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ON INTERVAL VALUED FUZZY PRIME IDEALS OF $\Gamma-$SEMIRINGS

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ABSTRACT. In this paper, we introduce the notion of interval valued fuzzy prime ideals of $\Gamma-$semirings.We study, some properties of prime ideals of a $\Gamma-$semirng in terms of interval valued fuzzy ideals.

1. INTRODUCTION

In 1995, M. Murali Krishna Rao [16, 17] introduced the notion of a $\Gamma-$semiring as a generalization of $\Gamma-$rings, rings, ternary semirings and semirings. Murali Krishna and his co-authors introduced many concepts in $\Gamma-$semirings(see [18, 19, 20, 21, 22]). Notion of a semiring was introduced by H.S.Vandiver[6] in 1934. A semiring is a well known Universal algebra. In [24], Allen Studied fundamental theorem of homomorphism on semirings. If in a ring, we do away with the requirement of having additive inverse of each element then the resulting algebraic structure becomes a semiring. As a generalization of a ring, the notion of a $\Gamma-$ring was introduced by N. Nobusawa [23] in 1964. In 1981, M. K. Sen [14] introduced the notion of a $\Gamma-$semigroup as a generalization of a semigroup. In 1965, Zadeh[7] introduced the fuzzy theory. The aim of this theory to develop theory which deals with problem of uncertainity. After 10 years Zadeh[8] introduced the notion of interval-valued fuzzy sets The fuzzification of algebraic structure was introduced by Rosenfeld [3] and he introduced the notion of fuzzy subgroups in 1971. The concept of interval valued fuzzy subset in algebra was initiated by Biswas. After the i-v fuzzy sets have been introduced (see [10, 9, 11, 4, 26, 27, 5, 25, 13, 12]), some theories related with i-v fuzzy sets have been developed. It is therefore important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive parts of fuzzy control is defuzzification. A.B.Saeid et al., introduced the $\Gamma - BCK-$algebras[1] and studied fuzzy ideals in $\Gamma - BCK-$algebras[2]. In this paper, we introduce the notion of interval valued fuzzy prime ideals of $\Gamma-$semirings. We study some properties of prime ideals of a $\Gamma-$semirng in terms of interval valued fuzzy ideals.

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2. PRELIMINARIES

In this section, we recall some definitions introduced by the pioneers in this field earlier.

Definition 2.1. [24] A set S together with two associative binary operations called addition and multiplication (denoted by + and $\cdot$ respectively) will be called semiring provided

(i). Addition is a commutative operation.
(ii). Multiplication distributes over addition both from the left and from the right.
(iii). There exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. [14] Let M and Γ be two non-empty sets. Then M is called a $\Gamma-$semigroup if it satisfies

(i) $x\alpha y \in M$,
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$, for all $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.3. [16] Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. A $\Gamma-$semigroup M is said to be a $\Gamma-$semiring M if it satisfies the following axioms, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$,
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$.

Every semiring M is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication

Definition 2.4. [16] A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$.

Definition 2.5. [15] A non-empty subset A of a $\Gamma-$semiring M is called

(i) a $\Gamma-$subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if A is a $\Gamma-$subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a left (right) ideal of M if A is a $\Gamma-$subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(iv) an ideal if A is a $\Gamma-$subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

Definition 2.6. [16] A function $f : R \to M$ where R and M are $\Gamma-$semirings is said to be a $\Gamma-$semiring homomorphism if $f(a+b) = f(a)+f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R, \alpha \in \Gamma$.

Definition 2.7. [16] Let M be a non-empty set. A mapping $f : M \to [0, 1]$ is called a fuzzy subset of a $\Gamma-$semiring M. If f is not a constant function then f is called a non-empty fuzzy subset

Definition 2.8. [15] Let f be a fuzzy subset of a non-empty set M, for $t \in [0, 1]$ the set $f_t = \{x \in M \mid f(x) \geq t\}$ is called a level subset of M with respect to f.

Definition 2.9. [15] Let M be a $\Gamma-$semiring.A fuzzy subset μ of M is said to be a fuzzy $\Gamma-$subsemiring of M if it satisfies the following conditions
(i) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.10. [15] A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy left (right) ideal of M if for all $x, y \in M, \alpha \in \Gamma$ it satisfies the following conditions
(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq \mu(y)\ (\mu(x))$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 2.11. [15] A fuzzy subset μ of a $\Gamma-$semiring M is called a fuzzy ideal of M if it satisfies the following conditions
(i) $\mu(x+y) \geq min\{\mu(x), \mu(y)\}$
(ii) $\mu(x\alpha y) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

3. Interval valued fuzzy prime ideals of $\Gamma-$semirings

In this section, we introduce the notion of interval-valued fuzzy prime ideals of $\Gamma-$semirings, interval valued Fuzzy prime ideals of $\Gamma-$semirings. And also we study, some properties of prime ideals of a $\Gamma-$semirng in terms of interval valued fuzzy prime ideals. An interval $[a^-, a^+]$, where $0 \leq a^- < a^+ \leq 1$ is called an interval number and it is denoted by $\overline{a}$. Let $D[0,1]$ denotes the family of closed subintervals of $[0,1]$ with the minimum element $\overline{0} = [0,0]$ and maximal element $\overline{1} = [1,1]$ according to partial order $[a,b] \leq [c,d] \Rightarrow a \leq c$ and $b \leq d$. Defined on $D[0,1]$ for all $[a,b], [c,d] \in D[0,1]$.

The interval min-norm is a function $\min_i : D[0,1] \times D[0,1] \to D[0,1]$ is defined by $\min_i[\overline{a}, \overline{b}\ ] = [\min(a^-, b^-), \min(a^+, b^+)]$, for all $[\overline{a}, \overline{b}\ ] \in D$, where $\overline{a} = [a^-, a^+], \overline{b} = [b^-, b^+]$.

The interval max-norm is a function $\max_i : D[0,1] \times D[0,1] \to D[0,1]$ is defined by $\max_i[\overline{a}, \overline{b}\ ] = [\max(a^-, b^-), \max(a^+, b^+)]$, for all $[\overline{a}, \overline{b}\ ] \in D$, where $\overline{a} = [a^-, a^+], \overline{b} = [b^-, b^+]$.

Let M be a non-empty set. A mapping $\overline{\mu} : M \to D[0,1]$ is called an interval valued fuzzy subset of M. For simplicity, interval valued fuzzy subset of M is denoted by IVFS of M.

Let $\overline{\mu}$ be an interval valued fuzzy subset of a $\Gamma-$semiring M then $\overline{\mu}(x)$ is an interval number, where $x \in M$. Suppose $\overline{\mu}(x) = [a,b]$, for some $x \in M$. Then $0 \leq a < b \leq 1$. We define two fuzzy subsets $\mu^-(x) = a$ and $\mu^+(x) = b$. Therefore, for all $x \in M$, $\overline{\mu}(x) = [\mu^-(x), \mu^+(x)]\ \leq [0,1]$.

Let A be a subset of a $\Gamma-$semiring M. Then the interval valued characteristic function $\chi_A : M \to D$ is defined by $\chi_A(x) = \begin{cases} \overline{1}, & \text{if } x \in A \\ \overline{0}, & \text{if } x \notin A. \end{cases}$.

Let $\overline{\mu}_1, \overline{\mu}_2$ be interval valued subsets of M. Then $\overline{\mu}_1$ is said to be subset of $\overline{\mu}_2$ if $\overline{\mu}_1(x) \leq \overline{\mu}_2(x)$, for all $x \in M$. It is denoted by $\overline{\mu}_1 \subseteq \overline{\mu}_2$. Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be an interval valued fuzzy subset of a $\Gamma-$semiring M.

Then $\overline{\mu}_1 \circ \overline{\mu}_2$ is defined as $\overline{\mu}_1 \circ \overline{\mu}_2(x) = \begin{cases} \sup\limits_{x=a\alpha b} \{\min\limits_i(\overline{\mu}_1(a), \overline{\mu}_2(b)\}, \\ 0, \text{ otherwise.} \end{cases}$

Let $\overline{\mu}$ be interval valued fuzzy subset of M and f be a mapping from a set M to a set N. Then the pre-image $f^{-1}(\overline{\mu})$ is defined by $f^{-1}(\overline{\mu})(x) = \overline{\mu}(f(x))$, for all $x \in M$.

The image $f(\overline{\mu})$ is an interval valued fuzzy subset of N defined by

$$f(\overline{\mu}) = \begin{cases} \inf\limits_{y \in \overline{f}} \mu(y), \text{ if } f^{-1}(x) \neq \phi \\ \overline{1}, \text{ otherwise.} \end{cases}.$$

Definition 3.1. A non-empty interval valued fuzzy subset $\bar{\mu}$ of a $\Gamma-$semiring M is said to be an interval valued fuzzy ideal of M if it satisfies the following conditions.

(i) $\bar{\mu}(x+y) \geq \min\limits_i\{\bar{\mu}(x),\ \bar{\mu}(y)\}$
(ii) $\bar{\mu}(x\alpha y) \geq \max\limits_i\{\bar{\mu}(x),\ \bar{\mu}(y)\}$, for all $x, y \in M, \alpha \in \Gamma$.

Definition 3.2. An interval valued fuzzy ideal $\bar{\mu}$ of a $\Gamma-$semiring M is said to be an interval valued fuzzy prime ideal of M if $\bar{\mu}$ is not a constant function and for any interval valued fuzzy ideals $\bar{\mu_1}$ and $\bar{\mu_2}$ of M, $\bar{\mu_1} \circ \bar{\mu_2} \subseteq \bar{\mu} \Rightarrow \bar{\mu_1} \subseteq \bar{\mu}$ or $\bar{\mu_2} \subseteq \bar{\mu}$.

Example 3.3. Let M be set of non-negative integers and Γ be set of all natural numbers. Then M is a $\Gamma-$semiring with respect to all usual addition and ternary operation and usual multiplication

$$\overline{\mu}(x) \ = \begin{cases} \bar{1}, & \text{if } x = 0 \\ [0.5, 0.6], & \text{if } n \ is \ odd \\ [0.7, 0.8], & \text{if } n \ is \ even \ . \end{cases}$$

Then $\bar{\mu}$ is an interval valued fuzzy ideal of M.

The following theorems are straight forward verification.

Theorem 3.1. *Let $\bar{\mu}$ be an interval valued fuzzy ideal of a $\Gamma-$semiring M. Then $\bar{\mu}(0) \geq \bar{\mu}(x)$, for all $x \in M$.*

Theorem 3.2. *A non-empty interval valued fuzzy subset $\bar{\mu}$ of a $\Gamma-$semiring M is an interval valued fuzzy ideal of M if and only if $\bar{U}(\bar{\mu}, [s, t])$ is an ideal of M, for all $[s, t] \in Im\bar{\mu}$*

Theorem 3.3. *Let I be a non-empty subset of a $\Gamma-$semiring M and $[a, b] \subseteq [c, d] \neq 0$ be any two interval numbers on $[0, 1]$. The interval valued fuzzy subset $\overline{\mu}$ of M is defined by*

$$\overline{\mu}(x) = \begin{cases} [c, d], & \textit{if } x \in I \\ [a, b], & \textit{otherwise} \ . \end{cases}$$

Then $\overline{\mu}$ is an interval valued ideal of a $\Gamma-$semiring M if and only if I is an ideal of a $\Gamma-$semiring M.

Proof. Suppose I is an ideal of a $\Gamma-$semiring M.
Then $\overline{\mu}(0) = [c, d], \ 0 \in I \Rightarrow \overline{\mu}$ is non-empty. Let $x, y \in M$ and $\alpha \in \Gamma$.
Case 1: Suppose $\max_i(\overline{\mu}(x), \overline{\mu}(x)) = [a, b]$. Then

$$\overline{\mu}(x) = [a, b] \text{ and } \overline{\mu}(y) = [a, b].$$

$$\begin{aligned} \overline{\mu}(x + y) &\geq [\overline{a}, \overline{b}] \\ &\geq \min_i(\overline{\mu}(x), \overline{\mu}(y)) \\ \text{and } \overline{\mu}(x\alpha y) &\geq [a, b] \\ &= \max_i(\overline{\mu}(x), \overline{\mu}(y)). \end{aligned}$$

Case 2: Suppose $\max_i(\overline{\mu}(x), \overline{\mu}(x)) = [c, d]$. Then

$$\begin{aligned} &\overline{\mu}(x) = [c, d] \text{ or } \overline{\mu}(y) = [c, d] \\ &\Rightarrow x \in I \text{ or } y \in I \\ &\Rightarrow x\alpha y \in I, \text{ since } I \text{ is an ideal.} \\ &\text{Then } \overline{\mu}(x\alpha y) = [c, d] = \max_i(\overline{\mu}(x), \overline{\mu}(y)) \\ &\Rightarrow \min_i(\overline{\mu}(x), \overline{\mu}(y)) = [a, b]\alpha[c, d]. \end{aligned}$$

Suppose $\min_i(\overline{\mu}(x), \overline{\mu}(y)) = [a, b]$. Then $\overline{\mu}(x + y) \geq \min_i(\overline{\mu}(x), \overline{\mu}(y))$.
Suppose $\min_i(\overline{\mu}(x), \overline{\mu}(y)) = [c, d]$. Then $\overline{\mu}(x) = [c, d]$ and $\overline{\mu}(y) = [c, d]$

$\Rightarrow x, y \in I$
$\Rightarrow x + y \in I$
$\Rightarrow \overline{\mu}(x + y) = [c, d] = \min\limits_i(\overline{\mu}(x), \overline{\mu}(y))$.
Hence $\overline{\mu}$ is an interval valued fuzzy subset of a $\Gamma-$semiring M.

Conversely, suppose $\overline{\mu}$ is an interval valued fuzzy subset of a $\Gamma-$semiring M. Then $\overline{\mu}$ is non-empty fuzzy sub set. So $\overline{\mu}$ is non zero
$\Rightarrow \overline{\mu}(x) = [c, d]$, for some $x \in M$. Let $x, y \in I$. Then $\overline{\mu}(x) = \overline{\mu}(y) = [c, d]$.
Since $\overline{\mu}$ is an interval valued fuzzy ideal of M, we have

$$\begin{aligned} \overline{\mu}(x + y) &\geq \min\limits_i(([c, d], [c, d]) = [c, d] \\ \Rightarrow \overline{\mu}(x + y) &\geq [c, d] \\ \Rightarrow \overline{\mu}(x + y) &= [c, d], \text{ by definition of } \overline{\mu}. \\ \Rightarrow x + y &\in I. \end{aligned}$$

Let $x \in I, y \in M$ and $\alpha \in \Gamma$. Then $\overline{\mu}(x) = [c, d]$
$\Rightarrow \overline{\mu}(x\alpha y) \geq \max\limits_i(\overline{\mu}(x), \overline{\mu}(y)) = [c, d]$
$\Rightarrow \overline{\mu}(x\alpha y) = [c, d]$, by definition $\overline{\mu}$
$\Rightarrow x\alpha y \in I$.
Similarly we can prove that $y\alpha x \in I$.
Hence the theorem. □

Corollary 3.4. *Let M be a $\Gamma-$semiring and I be a subset of M. Then I is an ideal of M if and only if μ_I is an interval valued fuzzy ideal of M.*

Theorem 3.5. *Let $\overline{\mu}$ be an interval valued fuzzy ideal of a $\Gamma-$semiring M. Then the set $\overline{\mu}_0 = \{x \in M \mid \overline{\mu}(x) = \overline{\mu}(0)\}$ is an ideal of M.*

Proof. Obviously $\overline{\mu}_0$ is non-empty. Let $x \in \overline{\mu}_0, \alpha \in \Gamma, y \in M$. Then

$$\begin{aligned} \overline{\mu}(x\alpha y) &\geq \max\limits_i(\overline{\mu}(x), \overline{\mu}(y)) \\ &= \max\limits_i(\overline{\mu}(0), \overline{\mu}(y)) \\ &= \overline{\mu}(0), \text{ since } \overline{\mu}(0) \geq \overline{\mu}(y), \text{ for all } y \in M. \\ \Rightarrow \overline{\mu}(x\alpha y) &= \overline{\mu}(0). \end{aligned}$$

Thus $x\alpha y \in \overline{\mu}_0$.

Similarly we can prove that $y\alpha x \in \overline{\mu}(0)$. Let $x, y \in \overline{\mu}(0)$. Then $\overline{\mu}(x) = \overline{\mu}(0)$ and $\overline{\mu}(y) = \overline{\mu}(0)$. We have

$$\begin{aligned} \overline{\mu}(y) &\geq \min\limits_i(\overline{\mu}(x), \overline{\mu}(y)) \\ &= \min\limits_i(\overline{\mu}(0), \overline{\mu}(y)) \\ &= \overline{\mu}(0) \\ \text{and } \overline{\mu}(x + y) &\leq \overline{\mu}(0) \\ \Rightarrow \overline{\mu}(x + y) &= \overline{\mu}(0) \\ \Rightarrow x + y &\in \overline{\mu}(0). \end{aligned}$$

Hence the theorem. □

Theorem 3.6. *If $\overline{\mu}$ is an interval valued fuzzy prime ideal of a $\Gamma-$semiring M then $\overline{\mu}(0) = \overline{1}$.*

Proof. Let $\overline{\mu}$ be an interval valued fuzzy prime ideal of a $\Gamma-$semiring M. Suppose $\overline{\mu}(0) \neq \overline{1}$. Since $\overline{\mu}$ is an interval valued prime ideal of a $\Gamma-$semiring M, there exists $a \in M$ such that $\overline{\mu}(0) > \overline{\mu}(a)$.

We define two fuzzy subsets $\overline{\mu}_1$ and $\overline{\mu}_2$ of M by

$$\overline{\mu}_1(x) \;=\; \begin{cases} \overline{1}, & \text{if } x \in \overline{\mu}(0) \\ \overline{0}, & \text{otherwise} \end{cases} \;;\; \overline{\mu}_2(x) = \overline{\mu}(0).$$

Then clearly, $\overline{\mu}_1$ is an interval valued fuzzy ideal of a $\Gamma-$semiring M. Since $\overline{\mu}_2$ is a constant function, $\overline{\mu}_2$ is an interval valued fuzzy ideal of a $\Gamma-$semiring M.

Let $x \in M, \overline{\mu}(x) = \overline{\mu}(0), y \in M$ and $\alpha \in \Gamma$. Then $\overline{\mu}_1(x) = T$.

$$\begin{aligned} \min_i(\overline{\mu}_1(x), \overline{\mu}_2(y)) &= \overline{\mu}_2(y) \\ &= \overline{\mu}(0) \\ \overline{\mu}(0) &\geq \overline{\mu}(x\alpha y) \\ &\geq \overline{\mu}(x) = \overline{\mu}(0) \\ \Rightarrow \overline{\mu}(x\alpha y) &= \overline{\mu}(0), \text{ for any } y \in M. \end{aligned}$$

Suppose $x \in M$ and $\overline{\mu}(x) \neq \overline{\mu}(0)$. Then $\overline{\mu}_1 = \overline{0}$.

$$\begin{aligned} \min_i(\overline{\mu}_1(x), \overline{\mu}_2(y)) &= \overline{0} \leq \overline{\mu}(x\alpha y) \\ \Rightarrow (\overline{\mu}_1 \circ \overline{\mu}_2)z &= \sup_{z=x\alpha y}\{\min_i(\overline{\mu}_1(x), \overline{\mu}_2(y))\} \\ &\leq \overline{\mu}(z), \text{ for all } z \in M. \end{aligned}$$

Therefore $\overline{\mu}_1 \circ \overline{\mu}_2 \subseteq \overline{\mu}_1$.

Since $\overline{\mu}$ is an interval valued fuzzy prime ideal of a $\Gamma-$semring, $\overline{\mu}_1 \subseteq \overline{\mu}$ or $\overline{\mu}_2 \subseteq \overline{\mu}$. Therefore, we have

$\overline{\mu}_1(x) \leq \overline{\mu}_1(x)$ and $\overline{\mu}_2(y) \leq \overline{\mu}(y) \Rightarrow \overline{0} \leq \overline{\mu}(x)$ and $\overline{\mu}(0) \leq \overline{\mu}(y)$, which is a contradition. Hence the theorem. □

Theorem 3.7. *Let $\overline{\mu}$ be an interval valued fuzzy prime ideal of a $\Gamma-$semiring M. Then $Im(\overline{\mu}) = 2$.*

Proof. Since $\overline{\mu}$ is a prime ideal, $\overline{\mu}$ is non-constant and $Im(\overline{\mu}) \geq 2$.

Suppose $Im(\overline{\mu}) > 2$. Let $r, s \in M$ and $\overline{1} > \overline{\mu}(r) > \overline{\mu}(s)$. Define two fuzzy subsets

$$\overline{\mu}_1(x) \;=\; \begin{cases} \overline{1}, & \text{if } x \in< r > \\ \overline{0}, & \text{otherwise} \end{cases} \;;\; \text{ and } \overline{\mu}_2(x) = \overline{\mu}_r(0), \text{ for all } x \in M.$$

Obviously $\overline{\mu}_1(x)$ and $\overline{\mu}_2(x)$ are interval valued fuzzy ideals of M.

Suppose $x \in< r >$ and $y \in M, \alpha \in \Gamma$. Then $\overline{\mu}_1(x) = T$ and $\overline{\mu}(x) \geq \overline{\mu}(r)$.

$$\begin{aligned} \overline{\mu}(x\alpha y) &\geq \overline{\mu}(r) \\ &= \overline{\mu}(r) \\ &= \min_i(\overline{\mu}_1(x), \overline{\mu}_1(y)). \end{aligned}$$

Suppose $x \in M$ and $x \notin< r >, \alpha \in \Gamma$. Then $\overline{\mu}_1(x) = \overline{0}$.

$\overline{\mu}(x\alpha y) = \min_i(\overline{\mu}_1(x), \overline{\mu}_1(y)) = \overline{0}$. Therefore

$$\begin{aligned}(\overline{\mu}_1 \circ \overline{\mu}_2)(z) &= \sup_{z=x\alpha y}\{\min_i(\overline{\mu}_1(x), \overline{\mu}_2(y))\}\\ &\leq \overline{\mu}(z), \text{ for all } z \in M\\ \overline{\mu}_1 \circ \overline{\mu}_2 &\subseteq \overline{\mu}.\end{aligned}$$

Since $\overline{\mu}$ is an interval valued fuzzy prime ideal.
We have $\overline{\mu}_1 \subseteq \overline{\mu}$ or $\overline{\mu}_2 \subseteq \overline{\mu}$. But
$T > \overline{\mu}(r) \Rightarrow \overline{\mu}_1(x) > \overline{\mu}(r)$ and $\overline{\mu}_2(y) = \overline{\mu}(r) > \overline{\mu}(s)$, which is a contradiction.
Hence $Im(\overline{\mu}) = 2$. □

The following theorem characterizes to interval valued fuzzy prime ideal of a $\Gamma-$semiring M.

Theorem 3.8. *Let I be a prime ideal of a $\Gamma-$semiring M and $[a, b] \in D[0, 1] \setminus \{\overline{1}\}$. and $\overline{\mu}$ be an interval valued fuzzy subset of a $\Gamma-$semiring M, defined by,*

$$\overline{\mu}_1(x) \ = \begin{cases}\overline{1}, & \textit{if } x \in I\\ [a, b], & \textit{otherwise}\end{cases}.$$

Then $\overline{\mu}$ is an interval valued fuzzy prime ideal of a $\Gamma-$semiring M.

Proof. Clearly, $\overline{\mu}$ is an interval valued fuzzy ideal of a $\Gamma-$semiring M.
Let $\overline{\mu}_1$ and $\overline{\mu}_2$ be two interval valued fuzzy subsets of a $\Gamma-$semring M such that $\overline{\mu}_1 \circ \overline{\mu}_2 \subseteq \overline{\mu}$.
Suppose $\overline{\mu}_1 \nsubseteq \overline{\mu}$ and $\overline{\mu}_2 \nsubseteq \overline{\mu}$. Then there exist $s, t \in M$ such that
$\overline{\mu}_1(s) \nleq \overline{\mu}(s)$ and $\overline{\mu}_2(t) \nleq \overline{\mu}(t)$. Suppose $\overline{\mu}_1(s) > \overline{\mu}(s)$ and $\overline{\mu}_2(t) > \overline{\mu}(t)$.
Then $\overline{\mu}(s), \overline{\mu}(t) \neq \overline{1}$. Therefore $\overline{\mu}(s) = \overline{\mu}(t) = [a, b]$. Hence $s, t \notin I$.
since I is a prime ideal of a $\Gamma-$semring M, there exist $\alpha, \beta \in \Gamma, y \in M$ such that $x\alpha y\beta t \notin I \Rightarrow \overline{\mu}(s\alpha y\beta t) = [a, b]$.

$$\begin{aligned}\overline{\mu}_1 \circ \overline{\mu}_2(s\alpha y\beta t) &= \sup_{s\alpha y\beta t = c\gamma d}\{\min_i(\overline{\mu}_1(c), \overline{\mu}_2(d))\}\\ &\geq \min_i(\overline{\mu}_1(s), \overline{\mu}_2(y\beta t))\\ &\geq \min_i(\overline{\mu}_1(s), \overline{\mu}_2(t))\\ &> \min_i(\overline{\mu}(s), \overline{\mu}(t))\\ &= \min_i([a, b], [a, b])\\ &= [a, b]\\ &= \overline{\mu}(s\alpha y\beta t).\end{aligned}$$

This is a contradiction. Therefore $\overline{\mu}_1 \subseteq \overline{\mu}$ or $\overline{\mu}_2 \subseteq \overline{\mu}$.
Hence $\overline{\mu}$ is an interval valued fuzzy prime ideal of a $\Gamma-$semiring M. □

Example 3.4. Let M be a $\Gamma-$semiring, where M is the set of all non-negative integers and Γ be the set of natural numbers. Then $(M, +)$ and $(\Gamma, +)$ are semigroups with respect to usual addition ternary operation is defined as usual multiplication.

Let $\overline{\mu}$ be interval valued fuzzy subset of a $\Gamma-$semiring M, defined by

$$\overline{\mu}(x) \ = \begin{cases}\overline{1}, & \text{if } x \in \{0, 3, 6, 9, \cdots\}\\ [.4, .5], & \text{otherwise}\end{cases}.$$

Then $\overline{\mu}$ is an interval valued fuzzy prime ideal of M.

Theorem 3.9. *If $\overline{\mu}$ is an interval valued fuzzy prime ideal of a $\Gamma-$semiring M. Then the set $\overline{\mu}_0 = \{x \in M \mid \overline{\mu}(x) = \overline{\mu}(0)\}$ is a prime ideal of a $\Gamma-$semiring M.*

Proof. Clearly, $\overline{\mu}_0$ is an ideal of a $\Gamma-$semiring M.
Let A, B be ideal of a $\Gamma-$semiring M such that $A\Gamma B \subseteq \overline{\mu}_0$.
Since A, B are ideals of a $\Gamma-$semiring M, $\overline{\chi}_A, \overline{\chi}_B$ are interval valued fuzzy ideals of a $\Gamma-$semiring M. Let $x \in M$. Suppose $\overline{\chi}_A \circ \overline{\chi}_B \neq \overline{0}$.

$$\begin{aligned} &\overline{\chi}_A \circ \overline{\chi}_B(x) = \sup_{x=a\alpha b}\{\min_i(\overline{\chi}_A(a), \overline{\chi}_B(b))\} \neq \overline{0} \\ &\Rightarrow \sup_{x=a\alpha b}\{\min_i(\overline{\chi}_A(a), \overline{\chi}_B(b))\} = \overline{1} \\ &\Rightarrow \min_i(\overline{\chi}_A(a), \overline{\chi}_B(b)) = \overline{1} \\ &\Rightarrow \overline{\chi}_A(a) = \overline{1} \text{ and } \overline{\chi}_B(b) = \overline{1} \\ &\Rightarrow a \in A, b \in B \\ &\Rightarrow a\alpha b \in A\Gamma B \\ &\Rightarrow x = a\alpha b \in \overline{\mu}_0. \end{aligned}$$

$$\begin{aligned} &\overline{\chi}_{\overline{\mu}_0} = \overline{1} \\ &\Rightarrow \overline{\chi}_A \circ \overline{\chi}_B(x) = \overline{\chi}_{\overline{\mu}_0}(x) = \overline{1} \\ &\Rightarrow \chi_A \circ \chi_B(x) \leq \chi_{\overline{\mu}_0}(x). \text{ for all } x \in M \\ &\Rightarrow \chi_A \circ \chi_B \subseteq \chi_{\overline{\mu}_0}. \end{aligned}$$

Let $y \in M$. Suppose $\chi_{\overline{\mu}_0} = \overline{0} \Rightarrow \overline{\chi}_{\overline{\mu}_0}(y) \leq \overline{\mu}(y)$.
Suppose $\overline{\chi}_{\overline{\mu}_0}(y) = \overline{1} \Rightarrow y \in \overline{\mu}_0$

$$\begin{aligned} &\Rightarrow \overline{\mu}(y) = \overline{\mu}(0), \text{ since } \overline{\mu} \text{ is an interval valued fuzzy prime ideal} \\ &\Rightarrow \overline{\chi}_{\overline{\mu}_0}(y) = \overline{\mu}(y) \\ &\Rightarrow \overline{\chi}_{\overline{\mu}_0}(y) \leq \overline{\mu}(y) \\ &\Rightarrow \overline{\chi}_{\overline{\mu}_0} \subseteq \overline{\mu} \\ &\Rightarrow \overline{\chi}_A \circ \overline{\chi}_B \subseteq \overline{\chi}_{\overline{\mu}_0} \subseteq \overline{\mu} \\ &\Rightarrow \overline{\chi}_A \subseteq \overline{\mu} \text{ or } \overline{\chi}_B \subseteq \overline{\mu}. \end{aligned}$$

Suppose $\overline{\chi}_A \subseteq \overline{\mu}$ and $z \in A$. Then $\overline{\chi}_A(z) = \overline{1} \Rightarrow \overline{\mu}(z) = \overline{1} = \overline{\mu}(0_s)$.
But we have $\overline{\mu}$ is an interval valued then $z \in \overline{\mu}_0$.
Therefore $A \subseteq \overline{\mu}_0$.
Similarly, we can prove that $B \subseteq \overline{\mu}_0$.
Hence $\overline{\mu}_0$ is a prime ideal of a $\Gamma-$semiring M. □

Corollary 3.10. *Let $\overline{\mu}$ is an interval valued fuzzy subset of a $\Gamma-$semiring M. Then $\overline{\mu}$ is an interval valued fuzzy prime ideal of a $\Gamma-$semiring M if and only if $Im\overline{\mu} = \{\overline{1}, [a, b]\}$, when $[\overline{a}, \overline{b}] \in D[0, 1] \setminus \{\overline{1}\}$ and $\overline{\mu}_0$ is a prime ideal.*

Corollary 3.11. *Let I be a proper ideal of a $\Gamma-$semiring M. Then I is a prime ideal of M if and only if $\overline{\chi}_I$ is an interval valued fuzzy prime ideal of M.*

Theorem 3.12. *Let $\overline{\mu}$ be an interval valued fuzzy prime ideal of a $\Gamma-$semiring M. Then* $\max_i(\overline{\mu}(a), \overline{\mu}(b)) = \inf\{\overline{\mu}(a\alpha s\beta b) \mid s, a, b \in M, \alpha, \beta \in \Gamma\}$, *for any $a, b \in M$.*

Proof. Let $\overline{\mu}$ be an interval valued fuzzy prime ideal of a $\Gamma-$semiring M.
Clearly, $Im\overline{\mu} = \{\overline{1}, [x, y]\}$, where $[a, b] \in D[0, 1] \setminus \{\overline{1}\}$.
Case 1: Suppose $\max_i(\overline{\mu}(a), \overline{\mu}(b)) = \overline{1}$

$$\begin{aligned}
&\Rightarrow \overline{\mu}(a) = \overline{1} \text{ or } \overline{\mu}(b) = \overline{1} \\
&\Rightarrow \overline{\mu}(a) = \overline{\mu}(0) \text{ or } \overline{\mu}(b) = \overline{\mu}(0) \\
&\Rightarrow a \in \overline{\mu}(0) \text{ or } b \in \overline{\mu}(0) \\
&\Rightarrow a\alpha s\beta b \in \overline{\mu}(0) \\
&\Rightarrow \overline{\mu}(a\alpha s\beta b) = \overline{\mu}(0), \text{ for all } s \in M.
\end{aligned}$$

Therefore $\max_i(\overline{\mu}(a), \overline{\mu}(b)) = \inf\{\overline{\mu}(a\alpha s\beta b) \mid s, a, b \in M, \alpha, \beta \in \Gamma\}$.
Case 2: Suppose $\max_i(\overline{\mu}(a), \overline{\mu}(b)) = [x, y]$. Then
$\overline{\mu}(a) = [x, y]$ and $\overline{\mu}(b) = [x, y] \Rightarrow a, b \notin \overline{\mu}_0$.
Since $\overline{\mu}_0$ is a prime ideal of a $\Gamma-$semiring M, there exist $\alpha, \beta \in \Gamma$ and $s \in M$ such that $a\alpha s\beta b \notin \overline{\mu}_0 \Rightarrow \overline{\mu}(a\alpha s\beta b) = [x, y]$. Therefore
$\max_i(\overline{\mu}(a), \overline{\mu}(b)) = \inf\{\overline{\mu}(a\alpha s\beta b) \mid s, a, b \in M, \alpha, \beta \in \Gamma\}$.
Hence the theorem. □

Theorem 3.13. *Let $\overline{\mu}$ be an interval valued fuzzy prime ideal of a $\Gamma-$semiring M. Then* $\overline{\mu}(a\alpha b) = \max_i(\overline{\mu}(a), \overline{\mu}(b))$.

Proof. Let $a, b \in M$ and $\alpha \in \Gamma$. Then,
$\max_i(\overline{\mu}(a), \overline{\mu}(b)) = \inf\{\overline{\mu}(a\alpha s\beta b) \mid s, a, b \in M, \alpha, \beta \in \Gamma\} \geq \overline{\mu}(\overline{a}\alpha\overline{b}) \cdots (1)$
We have $\overline{\mu}(\overline{a}\alpha\overline{b}) \geq \max_i(\overline{\mu}(a), \overline{\mu}(b)) \cdots (2)$
Therefore, from (1) and (2), $\overline{\mu}(a\alpha b) = \max_i(\overline{\mu}(a), \overline{\mu}(b))$. □

Now we defined the interval valued fuzzy prime ideal $\overline{\mu}$ of a $\Gamma-$semiring M. A non-empty set interval valued fuzzy subset $\overline{\mu}$ of a $\Gamma-$semiring M is said to be an interval valued fuzzy prime ideal of M if

(i) $\overline{\mu}(x + y) \geq \min_i(\overline{\mu}(x), \overline{\mu}(y))$
(ii) $\overline{\mu}(x\alpha y) = \max_i(\overline{\mu}(x), \overline{\mu}(y))$, for all $x, y \in M, \alpha \in \Gamma$.

Theorem 3.14. *Let $f : M \to N$ be a homomorphism of $\Gamma-$seemirings M and N. If $\overline{\mu}$ is an interval valued fuzzy prime ideal of N then $f^{-1}(\overline{\mu})$ is an interval valued fuzzy prime ideal of M.*

Proof. Let $\overline{\mu}$ be an interval valued fuzzy prime ideal of $N,\ x, y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} f^{-1}(\overline{\mu})(x+y) &= \overline{\mu}(f(x+y)) \\ &= \overline{\mu}(f(x)+f(y)) \\ &\geq \min_i(\overline{\mu}f(x), \overline{\mu}f(y)) \\ &= \min_i(f^{-1}(\overline{\mu})(x), f^{-1}(\overline{\mu})(y)) \\ f^{-1}(\overline{\mu})(x\alpha y) &= \overline{\mu}(f(x\alpha y)) \\ &= \overline{\mu}(f(x)\alpha f(y)) \\ &\geq \max_i(\overline{\mu}f(x), \overline{\mu}f(y)), \text{ since } \overline{\mu} \text{ is an interval valued fuzzy prime ideal of } M. \\ &= \max_i(f^{-1}(\overline{\mu})(x), f^{-1}(\overline{\mu})(y)). \end{aligned}$$

Hence $f^{-1}(\overline{\mu})$ is an interval valued fuzzy prime ideal of M. □

4. Conclusion

In this paper, we introduced the notion of an interval valued fuzzy prime ideal and discussed the algebraic properties of interval valued fuzzy prime ideals of $\Gamma-$semirings. We proved that if $\overline{\mu}(a\alpha b) = \max_i(\overline{\mu}(a), \overline{\mu}(b)),\ \alpha \in \Gamma,\ a, b \in M$ and $Im(\overline{\mu}) = 2$. One can extend this work by studying the other (ordered) algebraic structures.

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