsymbol{v}_1\right) - j\left(\boldsymbol{u}_2, \boldsymbol{v}_2\right) \\ &= \int_{\Gamma_3}\left(p\left(u_{1\nu} - g\right) - p\left(u_{2\nu} - g\right)\right)\left(v_{2\nu} - v_{1\nu}\right) d\Gamma \\ &+ \int_{\Gamma_3}\left(\mu\left(\|\boldsymbol{u}_{1\tau}\|\right) p\left(u_{1\nu} - g\right) - \mu\left(\|\boldsymbol{u}_{2\tau}\|\right) p\left(u_{2\nu} - g\right)\right)\left(\|\boldsymbol{v}_{2\tau}\| - \|\boldsymbol{v}_{1\tau}\|\right) d\Gamma \\ &\leq L_p\|\gamma\| \left\|u_{1\nu} - u_{2\nu}\right\|_{L^2(\Gamma_3)} \left\|\boldsymbol{v}_1 - \boldsymbol{v}_2\right\|_V \\ &\quad + \mu^* L_p\|\gamma\| \left\|u_{1\nu} - u_{2\nu}\right\|_{L^2(\Gamma_3)} \left\|\boldsymbol{v}_1 - \boldsymbol{v}_2\right\|_V \\ &\quad + L_\mu p^*\|\gamma\|^2 \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V \left\|\boldsymbol{v}_1 - \boldsymbol{v}_2\right\|_V\end{aligned} \tag{4.13}$$

For all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1$ and $\boldsymbol{v}_2 \in V$, which shows that the functional j satisfies the condition (3.14) (b) on V. We note that $f_\eta \in C([0,T];V)$. Finally, note that (3.31) shows that the condition (3.17) is also satisfied. And from it, the conditions of the Theorem 3.1 are satisfied, which means that there is a single weak solution $\boldsymbol{u}_\eta \in C^1(0,T;V)$. □

In the second step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.2, to construct the following variational problem for the an electrical potential.

Problem $\mathcal{P}^2_\eta$. Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$\begin{aligned}(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\zeta)_H + (h(\boldsymbol{u}_\eta(t), \varphi_\eta(t)), \zeta)_W \\ = (q(t), \zeta)_W, \text{ for all } \zeta \in W, t \in (0,T).\end{aligned} \tag{4.14}$$

We have the following result for problem $\mathcal{P}^2_\eta$

Lemma 4.3. *Problem (4.14) has unique solution φ_η which satisfies the regularity (4.3). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (4.14) corresponding to $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0,T];\mathcal{H})$, then there exists $C > 0$ such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq C\,\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V \quad \forall t \in [0,T]. \tag{4.15}$$

To prove the above lemma, we use an abstract existence and unique result which may be found in [20] .

In the third step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.2 to consider the following variational problem.

Problem $\mathcal{P}_\chi$. Find the temperature field $\theta_\chi : (0,T) \to L^2(\Omega)$

$$\begin{aligned}\left(\dot{\theta}_\chi(t), \mathbf{v}\right)_{L^2(\Omega)} + a_0(\theta_\chi(t), \mathbf{v}) = (\chi(t) + \rho(t), \mathbf{v})_{L^2(\Omega)}, \\ \forall \mathbf{v} \in L^2(\Omega), \text{ a.e. } t \in (0,T),\end{aligned} \tag{4.16}$$

$$\theta_\chi(0) = \theta_0, \text{ in } \Omega. \tag{4.17}$$

Lemma 4.4. *There exists a unique solution θ_χ to the auxiliary problem $\mathcal{P}_\chi$ satisfying (4.4).*

Proof. The inclusion of the trace of $(\mathcal{V}; \|.\|_{\mathcal{V}})$ in $(L^2(\Omega), \|.\|_{L^2(\Omega)})$ is continuous and dense, we can write the Gelfand Triplet

$$\mathcal{V} \subset L^2(\Omega) \subset \mathcal{V}'.$$

We use the notation $(.,.)_{\mathcal{V}'\times\mathcal{V}}$ to denote the duality product between $\mathcal{V}'$ and $\mathcal{V}$, we have

$$(\zeta, \xi)_{\mathcal{V}'\times\mathcal{V}} = (\zeta, \xi)_{L^2(\Omega)}.$$

We consider the linear operator $A_0 : \mathcal{V} \to \mathcal{V}'$ defined by

$$(A_0\zeta, \xi)_{\mathcal{V}'\times\mathcal{V}} = \mathfrak{a}_0(\zeta, \xi), \quad \forall \zeta, \xi \in \mathcal{V}.$$

we use this last equality and the de definition (3.34), we can write for all $\zeta, \xi \in \mathcal{V}$

$$\left|(A_0\zeta, \xi)_{\mathcal{V}'\times\mathcal{V}}\right| \leq k_0 \int_\Omega |\nabla\zeta.\nabla\xi| dx + k_2 \int_\Gamma |\zeta\xi| da,$$

by using Hölder's inequality, we obtain

$$\left|(A_0\zeta, \xi)_{\mathcal{V}'\times\mathcal{V}}\right| \leq k_0\|\nabla\zeta\|_{L^2(\Omega)^d}\|\nabla\xi\|_{L^2(\Omega)^d} + k_2\|\zeta\|_{L^2(\Gamma)}\|\xi\|_{L^2(\Gamma)}. \tag{4.18}$$

Keeping in mind Sobolev trace Theorem, the inequality (4.18) becomes

$$\|A_0\zeta\|_{\mathcal{V}'} \leq C\|\zeta\|_{\mathcal{V}},$$

which shows that $A_0 : \mathcal{V} \to \mathcal{V}'$ is continuous and therefore hemicontinuous. Easy to make sure that

$$(A_0\zeta, \zeta)_{\mathcal{V}'\times\mathcal{V}} \geq 0,$$

i.e, that $A_0 : \mathcal{V} \to \mathcal{V}'$ is a monotone operator.

On the other hand, by means of (4.18), as $k_2 > 0$ and for all $\zeta \in \mathcal{V}$ we obtain

$$(A_0\zeta, \zeta)_{\mathcal{V}'\times\mathcal{V}} \geq k_0\|\nabla\zeta\|^2_{L^2(\Omega)^d},$$

and from where

$$(A_0\zeta, \zeta)_{\mathcal{V}'\times\mathcal{V}} \geq k_0\|\zeta\|^2_{\mathcal{V}} - k_0\|\zeta\|^2_{L^2(\Omega)}.$$

Then, A_0 satisfies condition (3.19) of Theorem 3.2 with $w = k_0$ and $\varsigma = -k_0\|\xi\|^2_{L^2(\Omega)}$, according to the above we have also A_0 satisfies the condition (3.20). It now follows from the regularity (3.31) and $\chi \in L^2(0, T; \mathcal{V}')$ that $\rho_\chi = \chi + \rho \in L^2(0, T; L^2(\Omega))$ and $\theta_0 \in L^2(\Omega)$. Finally, we notice that all the conditions of Theorem 3.2 hold, so we conclude that there exists a unique function θ_χ which satisfies

$$\theta_\chi \in L^2(0, T; \mathcal{V}) \cap C\left([0, T]; L^2(\Omega)\right), \quad \dot{\theta}_\chi \in L^2\left(0, T; \mathcal{V}'\right), \tag{4.19}$$

$$\dot{\theta}_\chi(t) + A_0\theta_\chi(t) = \rho_\chi(t) \quad \text{a.e.t } \in (0, T), \tag{4.20}$$

$$\theta_\chi(0) = \theta_0. \tag{4.21}$$

□

In the fourth step we let $\lambda \in C(0, T; L^2(\Omega))$

Problem $\mathcal{P}_\lambda$. Find the damage field $\alpha_\lambda : (0, T) \to L^2(\Omega)$ such that $\alpha_\lambda(t) \in \mathcal{Y}$ and

$$\begin{aligned}(\dot{\alpha}_\lambda(t), \xi - \alpha_\lambda)_{L^2(\Omega)} &+ a\left(\alpha_\lambda(t), \xi - \alpha_\lambda(t)\right) \\ &\geq (\lambda(t), \xi - \alpha_\lambda(t))_{L^2(\Omega)} \quad \forall\xi \in K, \text{ a.e. } t \in (0, T),\end{aligned} \tag{4.22}$$

$$\alpha_\lambda(0) = \alpha_0. \tag{4.23}$$

For the study of problem $\mathcal{P}_\lambda$, we have the following result.

Lemma 4.5. *There exists a unique solution α_λ to the auxiliary problem $\mathcal{P}_\lambda$ satisfying (4.5).*

Proof. The inclusion mapping of $\left(H^1(\Omega), \|\cdot\|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset \left(H^1(\Omega)\right)'.$$

We use the notation $(.,.)_{(H^1(\Omega))'\times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $(H^1(\Omega))$. We have

$$(\alpha, \varsigma)_{(H^1(\Omega))'\times H^1(\Omega)} = (\alpha, \varsigma)_{L^2(\Omega)}, \quad \forall\alpha \in L^2(\Omega), \varsigma \in H^1(\Omega),$$

and we note that K is a closed convex set in $(H^1(\Omega))$, using the definition (3.35) of the bilinear form a, for all $\vartheta, \varsigma \in H^1(\Omega)$, we have $a(\vartheta, \varsigma) = a(\varsigma, \vartheta)$ and

$$|a(\vartheta, \varsigma)| \leq k_0\|\nabla\vartheta\|_H\|\nabla\varsigma\|_H \leq c\|\vartheta\|_{H^1(\Omega)}\|\varsigma\|_{H^1(\Omega)},$$

Therefore, a is continuous and symmetric. Thus, for all $\vartheta \in H^1(\Omega)$, we have

$$a(\vartheta, \vartheta) = k_0\|\nabla\vartheta\|^2_H,$$

so

$$a(\vartheta, \vartheta) + (k_0 + 1)\|\vartheta\|^2_{L^2(\Omega)} \geq k_0\left(\|\nabla\vartheta\|^2_H + \|\vartheta\|^2_{L^2(\Omega)}\right),$$

which implies

$$a(\vartheta,\vartheta)+c_0l\|\vartheta\|^2_{L^2(\Omega)}\geq c_1\|\vartheta\|^2_{H^1(\Omega)} \text{ with } c_0=k_0+1 \text{ and } c_1=k_0.$$

Finally, we use that condition $\alpha_0\in K$, and we use a standard result for parabolic variational inequalities (see [3], p. 124), we find that there exists a unique function $\alpha\in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$, such that $\alpha(0)=\alpha_0$, $\alpha(t)\in K$ for all $t\in[0,T]$ and for almost all $t\in(0,T)$

$$(\dot\alpha_\lambda(t),\varsigma-\alpha_\lambda)_{(H^1(\Omega))'\times H^1(\Omega)}+a\left(\alpha_\lambda(t),\varsigma-\alpha_\lambda(t)\right)\geq\left(\lambda(t),\varsigma-\alpha_\lambda(t)\right)_{L^2(\Omega)},$$
$$\forall\varsigma\in K,$$

□

In the fifth step, we use $\boldsymbol{u}_\eta$, φ_η, θ_η and α_λ obtained in Lemmas 4.2, 4.3, 4.4 and 4.5, respectively to construct the following Cauchy problem for the stress field.

Problem $\mathcal{P}_{\eta,\chi,\lambda}$**.** Find the stress field $\boldsymbol{\sigma}_{\eta,\lambda,\mu}:[0,T]\to\mathcal{H}$ which is a solution of the problem

$$\sigma_{\eta,\chi,\lambda}(t)=\mathcal{B}\varepsilon\left(u_\eta(t)\right)+\int_0^t\mathcal{G}\left(\sigma_{\eta,\chi,\lambda}(s),\varepsilon\left(u_\eta(s)\right),\theta_\chi(s),\alpha_\lambda(s)\right)ds\quad\forall t\in[0,T]. \tag{4.24}$$

Lemma 4.6. *$\mathcal{P}_{\eta,\chi,\lambda}$ has a unique solutions $\boldsymbol{\sigma}_{\eta,\chi,\lambda}\in C(0,T;\mathcal{H})$. Moreover, if $\boldsymbol{\sigma}_{\eta_i,\chi_i,\lambda_i}$, $\boldsymbol{u}_{\eta_i}$, θ_{χ_i} and α_{λ_i} represent the solutions of Problems $\mathcal{P}_{\eta,\chi,\lambda}$, $\mathcal{P}^1_\eta$, $\mathcal{P}^2_\eta$ $\mathcal{P}_\chi$ and, $\mathcal{P}_\lambda$ respectively, for $(\boldsymbol{\eta}_i,\chi_i,\lambda_i)\in C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$, $i=1,2$, then there exists $C>0$ such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\chi_1,\lambda_1}(t)-\boldsymbol{\sigma}_{\eta_2,\chi_2,\lambda_2}(t)\|^2_{\mathcal{H}}\leq &C\Big(\|\boldsymbol{u}_{\eta_1}(t)-\boldsymbol{u}_{\eta_2}(t)\|^2_V\\ &+\int_0^t\|\alpha_{\lambda_1}(s)-\alpha_{\lambda_2}(s)\|^2_{L^2(\Omega)}ds\\ &+\int_0^t\|\theta_{\chi_1}(s)-\theta_{\chi_2}(s)\|^2_{L^2(\Omega)}ds\\ &+\int_0^t\|\boldsymbol{u}_{\eta_1}(s)-\boldsymbol{u}_{\eta_2}(s)\|^2_V\,ds\Big).\end{aligned} \tag{4.25}$$

Proof. Let $\boldsymbol{\Sigma}_{\eta,\chi,\lambda}:C(0,T;\mathcal{H})\to C(0,T;\mathcal{H})$ be the operator given by

$$\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}(t)=\mathcal{B}\varepsilon\left(\boldsymbol{u}_\eta(t)\right)+\int_0^t\mathcal{G}\left(\boldsymbol{\sigma}_{\eta,\chi,\lambda,\mu}(s),\varepsilon\left(\boldsymbol{u}_\eta(s)\right),\theta_\chi(s),\alpha_\lambda(s)\right)ds, \tag{4.26}$$

Let $\boldsymbol{\sigma}_i\in W^{1,\infty}(0,T;\mathcal{H})$, $i=1,2$ and $t_1\in(0,T)$. Using hypothesis (3.23) and Holder's inequality, we find

$$\|\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_1(t_1)-\Sigma_{\eta,\chi,\lambda}\boldsymbol{\sigma}_2(t_1)\|^2_{\mathcal{H}}\leq L^2_{\mathcal{G}}T\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,ds.$$

Integration on the time interval $(0,t_2)\subset(0,T)$, it follows that

$$\int_0^{t_2}\|\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_1(t_1)-\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_2(t_1)\|^2_{\mathcal{H}}\,dt_1\leq L^2_{\mathcal{G}}T\int_0^{t_2}\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,dsdt_1.$$

Therefore

$$\|\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_1(t_2)-\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_2(t_2)\|^2_{\mathcal{H}}\leq L^4_{\mathcal{G}}T^2\int_0^{t_2}\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\,dsdt_1.$$

For $t_1, t_2, ..., t_n \in (0,T)$, we generalize the procedure above by recurrence on n. We obtain the inequality

$$\begin{aligned}&\|\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_1(t_n) - \boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_2(t_n)\|^2_{\mathcal{H}} \\ &\qquad \leq L_{\mathcal{G}}^{2n}T^n \int_0^{tn} \cdots \int_0^{t_2}\int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds dt_1 \dots dt_{n-1}.\end{aligned}$$

Which implies

$$\|\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_1(t_n) - \boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_2(t_n)\|^2_{\mathcal{H}} \leq \frac{L_{\mathcal{G}}^{2n}T^{n+1}}{n!}\int_0^T \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\|\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_1 - \boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma}_2\|^2_{C(0,T;\mathcal{H})} \leq \frac{L_{\mathcal{G}}^{2n}T^{n+2}}{n!}\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|^2_{C(0,T;\mathcal{H})}.$$

It follows from this inequality that for large n enough, the operator $\boldsymbol{\Sigma}^n_{\eta,\chi,\lambda}$ is a contraction on the Banach space $C(0,T;\mathcal{H})$, and therefore there exists a unique element $\boldsymbol{\sigma} \in C(0,T;\mathcal{H})$ such that $\boldsymbol{\Sigma}_{\eta,\chi,\lambda}\boldsymbol{\sigma} = \boldsymbol{\sigma}$. Moreover, $\boldsymbol{\sigma}$ is the unique solution of Problem $\mathcal{P}_{\eta,\chi,\lambda}$. Consider now $(\boldsymbol{\eta}_1, \chi_1, \lambda_1), (\boldsymbol{\eta}_2, \chi_2, \lambda_2) \in C(0,T;\mathcal{H}\times L^2(\Omega))$ and for $i = 1, 2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\theta_{\chi_i} = \theta_i$, $\alpha_{\lambda_i} = \alpha_i$ and $\boldsymbol{\sigma}_{\eta_i,\chi_i,\lambda_i} = \boldsymbol{\sigma}_i$. We have

$$\begin{aligned}\boldsymbol{\sigma}_i(t) = \mathcal{B}\varepsilon(\boldsymbol{u}_i(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_i(s), \varepsilon(\boldsymbol{u}_i(s)), \theta_i(s), \alpha_i(s))\, ds,\\ \text{a.e. } t \in (0,T).\end{aligned} \tag{4.27}$$

and using the properties (3.22) and (3.23) of $\mathcal{B}$ and $\mathcal{G}$ we find

$$\begin{aligned}&\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}} \leq C\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V \\ &+ \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)}\, ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|^2_L (\Omega) ds \\ &+ \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V\, ds + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}}\, ds\Big), \quad \forall t \in [0,T].\end{aligned} \tag{4.28}$$

We use Gronwall argument in the previous inequality to deduce (4.25), which concludes the proof of Lemma 4.6. □

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$ the operator $\mathcal{E}$, the functions Θ and S for $t \in (0,T)$, we consider the element

$$\Lambda(\eta,\chi,\lambda)(t) = \left(\Lambda^1(\eta,\chi,\lambda)(t), \Lambda^2(\eta,\chi,\lambda)(t), \Lambda^3(\eta,\chi,\lambda)(t)\right) \in \mathcal{H}\times L^2(\Omega)\times L^2(\Omega), \tag{4.29}$$

defined by

$$\begin{aligned}&\left(\Lambda^1(\eta,\chi,\lambda)(t), \boldsymbol{v}\right)_{\mathcal{H}\times V} = (\mathcal{E}^*\nabla\varphi_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &\qquad + \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}_{\eta,\chi,\lambda}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \theta_\chi(t), \alpha_\lambda(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}}, \forall \boldsymbol{v} \in V,\end{aligned} \tag{4.30}$$

$$\Lambda^2(\eta,\chi,\lambda)(t) = \Theta(\boldsymbol{\sigma}_{\eta,\chi,\lambda}, \varepsilon(\boldsymbol{u}_\eta(t)), \theta_\chi(t), \alpha_\lambda(t)). \tag{4.31}$$

$$\Lambda^3(\eta,\chi,\lambda)(t) = S(\boldsymbol{\sigma}_{\eta,\chi,\lambda}, \varepsilon(\boldsymbol{u}_\eta(t)), \theta_\chi(t), \alpha_\lambda(t)). \tag{4.32}$$

Here, for every $(\eta,\chi,\lambda) \in C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$. $\boldsymbol{u}_\eta$, φ_η, θ_χ, α_λ and $\boldsymbol{\sigma}_\eta$ represent the displacement field, the electric potential field, the temperature field, the damage field and the stress field , obtained in Lemmas 4.2, 4.3, 4.4,4.5 and 4.6 respectively. We have the following result.

Lemma 4.7. *The mapping Λ has a fixed point $(\eta,\chi,\lambda)\in C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$, such that $\Lambda(\eta^*,\chi^*,\lambda^*)=(\eta^*,\chi^*,\lambda^*)$.*

Proof. Let $t\in(0,T)$ and $(\eta_1,\chi_1,\lambda_1),(\eta_2,\chi_2,\lambda_2)\in C\big(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega)\big)$. We use the notation that $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i}=\boldsymbol{v}_{\eta_i}=\boldsymbol{v}_i$, $\theta_{\chi_i}=\theta_i$, $\varphi_{\eta_i}=\varphi_i$, $\alpha_{\lambda_i}=\alpha_i$ and $\boldsymbol{\sigma}_{\eta_i,\theta_i}=\boldsymbol{\sigma}_i$ for $i=1,2$.

Let us start by using (3.23), (3.28) and (4.25), we have

$$\begin{aligned}
&\|\Lambda^1(\eta_1,\chi_1,\lambda_1)(t)-\Lambda^1(\eta_2,\chi_2,\lambda_2)(t)\|^2_{\mathcal{H}}\\
&\le \|\mathcal{E}^*\nabla\varphi_1(t)-\mathcal{E}^*\nabla\varphi_2(t)\|^2_{\mathcal{H}}\\
&+\int_0^t \|\mathcal{G}(\boldsymbol{\sigma}_1(s),\varepsilon(u_1(s)),\theta_1(s),\alpha_1(s))-\mathcal{G}(\boldsymbol{\sigma}_2(s),\varepsilon(u_2(s)),\theta_2(s),\alpha_2(s))\|^2_{\mathcal{H}}\,ds,\\
&\le C\Bigg(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V\,ds\\
&+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\Bigg).
\end{aligned}\tag{4.33}$$

By similar arguments, from (4.31), (3.24) we obtain

$$\begin{aligned}
&\left\|\Lambda^2(\eta_1,\chi_1,\lambda_1)(t)-\Lambda^2(\eta_2,\chi_2,\lambda_2)(t)\right\|^2_{\mathcal{H}}\\
&\le C\Big(\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_V+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V\\
&\quad+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big),\quad \text{a.e. } t\in(0,T).
\end{aligned}\tag{4.34}$$

Similarly, using (4.32), (3.25) implies

$$\begin{aligned}
&\left\|\Lambda^3(\eta_1,\chi_1,\lambda_1)(t)-\Lambda^3(\eta_2,\chi_2,\lambda_2)(t)\right\|^2_{\mathcal{H}}\\
&C\Big(\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_V+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V\\
&\quad+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\Big),\quad \text{a.e. } t\in(0,T).
\end{aligned}\tag{4.35}$$

It follows now from (4.33), (4.34) and (4.35) that

$$\begin{aligned}
&\|\Lambda(\eta_1,\chi_1,\lambda_1)(t)-\Lambda(\eta_2,\chi_2,\lambda_2)(t)\|^2_{\mathcal{H}}\\
&\qquad\le C\Big(\|\varphi_1(t)-\varphi_2(t)\|^2_W+\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|^2_{\mathcal{H}}\\
&\qquad+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|^2_V\,ds\\
&\qquad+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds\\
&\qquad+\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\Big).
\end{aligned}\tag{4.36}$$

We use estimates (4.15), (4.25)to obtain

$$\begin{aligned}
&\|\Lambda(\eta_1,\chi_1,\lambda_1)(t)-\Lambda(\eta_2,\chi_2,\lambda_2)(t)\|_{\mathcal{H}}^2\\
&\qquad\le C\left(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds\right.\\
&\qquad+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2\,ds\\
&\qquad\left.+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2\,ds\right).
\end{aligned}\tag{4.37}$$

On the other hand, since $\boldsymbol{u}_i(t)=\int\limits_0^t\dot{\boldsymbol{u}}_i(s)\,ds+\boldsymbol{u}_0$, we know that for a.e. $t\in(0,T)$,

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2\le\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V^2\,ds,\tag{4.38}$$

and using this inequality in (4.9) yields

$$\|\dot{\boldsymbol{u}}_1(t)-\dot{\boldsymbol{u}}_2(t)\|_V\le C\left(\|\eta_1(t)-\eta_2(t)\|_{\mathcal{H}}+\int_0^t|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)|_V\,ds\right).\tag{4.39}$$

Next, we apply Gronwall's inequality to deduce

$$\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds\le c\int_0^t\|\eta_1(t)-\eta_2(t)\|_Q\,ds,\ \text{for all } t\in[0;T],\tag{4.40}$$

which also implies, using a variant of (4.38), that

$$\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\le C\int_0^t\|\eta_1(s)-\eta_2(s)\|_Q^2\,ds,\quad \text{a.e. } t\in[0,T].\tag{4.41}$$

For the temperature, if we take the substitution $\chi=\chi_1,\chi=\chi_2$ in (4.16) and subtracting the two obtained equations, we deduce by choosing $v=\theta_1-\theta_2$ as test function and $t\in[0,T]$, such that

$$\begin{aligned}
&\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+C_1\int_0^t\|\theta_1(t)-\theta_2(t)\|_{\mathcal{V}}^2\\
&\qquad\qquad\le\int_0^t\|\chi_1(s)-\chi_2(s)\|_{\mathcal{V}'}\|\theta_1(s)-\theta_2(s)\|_{\mathcal{V}}\,ds.
\end{aligned}$$

We use the inclusion $L^2(\Omega)\subset\mathcal{V}$, we get

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2\,ds\le C\int_0^t\|\chi_1(s)-\chi_2(s)\|_{\mathcal{V}'}^2\,ds.$$

From this inequality, combined with Gronwall's inequality, we deduce that

$$\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2\le C\int_0^t\|\chi_1(s)-\chi_2(s)\|_{\mathcal{V}'}^2\,ds,\tag{4.42}$$

Form (4.22), deduced that

$$\begin{aligned}
&(\dot\alpha_1-\dot\alpha_2,\alpha_1-\alpha_2)_{L^2(\Omega)}+a_1(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\\
&\qquad\qquad\le(\lambda_1-\lambda_2,\alpha_1-\alpha_2)_{L^2(\Omega)},\ \text{a.e. } t\in(0,T).
\end{aligned}$$

integrate inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$, and inequality $a_1(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\ge0$, we find

$$\tfrac{1}{2}\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \leq C\int_0^t \left(\lambda_1(s)-\lambda_2(s),\alpha_1(s)-\alpha_2(s)\right)_{L^2(\Omega)} ds,$$

which implies

$$\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \leq C\left(\int_0^t \left\|\lambda_1(s)-\lambda_2(s)\right\|^2_{L^2(\Omega)} ds + \int_0^t \left\|\alpha_1(s)-\alpha_2(s)\right\|^2_{L^2(\Omega)} ds\right).$$

This inequality combined with the Gronwall inequality leads to

$$\left\|\alpha_1(t)-\alpha_2(t)\right\|^2_{L^2(\Omega)} \leq C\int_0^t \left\|\lambda_1(s)-\lambda_2(s)\right\|^2_{L^2(\Omega)} ds \ ,\forall t\in[0,T]\,. \tag{4.43}$$

Form the previous inequality and estimates (4.42), (4.41) and (4.37) it follows now that

$$\begin{aligned}&\left\|\Lambda(\eta_1,\chi_1,\lambda_1)(t)-\Lambda(\eta_2,\chi_2,\lambda_2)(t)\right\|^2_{\mathcal{H}\times L^2(\Omega)\times L^2(\Omega)}\\ &\qquad\leq C\int_0^T \left\|(\eta_1,\chi_1,\lambda_1)(s)-(\eta_2,\chi_2,\lambda_2)(s)\right\|^2_{\mathcal{H}\times L^2(\Omega)\times L^2(\Omega)} ds.\end{aligned} \tag{4.44}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\left\|\Lambda^m(\eta_1,\chi_1,\lambda_1)-\Lambda^m(\eta_2,\chi_2,\lambda_2)\right\|^2_{C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))}\\ &\qquad\leq \frac{C^mT^m}{m!}\left\|(\eta_1,\chi_1,\lambda_1)-(\eta_2,\chi_2,\lambda_2)\right\|^2_{C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now we have every thing that is required to prove Theorem 4.1.

Existence. Let $(\eta^*,\chi^*,\lambda^*)\in C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$ be the fixed point of Λ defined by (4.29)-(4.32) and denote

$$\boldsymbol{u}=\boldsymbol{u}_{\eta^*},\quad \varphi_{\eta^*}=\varphi,\quad \boldsymbol{\sigma}=\mathcal{A}\varepsilon(\dot{\boldsymbol{u}})+\boldsymbol{\sigma}_{\eta^*,\chi^*,\lambda^*}. \tag{4.45}$$

$$\theta=\theta_{\chi^*},\quad \alpha=\alpha_{\lambda^*}. \tag{4.46}$$

$$\boldsymbol{D}=\mathcal{E}\varepsilon(\boldsymbol{u})+\boldsymbol{B}\nabla(\varphi). \tag{4.47}$$

We prove that $(\boldsymbol{u},\boldsymbol{\sigma},\theta,\alpha)$ satisfies (3.41)-(3.46) and (4.1)-(4.6). Indeed, we write (4.24) for $\eta^*=\eta,\quad \chi^*=\chi,\quad \lambda^*=\lambda$ and use (4.45)-(4.46) to obtain that (3.41) is satisfied. Now we consider (4.7) for $\eta^*=\eta$ and use the first equality in (4.45) to find

$$\begin{gathered}(\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)),\varepsilon(\boldsymbol{v}-\dot{\mathbf{u}}(t)))_{\mathcal{H}}+(\mathcal{B}\varepsilon(\mathbf{u}(t)),\varepsilon(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}+(\eta^*(t),\varepsilon(\boldsymbol{v}-\dot{\boldsymbol{u}}(t)))_{\mathcal{H}}\\ +j\left(\dot{\boldsymbol{u}}(t),\boldsymbol{v}\right)-j\left(\dot{\boldsymbol{u}}(t),\dot{\boldsymbol{u}}(t)\right)\geq\left(f,\boldsymbol{v}-\dot{\boldsymbol{u}}(t)\right)_{V'\times V},\\ \text{a.e. } t\in(0,T),\qquad \text{for all } \boldsymbol{v}\in V.\end{gathered} \tag{4.48}$$

The equalities $\Lambda^1(\eta^*, \chi^*, \lambda^*) = \eta^*$, $\Lambda^2(\eta^*, \chi^*, \lambda^*) = \chi^*$ and $\Lambda^3(\eta^*, \chi^*, \lambda^*) = \lambda^*$ combined with (4.30)-(4.32) and (4.45)-(4.46) show that for all $\boldsymbol{v} \in V$,

$$(\boldsymbol{\eta}^*(t), \boldsymbol{v})_{\mathcal{H}\times V} = \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}(s)), \theta(s), \alpha(s))\, ds, \varepsilon(\boldsymbol{v})\right)_{\mathcal{H}}. \tag{4.49}$$

$$\chi^*(t) = \chi(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \theta(t), \alpha(t)). \tag{4.50}$$

$$\lambda^*(t) = \Theta(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \theta(t), \alpha(t)). \tag{4.51}$$

We now substitute (4.48) in (4.47) and use (3.41) to see that (3.42) is satisfied.

We write now (4.14) for $\eta = \eta^*$ and use (4.45) to find (3.43).

We write (4.16) for $\chi = \chi^*$ and use (4.45)-(4.46) and (4.49) to find that (3.44) is satisfied.

Also We write (4.22) for $\lambda = \lambda^*$ and using (4.45)-(4.46) and (4.50) to find that (3.45) is satisfied. Next (3.46), and regularities (4.1), (4.2), (4.3) and (4.4) follow Lemmas 4.2,4.3 and 4.4. The regularity $\boldsymbol{\sigma} \in C(0, T; \mathcal{H})$ follows from Lemma 4.5.

Let now $t_1, t_2 \in [0, T]$, from (3.8), (3.27), (3.28) and (4.47), we conclude that there exists a positive constant $C > 0$ verifying

$$\|\boldsymbol{D}(t_1) - \boldsymbol{D}(t_2)\|_H \leq C\left(\|\varphi(t_1) - \varphi(t_2)\|_W + \|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_V\right).$$

The regularity of $\boldsymbol{u}$ and φ given by (4.1) and (4.3) implies

$$\boldsymbol{D} \in C(0, T; \mathcal{W}). \tag{4.52}$$

We choose $\phi \in D(\Omega)^d$ in (3.39) and using (3.43) we find

$$\operatorname{div} \boldsymbol{D}(t) = q_0(t), \quad \forall t \in [0, T], \tag{4.53}$$

Property (4.6) follows from (3.32),(4.52) and (4.53) which concludes the existence part the Theorem 4.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ.

5. Conclusion

This study investigates a specific category of evolutionary variational problems, where the model integrates the displacement field, an electric potential field, a temperature field, and the damage field. The paper explores the analysis of quasi-static contact conditions involving normal compliance and adhering to the Coulomb law of dry friction between a thermo-piezoelectric body and an electrically conductive foundation. Notably, it establishes the existence of a unique weak solution to this problem. As a suggestion for future exploration, researchers may consider employing a numerical approach to this problem, utilizing a fully discrete finite-elements scheme with explicit time discretization. This approach could provide valuable insights into the practical implementation and behavior of the proposed model.

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