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$e^{\star}$-LOCAL FUNCTIONS AND $\psi_{e^{\star}}$-OPERATOR IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT. The main goal of this paper is to introduce another local function to give the possibility of obtaining a Kuratowski closure operator. On the other hand, $e^{\star}$-local functions defined for ideal topological spaces have not been found in the current literature. $e^{\star}$-local functions for the ideal topological spaces have been described within this work. Moreover, with the help of $e^{\star}$-local functions Kuratowski closure operators $cl_I^{*e^{\star}}$ and $\tau^{*e^{\star}}$ topology are obtained. Many theorems in the literature have been revised according to the definition of $e^{\star}$-local functions.

1. INTRODUCTION

The study of ideal topological spaces has garnered significant attention from mathematicians worldwide, leading to numerous advancements and insights. Hamlett and Jankovic, as referenced in [6], introduced a groundbreaking closure operator using local functions, thereby introducing a novel topology into the field.

In recent times, there has been a growing interest in exploring local functions within spaces where general topology is replaced by generalized open sets. This shift has been fueled by the collaborative efforts of several mathematicians, as indicated by the collective works cited in [1, 8, 14].

Jain's introduction of totally continuous functions in classical topology served as a pivotal moment, offering a broader perspective on continuous functions and leading to the development of $e^{\star}$-open sets, $e^{\star}$-continuous functions, and $e^{\star}$-compactness, as meticulously studied by Ekici [3, 4, 5]. These concepts not only expanded the scope of classical topology but also unveiled profound connections between different branches of mathematical inquiry.

The practical implications of $e^{\star}$-local functions in ideal topological spaces quickly became evident, prompting a surge in research activities aimed at unraveling their intricacies and exploring their applications. This paper represents a significant contribution to this evolving field by focusing on spaces where the conventional notion of topology is replaced by the family of $e^{\star}$-open sets.

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Through an in-depth exploration of $e^{\star}$-local functions and their relevance in ideal topological spaces, this paper aims to elucidate their significance, uncover new insights, and stimulate further progress in this fascinating area of mathematical research. By examining these concepts comprehensively, we seek not only to deepen our understanding of ideal topological spaces but also to identify new avenues for exploration and discovery within this rich and evolving field.

2. Preliminaries

Let $(X,\ \tau)$ be a topological space with no separation properties assumed. For a subset A of a topological space $(X,\ \tau)$, $cl(A)$ and $int(A)$ denote the closure and interior of A in $(X,\ \tau)$, respectively.

An ideal I on a topological space $(X,\ \tau)$ is a non-empty collection of subsets of X which satisfies the following properties[9]:

(1) $A \in I$ and $B \subseteq A$ implies $B \in I$ (heredity),
(2) $A \in I$ and $B \in A$ implies $A \cup B \in I$ (finite additivity).

An ideal topological space is a topological space $(X,\ \tau)$ with an ideal I on X and is denoted by $(X,\ \tau,\ I)$. For a subset $A \in X$, $A^*(I,\ \tau) = \{x \in X :\ A \cap U \notin I,$ for every $U \in \tau(X,\ x)\}$ is called the local function of A with respect to I and τ, where $\tau(X,\ x) = \{U \in \tau :\ x \in U\}$ [7]. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. For every ideal topological space $(X,\ \tau,\ I)$, there exists a topology $\tau^*(I)$, finer than τ, generated by the base $\beta(I,\ \tau) = \{U - J :\ U \in \tau$ and $J \in I\}$. It is known in [7] that $\beta(I.\ \tau)$ is not always a topology. When there is no ambiguity, $\tau^*(I)$ is denoted by τ^*. For a subset $A \subseteq X$, $cl^*(A)$ and $int^*(A)$ will, respectively, denote the closure and interior of A in $(X,\ \tau^*)$.

Definition 2.1. [16] Let $(X,\ \tau)$ be a topological space. A subset A of X is said to be regular open if $A = int(cl(A))$. The complement of a regular open set is said to be regular closed. The collection of all regular open (resp. regular closed) sets in X is denoted by $RO(X)$ (resp. $RC(X)$.) The regular closure of A in $(X,\ \tau)$ is denoted by the intersection of all regular closed sets containing A and is denoted by $rcl(A)$.

Definition 2.2. [17] Let $(X,\ \tau)$ be a topological space. The δ-interior of a subset A of X is the union of all regular open set of X contained in A and is denoted by $Int_\delta(A)$. The subset A is called δ-open if $A = Int_\delta(A)$. i.e., a set is δ-open if it is the union of regular open sets. The complement of a δ-open is called δ-closed. Alternatively a set $A \subseteq (X,\ \tau)$ is called δ-closed if $A = Cl_\delta(A)$ where $Cl_\delta(A) = \{x \in X : Int(Cl(U)) \cap A \neq \phi,\ U \in \tau$and $x \in U\}$.

Definition 2.3. [10, 12] Let $(X,\ \tau)$ be a topological space. A subset A of X is said to be semi open if there exists an open set U in X such that $U \subseteq X \subseteq cl(U)$. The other definition of semi open set is that: A subset A of X is said to be semi open if $A \subseteq cl(int(A))$. The complement of a semi open set is said to be semi closed. The collection of all semi open(resp. semi closed) sets in X is denoted by $SO(X)$(resp. $SC(X)$).

Definition 2.4. [3] Let $(X,\ \tau)$ be a topological space. A subset A of a space X is said to be $e^{\star}$-open if $A \subseteq clintcl_\delta(A)$.

Definition 2.5. [2] Let $(X,\ \tau,\ I)$ be an Ideal topological spaces and $A \subseteq X$. If $\tau \cap I = \{\phi\}$ then we say the I is codense ideal.

Definition 2.6. [13] Let $(X,\ \tau,\ I)$ be an ideal topological spaces. We say the τ is compatible with the ideal I,denoted $\tau \sim I$ if the following holds for every $A \subseteq X$, if for every $x \in A$ there exists $U \in \tau(x)$ such that $U \cap A \in I$, then $A \in I$.

Definition 2.7. [8] Let $(X,\ \tau,\ I)$ be a Ideal space and A a subset of X. Then $A_*(I,\ \tau) = \{x \in X :\ A \cap U \notin I$ for every $U \in SO(X,\ x)\}$ is called the semi local function of A with respect to I and τ where $SO(X,\ x) = \{U \in SO(X) :\ x \in U\}$. .

When there is no ambiguity, we will write simply A_* for $A_*(I,\ \tau)$.

3. $e^{\star}$-LOCAL FUNCTIONS

In this section we shall introduce $e^{\star}$-ideal space and $()^{*e^{\star}}$ operator and discuss various properties of this operator.

Let $(X,\ \tau)$ be a topological space and I be an ideal on X, then $(X,\ e^{\star}O(X,\ \tau),\ I)$ is called $e^{\star}$-ideal space.

Now we shall define the operator $()^{*e^{\star}}$.

Definition 3.1. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal space and A a subset of X. Then $A^{*e^{\star}}(I,\ e^{\star}O(X,\ \tau)) = \{x \in X :\ A \cap U \notin I$ for every $U \in e^{\star}O(X,\ x)\}$ is called the $e^{\star}$-local function of A with respect to I and τ where $e^{\star}O(X,\ x) = \{U \in e^{\star}O(X) :\ x \in U\}$.

When there is no ambiguity, we will write simple $A^{*e^{\star}}$ for $A^{*e^{\star}}(I,\ \tau)$.

Theorem 3.1. Let $(X, e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal space and A a subset of X.

(i) $A^{*e^{\star}} \subseteq A^{*} \subseteq A^{*r}$ for every $A \subseteq X$.
(ii) $A_* = A^{*e^{\star}}$ if $O(X,\ \tau) = e^{\star}O(X,\ \tau)$.
(iii) If $A \in I$, then $A^{*e^{\star}} = \phi$.
(iv) $(\phi)^{*e^{\star}} = \phi$.

Proof. (i). Let $x \in A^{*}(I,\ \tau)$. Then, $A \cap U \notin I$ for every $U \in \tau$. Since every open set is $e^{\star}$-open, therefore $x \in A^{*e^{\star}}(I,\ \tau)$. Converse is not true in general, it is shown in Example 3.1.

(ii). It is obvious from definition of local and $e^{\star}$-local functions.

(iii). Let $A \in I$ and $x \in A^{*e}$. Then for every $e^{\star}$-open set U containing x, $U \cap A \notin I$. On the other hand X is also $e^{\star}$-open set. So $X \cap A = A \notin I$. It is contradiction.

(iv). Because of Theorem 3.1(iii). it is obvious.

Example 3.1. Let $X = \{1,\ 2,\ 3\}$ and $\tau = \{\phi,\ \{3\},\ \{1,\ 2\},\ X\}$ with $I = \{\phi,\ \{3\}\}$. Let $A = \{1\}$ then $A^{*} = \{1,\ 2\} = Cl(A^{*})$ and $A^{*e^{\star}} = \{1\} = e^{\star}Cl(A^{*e^{\star}})$. So $A^{*} \nsubseteq A^{*e^{\star}}$.

Remark 3.1. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal space and A a subset of X. Neither $A \subseteq A^{*e^{\star}}$ nor $A^{*e^{\star}} \subseteq A$ in general.

The following is an example that supports this remark.

Example 3.2. Let $X = \{1,\ 2,\ 3,\ 4\}$ and $\tau = \{\phi,\ \{2\},\ \{3,\ 4\},\ \{2,\ 3,\ 4\},\ X\}$ with $I = \{\phi,\ \{1\}\}$. For $A = \{1\}$, $A^{*e^{\star}} = \phi$ and so $A^{*e^{\star}} \subset A$. For $A = \{2,\ 3\}$, $A^{*e^{\star}} = \{1,\ 2,\ 3\}$ and so $A \subset A^{*e^{\star}}$. For $A = \{1,\ 2,\ 3\}, A^{*e^{\star}} = \{1,\ 2,\ 3\}$ and so $A^{*e^{\star}} = A$.

Theorem 3.2. Let $(X,\ \tau,\ I)$ be an ideal topological space and $A,\ B$ subsets of X. Then, for $e^{\star}$-local functions, the following properties hold:

(i) If $A \subseteq B$, then $A^{*e^{\star}} \subseteq B^{*e^{\star}}$,
(ii) If $I,\ J$ ideal on X and $I \subseteq J$, then $A^{*e^{\star}}(J) \subseteq A^{*e^{\star}}(I)$.

Proof. (i). Let $x \in A^{*e^\star}$. Then for every $e^\star$-open set U_x containing x, $U_x \cap A \notin I$. Since $U_x \cap A \subseteq U_x \cap B$, then $U_x \cap B \notin I$.

(ii). Let $x \in A^{*e^\star}(J)$. Then $A \cap U \notin J$, for every $U \in e^\star O(X,\ x)$. Since $J \supseteq I,\ A \cap U \notin I$ and hence $x \in A^{*e^\star}(I)$.

Theorem 3.3. Let $(X,\ e^\star O(X,\ \tau),\ I)$ be an $e^\star$-ideal space and $A,\ B$ subsets of X. Then, for $e^\star$-local functions, the following properties hold:

(i) $A^{*e^\star} = cl(A^{*e^\star}) \subseteq e^\star cl(A)$ and $A^{*e^\star}$ is closed in $(X,\ \tau)$.
(ii) $(A^{*e^\star})^{*e^\star} \subseteq A^{*e^\star}$.
(iii) $A^{*e^\star} \cup B^{*e^\star} = (A \cup B)^{*e^\star}$.
(iv) $(A \cap B)^{*e^\star} \subseteq A^{*e^\star} \cap B^{*e^\star}$.
(v) $A^{*e^\star} \backslash B^{*e^\star} = (A \backslash B)^{*e^\star} \backslash B^{*e^\star} \subseteq (A \backslash B)^{*e^\star}$,

Proof. (i) We have $A^{*e^\star} \subseteq cl(A^{*e^\star})$ in general. Let $x \in cl(A^{*e^\star})$. Also given the set $U \in e^\star O(X,\ x)$. Then $A^{*e^\star} \cap T \neq \phi,\ T \in \tau(x)$. Since U is open, $A^{*e^\star} \cap U \neq \phi$. Therefore, there exists some $y \in (A^{*e^\star} \cap U)$ and $U \in e^\star O(X,\ y)$. Since $y \in A^{*e^\star},\ A \cap U \notin I$ and hence $x \in A^{*e^\star}$. Hence we have $cl(A^{*e^\star}) \subseteq A^{*e^\star}$ and hence $A^{*e^\star} = cl(A^{*e^\star})$. Again, let $x \in A^{*e^\star} = cl(A^{*e^\star})$, then $A \cap U \notin I$, for every $U \in e^\star O(X.\ x)$. This implies $A \cap U \neq \phi$ for every $U \in e^\star O(X,\ x)$. Therefore, $x \in e^\star cl(A)$. This shows that $A^{*e^\star} = cl(A^{*e^\star}) \subseteq e^\star cl(A^{*e^\star})$. Since $A^{*e^\star} = cl(A^{*e^\star}),\ A^{*e^\star}$ is closed.

(ii) Let $x \in (A^{*e^\star})^{*e^\star}$. Then for every $U \in e^\star O(X,\ x),\ U \cap A^{*e^\star} \notin I$ and hence $U \cap A^{*e^\star} \neq \phi$. Let $y \in U \cap A^{*e^\star}$. Then $U \in e^\star O(X,\ y)$ and $y \in A^{*e^\star}$. Hence we have $U \cap A \neq I$ and $x \in A^{*e^\star}$. This shows that $(A^{*e^\star})^{*e^\star} \subseteq A^{*e^\star}$.

(iii)By Theorem3.2(i)., we have $A^{*e^\star} \cup B^{*e^\star} \subseteq (A \cup B)^{*e^\star}$. To prove the reverse inclusion, let $x \notin A^{*e^\star} \cup B^{*e^\star}$. Then x belongs neither to $A^{*e^\star}$ nor to $B^{*e^\star}$. Therefore there exists $U_x,\ V_x \in e^\star O(X,\ x)$ such that $A \cap U_x \in I$ and $B \cap V_x \in I$. Since I is additive, $(A \cap U_x) \cup (B \cap V_x) \in I$.

$(A \cap U_x) \bigcup (B \cap V_x) = [(A \cap U_x) \bigcup V_x] \bigcap [(A \cap U_x) \bigcup B]$
$= (U_x \cup Vx) \bigcap (A \cup V_x) \bigcap (U_x \cup B) \bigcap (A \cup B)$

On the other hand since $U_x \cap V_x \subseteq U_x \cup V_x,\ V_x \subseteq A \cup V_x$ and $U_x \subseteq B \cup U_x$, we have
$(U_x \cup V_x) \bigcap (A \cup V_x) \bigcap (U_x \cup B) \bigcap (A \cup B) \supseteq (U_x \cap V_x) \bigcap (A \cup B)$

Since I is heredity, $(U_x \bigcap V_x) \bigcap (A \cup B) \in I$. Since $e^\star$-open sets closed under the finite intersections, $U_x \cap V_x \in e^\star O(X,\ x)$ and so $x \notin (A \cup B)^{*e^\star}$. Hence $(X \backslash A^{*e^\star}) \bigcap (X \backslash B^{*e^\star}) \subseteq X \backslash (A \cup B)^{*e^\star}$ or $(A \cup B)^{*e^\star} \subseteq A^{*e^\star} \cup B^{*e^\star}$.

(iv)By Theorem3.2(i)., $(A \cap B)^{*e^\star} \subseteq A^{*e^\star}$ and $(A \cap B)^{*e^\star} \subseteq B^{*e^\star}$ so $(A \cap B)^{*e^\star} \subseteq A^{*e^\star} \cup B^{*e^\star}$.

(v)We have by Theorem3.3(iii)., $A^{*e^\star} = [(A \backslash B) \bigcup (A \cap B)]^{*e^\star} = (A \backslash B)^{*e^\star} \bigcup (A \cap B)^{*e^\star} \subseteq (A \backslash B)^{*e^\star} \bigcup B^{*e^\star}$. Thus $A^{*e^\star} \backslash B^{*e^\star} \subseteq (A \backslash B)^{*e^\star} \backslash B^{*e^\star}$.

On the other hand, by Theorem3.2(i)., $(A \backslash B)^{*e^\star} \subseteq A^{*e^\star}$ and hence $(A \backslash B)^{*e^\star} \backslash B^{*e^\star} \subseteq A^{*e^\star} \backslash B^{*e^\star}$. Hence $A^{*e^\star} \backslash B^{*e^\star} = (A \backslash B)^{*e^\star} \backslash B^{*e^\star} \subseteq (A \backslash B)^{*e^\star}$.

Theorem 3.4. Let $(X,\ e^\star O(X,\ \tau),\ I)$ be an $e^\star$-ideal space and $A,\ B$ subsets of X. Then, for $e^\star$-local functions, the following properties hold:

(i) If $I_0 \in I$, then $(A \backslash I_0)^{*e^\star} = A^{*e^\star} = (A \cup I_0)^{*e^\star}$.
(ii) If $U \subseteq X$, then $U \cap (U \cap A)^{*e^\star} \subseteq U \cap A^{*e^\star}$.
(iii) If $A \subseteq X$ and $U \in e^\star O(X,\ \tau)$, then $U \cap A \in I \Rightarrow U \cap A^{*e^\star} = \phi$.
(iv) If $A \subseteq X$, then $(A \cap A^{*e^\star})^{*e^\star} \subseteq A^{*e^\star}$.

Proof. (i). Since $I_0 \in I$, by Theorem3.1(iii)., $I_0^{*e^\star} = \phi$. By 3.7(v)., $A^{*e^\star} = (A \backslash I_0)^{*e^\star}$ and by Theorem3.3(iii)., $(A \cup I_0)^{*e^\star} = A^{*e} \cup I_0^{*e^\star} = \phi \cup A^{*e^\star} = A^{*e^\star}$.

(ii). Since $U \cap A \subseteq A$, by Theorem3.2(i)., $(U \cap A)^{*e^{\star}} \subseteq A^{*e^{\star}}$ and hence $U \bigcap (U \cap A)^{*e^{\star}} \subseteq U \cap A^{*e^{\star}}$.

(iii). Let $U \cap A \in I$, then for every $x \in U,\ x \notin A^{*e^{\star}}$ because of $U \in e^{\star}O(X,\ \tau)$. So $U \cap A^{*e^{\star}} = \phi$.

(iv). By Theorem3.2(i) $(A \cap A^{*e^{\star}})^{*e^{\star}} \subseteq (A^{*e^{\star}})^{*e^{\star}}$. On the other hand, from 3.7(ii)., we have $(A \cap A^{*e^{\star}})^{*e^{\star}} \subseteq (A^{*e^{\star}})^{*e^{\star}} \subseteq A^{*e^{\star}}$.

In literature [7] for ideal topological spaces we will obtain $cl^* = A \cup A^*$ Kuratowski Closure operator. But in [8, 14] and [15] we are not able to define a Kuratowski Closure operator with the help of () local function. Because that functions do not provide a Theorem3.3(iii), given above for $()^{*e^{\star}}$-Operator.

We are able to define a closure operator with the help of $e^{\star}$-local function. Because the $()^{*e^{\star}}$ operator satisfy the conditions of Theorem3.1(iv)., Theorem3.3(ii). and Theorem3.3(iii). And thus $Cl_I^{*e^{\star}} : \wp(X) \to \wp(X),\ Cl_I^{*e^{\star}} = A \cup A^{*e^{\star}},\ \forall A \in \wp(X)$ is a Kuratowski closure operator. Hence it generates a $\tau^{*e^{\star}}$ topology:

$$\tau^{*e^{\star}}(I) = \{A \in \wp(X) :\ cl^{*e^{\star}}(X \backslash A) = X \backslash A\}$$

4. $\psi_{e^{\star}}$-OPERATOR

In topological space $cl(A) = X \backslash int(X \backslash A)$ [9] is remarkable result. Many useful result have been proved with the help of this result. This relation is the motivation of defining the operator $\psi_{e^{\star}}$.

Definition 4.1. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal space. An operator $\psi_{e^{\star}} : \wp(X) \to \tau$ is defined as: $\psi_{e^{\star}}(A) = \{x \in X | \exists U_x \in e^{\star}O(X,\ x) :\ U_x \backslash A \in I\}$, for every $A \in \wp(X)$.

We observe that $\psi_{e^{\star}}(A) = X \backslash (X \backslash A)^{*e^{\star}}$.

Theorem 4.1. Let $(X, e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal space and $A,\ B \in \wp(X)$.

(i) $\psi_{e^{\star}}(A) \supseteq e^{\star}int(A)$.
(ii) $\psi_{e^{\star}}(A)$ is open.
(iii) If $A \subseteq B$, then $\psi_{e^{\star}}(A) \subseteq \psi_{e^{\star}}(B)$.
(iv) $\psi_{e^{\star}}(A) \cup \psi_{e^{\star}}(B) \subseteq \psi_{e^{\star}}(A \cup B)$.
(v) $\psi_{e^{\star}}(A \cap B) = \psi_{e^{\star}}(A) \cap \psi_{e^{\star}}(B)$.
(vi) $\psi_{e^{\star}}(A) \subseteq \psi(A)$.

Proof. (i). $\psi_{e^{\star}}(A) = X \backslash (X \backslash A)^{*e^{\star}} \supseteq X \backslash e^{\star}cl(X \backslash A)$ by Theorem3.3(i). So $\psi_{e^{\star}}(A) \supseteq e^{\star}int(A)$.

(ii). Since $A^{*e^{\star}}$ is closed, then $(X \backslash A)^{*e^{\star}}$ is closed. So $X \backslash (X \backslash A)^{*e^{\star}} = \psi_{e^{\star}}(A)$ is a open set.

(iii). $A \subseteq B \Rightarrow X \backslash A \supseteq X \backslash B \Rightarrow (X \backslash A)^{*e^{\star}} \supseteq (X \backslash B)^{*e^{\star}}$
$\Rightarrow X \backslash (X \backslash A)^{*e^{\star}} \subseteq X \backslash (X \backslash B)^{*e^{\star}}$
$\Rightarrow \psi_{e^{\star}}(A) \subseteq \psi_{e^{\star}}(B)$

(iv). Proof is obvious from Theorem4.1(iii).

(v). $\psi_{e^{\star}}(A \cap B) = X \backslash [X \backslash (A \cap B)]^{*e^{\star}}$
$= X \backslash [(X \backslash A) \cup (X \backslash B)]^{*e^{\star}}$
$= X \backslash [(X \backslash A)^{*e^{\star}} \cup (X \backslash B)^{*e^{\star}}]$
$= [X \backslash (X \backslash A)^{*e^{\star}}] \cap [X \backslash (X \backslash B)^{*e^{\star}}]$
$= \psi_{e^{\star}}(A) \cap \psi_{e^{\star}}(B)$

(vi). From Theorem3.1(i)., we have that
$(X \backslash A)^* \subseteq (X \backslash A)^{*e^{\star}} \Rightarrow X \backslash (X \backslash A)^{*e^{\star}} \subseteq X \backslash (X \backslash A)^*$
$\Rightarrow \psi_{e^{\star}}(A) \subseteq \psi(A)$

Theorem 4.2. Let $(X, e^\star O(X,\ \tau),\ I)$ be a $e^\star$-ideal space and $A,\ B \in \wp(X)$.

(i) $\psi_{e^\star}(A) = \psi_{e^\star}(\psi_{e^\star}(A))$ if and only if $(X\backslash A)^{*e^\star} = [(X\backslash A)^{*e^\star}]^{*e^\star}$.
(ii) If $I_0 \in I$, then $\psi_{e^\star}(A\backslash I_0) = \psi_{e^\star}(A)$.
(iii) If $I_0 \in I$, then $\psi_{e^\star}(A \cup I_0) = \psi_{e^\star}(A)$.
(iv) If $(A\backslash B) \cup (B\backslash A) \in I$, then $\psi_{e^\star}(A) = \psi_{e^\star}(B)$.
(v) If $A \in e^\star O(X,\ \tau)$, then $A \subseteq \psi_{e^\star}(A)$.

Proof. (i). Proof is obvious from definition of $\psi_{e^\star}(A)$ and the fact:
$\psi_{e^\star}(\psi_{e^\star}(A)) = X\backslash[X\backslash(X\backslash(X\backslash A)^{*e^\star})]^{*e^\star} = X\backslash[(X\backslash A)^{*e^\star}]^{*e^\star}$.
(ii). By Theorem3.4(ii)., we have
$\psi_{e^\star}(A\backslash I_0) = X\backslash[X\backslash(A\backslash I_0)]^{*e^\star}$
$= X\backslash[(X\backslash A) \cup I_0]^{*e^\star}$
$= X\backslash(X\backslash A)^{*e^\star}$
$= \psi_{e^\star}(A)$
(iii). By Theorem3.4(ii)., we have
$\psi_{e^\star}(A \cup I_0) = X\backslash[X\backslash(A \cup I_0)]^{*e^\star}$
$= X\backslash[(X\backslash A)\backslash I_0]^{*e^\star}$
$= X\backslash(X\backslash A)^{*e^\star}$
$= \psi_{e^\star}(A)$
(iv). Assume $(A\backslash B) \cup (B\backslash A) \in I$. Let $A\backslash B = I_1$ and $B\backslash A = I_2$. Observe that by heredity $I_1,\ I_2 \in I$. Also observe that $B = (A\backslash I_1) \cup I_2$. Thus $\psi_{e^\star}(A) = \psi_{e^\star}(A\backslash I_1) = \psi[(A\backslash I_1) \cup I_2] = \psi_{e^\star}(B)$ by Theorem4.2(ii) and Theorem4.2(iii).
(v). Since $A \in e^\star O(X,\ \tau)$, $(X\backslash A) \in e^\star C(X,\ \tau)$. So $(X\backslash A) = e^\star cl(X\backslash A)$. From Theorem3.3(i), we have
$(X\backslash A)^{*e^\star} \subseteq e^\star cl(X\backslash A) = X\backslash A \Rightarrow (X\backslash A)^{*e^\star} \subseteq X\backslash A$
$\Rightarrow A \subseteq X\backslash(X\backslash A)^{*e^\star}$
$\Rightarrow A \subseteq \psi_{e^\star}(A)$.

Example 4.1. Let $X = \{1,\ 2,\ 3\}$ and $\tau = \{\phi, \{1\},\ \{2\},\ \{1,\ 2\},\ X\}$ with $I = \{\phi,\ 2,\ 3, \{2,3\}\}$. Then for $A = \{3\}$, we have$\psi_{e^\star}(A) = \{1,\ 2,\ 3\} \supseteq A$ but $A = \{3\}$ is not a $e^\star$-open set.

Theorem 4.3. Let $(X,\ e^\star O(X,\ \tau),\ I)$ be a $e^\star$-ideal space and $A \subseteq X$.

(i) $\psi_{e^\star}(A) = \bigcup\{U \in e^\star O(X,\ \tau) :\ U\backslash A \in I\}$.
(ii) $\psi_{e^\star}(A) \supseteq \bigcup\{U \in e^\star O(X,\ \tau) :\ (U\backslash A) \cup (A\backslash U) \in I\}$

Proof. (i). Proof is obvious from definition of $\psi_{e^\star}(A)$.
(ii). Since I is heredity, we have
$\bigcup\{U \in e^\star O(X,\ \tau) :\ (U\backslash A) \cup (A\backslash U) \in I\} \subseteq \bigcup\{U \in e^\star O(X,\ \tau) :\ U\backslash A \in I\} = \psi_{e^\star}(A)$.

5. $e^\star O$-CODENSE IDEAL

Definition 5.1. Let $(X,\ e^\star(X,\ \tau),\ I)$ be a $e^\star$-ideal spaces and $A \subseteq X$. If $e^\star O(X,\ \tau) \cap I = \{\phi\}$ then we say that I is $e^\star O$-codense ideal.

Theorem 5.1. Let $(X,\ e^\star O(X,\ \tau),\ I)$ be a $e^\star$-ideal spaces. If I is $e^\star O$-codense ideal with $e^\star O(X,\ \tau)$, then $X = X^{*e^\star}$.

Proof. It is obvious that $X^{*e^\star} \subseteq X$. Let $x \notin X^{*e^\star}$, for $x \in X$. Then there is atleast one $U_x \in e^\star O(X.\ \tau)$ that provide $U_x \cap X \in I$. Hence $U_x \cap X = U_x \in I$. But $e^\star O(X.\ \tau) \cap I = \{\phi\}$. It is a contradiction. So $X = X^{*e^\star}$.

Theorem 5.2. The following are equivalent for $(X,\ e^{\star}O(X,\ \tau),\ I)$ $e^{\star}$-ideal space.

(i) $e^{\star}O(X,\ \tau)\cap I=\{\phi\}$,
(ii) $\psi_{e^{\star}}(\phi)=\phi$,
(iii) If $I_0\in I, \psi_{e^{\star}}(I_0)=\phi$.

Proof. (i)$\Rightarrow$ (ii): Let $e^{\star}O(X,\ \tau)\cap I=\{\phi\}$. From definition of $\psi_{e^{\star}}$operator and Theorem 5.1., we have $\psi_{e^{\star}}(\phi)=X\backslash(X\backslash\phi)^{*e^{\star}}=X\backslash X^{*e^{\star}}=\phi$.

(ii)$\Rightarrow$ (iii): Let $I_0\in I$ and $\psi_{e^{\star}}(\phi)=\phi$. Also because of Theorem3.4(i), we have obtained $(X\backslash I_0)^{*e^{\star}}=X^{*e^{\star}}$. So we have

$\psi_{e^{\star}}(I_0)=X\backslash(X\backslash I_0)^{*e^{\star}}=X\backslash X^{*e^{\star}}=\psi_{e^{\star}}(\phi)=\phi$.

(iii)$\Rightarrow$ (i): Let $A\in e^{\star}O(X.\ \tau)\cap I$. Then because of $A\in I$ and Theorem5.2(iii)., we have $\psi_{e^{\star}}(A)=\phi$. Also $A\subseteq\psi_{e^{\star}}(A)=\phi$ since $A\in e^{\star}O(X,\ \tau)$ and $A\subseteq\psi_{e^{\star}}(A)$. And so $A=\phi$. Hence we have $e^{\star}O(X,\ \tau)=\{\phi\}$.

Theorem 5.3. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal spaces. If I is $e^{\star}O$-codense ideal with $e^{\star}O(X,\ \tau)$, then $\psi_{e^{\star}}(A)\subseteq A^{*e^{\star}}$ for every $A\subseteq X$.

Proof. Let $x\in\psi_{e^{\star}}(A)$ and $x\notin A^{*e^{\star}}$ for a least one $x\in X$. Then we obtain $x\notin A^{*e^{\star}}\Rightarrow\exists T_x\in e^{\star}O(X,\ x);\ T_x\cap A\in I$.

Since $x\in\psi_{e^{\star}}(A)$, we have $x\in\bigcup\{U\in e^{\star}O(X,\ \tau):\ U\backslash A\in I\}$ from Theorem4.3(i). Hence there is $V\in e^{\star}O(X,\ \tau)$ which satisfy $x\in V$ and $V\backslash A\in I$. Since $x\in T_x\cap V$ is a $e^{\star}$-open set, we obtain $(T_x\cap V)\cap A\in I$ and $(T_x\cap V)\backslash A\in I$ from heredity of I. Also since I is finite additivity, we obtain $T_x\cap V=[(T_x\cap V)\cap A]\bigcup[(T_x\cap V)\backslash A]\in I$

Since $T_x\cap V\neq\phi$ is a $e^{\star}$-open set, $I\cap e^{\star}O(X,\ \tau)\neq\phi$. But it contradict with the fact I is $e^{\star}O$-codense. So $x\in A^{*e^{\star}}$.

Remark 5.1. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal spaces and $A\subseteq X$. If I is $e^{\star}O$-codense ideal, then $\psi_{e^{\star}}(A)\subseteq e^{\star}cl(A)$.

6. $e^{\star}$-COMPATIBILITY TOPOLOGY WITH AN IDEAL

Definition 6.1. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be an $e^{\star}$-ideal spaces. We say the τ is $e^{\star}$-compatible with the ideal I, denoted $\tau\sim_{e^{\star}}I$, if the following holds for every $A\subseteq X$, if for every $x\in A$ there exists $U\in e^{\star}O(X,\ x)$ such that $U\cap A\in I$, then $A\in I$.

Theorem 6.1. Let $(X,\ e^{\star}O(X,\ \tau),\ I)$ be a $e^{\star}$-ideal space. $\tau\sim_{e^{\star}}I$ if and only if $\psi_{e^{\star}}(A)\backslash A\in I$, for every $A\subseteq X$.

Proof. Let $\tau\sim_{e^{\star}}I$ and $\psi_{e^{\star}}(A)\backslash A\in I$, for every $A\subseteq X$. Then we have

$x\in\psi_{e^{\star}}(A),\ x\notin A\Rightarrow x\in\backslash(X\backslash A)^{*e^{\star}},\ x\notin A$

$\Rightarrow x\notin(X\backslash A)^{*e^{\star}},\ x\notin A$

$\Rightarrow\exists U\in e^{\star}O(X,\ x);\ U\cap(X\backslash A)\in I,\ x\notin A$

$\Rightarrow X\backslash A\in I,\ x\in X\backslash A$

So $\psi_{e^{\star}}(A)\backslash A\subseteq X\backslash A\in I$.

Conversely, let $\psi_{e^{\star}}(A)\backslash A\in I$ for every $A\subseteq X$. Also there is $U\in e^{\star}O(X,\ x)$ which $U\cap A\in I$ for every $x\in A$. Then

$x\notin A^{*e^{\star}}\Rightarrow x\in X\backslash A^{*e^{\star}}\Rightarrow A\subseteq X\backslash A^{*e^{\star}}$.

Hence because of the following equation and the fact $A\subseteq X\backslash A^{*e^{\star}}$ we have $\psi_{e^{\star}}(X\backslash A)\backslash(X\backslash A)=A$.

$\psi_{e^{\star}}(X\backslash A)\backslash(X\backslash A)=[X\backslash(X\backslash(X\backslash A))^{*e^{\star}}]\backslash(X\backslash A)=(X\backslash A^{*e^{\star}})\cap A$

Since $\psi_{e^{\star}}(A)\backslash A\in I$ for every $A\subseteq X$, $\psi_{e^{\star}}(X\backslash A)\backslash(X\backslash A)=A\in I$.

From the above theorem we will give the following remark.

Remark 6.1. Let $(X,\, e^{\star}O(X,\, \tau),\, I)$ be a $e^{\star}$ ideal space and $\tau \sim_{e^{\star}} I$. Then $\psi_{e^{\star}}(\psi_{e^{\star}}(A)) = \psi_{e^{\star}}(A)$, for every $A \subseteq X$.

7. Conclusions and Discussions

The contributions of the $\psi_{e^{\star}}$-operator in ideal topological spaces have initiated the generalization of some important properties in general topology via topological ideals. Properties such as decomposition of continuity, separation axioms, connectedness, compactness, and resolvability are to be studied in the future. In the usual topology, open sets are defined as subsets of the space that satisfy certain properties, such as being closed under unions and finite intersections. The e*-ideal topology, on the other hand, is defined based on a specific type of ideal and may not coincide with the open sets in the usual topology.

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