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APPLICATIONS OF SOFT SET THEORY TO THE SUBALGEBRAS OF CI-ALGEBRAS

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ABSTRACT. In this paper, the applicability of soft set theory to subalgebras of CI-algebras is investigated. CI-algebras are utilized in algebraic logic and computer science. Soft set theory is a framework for dealing with ambiguous or imprecise information. We employ soft sets to investigate the intersection and union of subalgebras, among other properties of subalgebras of CI-algebras. We demonstrate that soft set theory is a valuable tool for analyzing subalgebras of CI-algebras and developing new results in this domain. This paper contributes to the comprehension of soft set theory and its applications in CI-algebras with the findings presented herein.

1. INTRODUCTION

Soft set theory, introduced by Molodtsov in 1999 [17], is a mathematical framework that provides a flexible and intuitive way to handle uncertain or imprecise information. It has gained significant attention in various fields, including decision-making, data mining, pattern recognition, and artificial intelligence. Soft set theory allows for the representation and analysis of vague or incomplete information through the concept of a soft set, which is a generalization of classical set theory.

In recent years, there has been a growing interest in combining different mathematical theories to gain new insights and solve complex problems. The combination of soft set theory and BCI-algebras has the potential to provide a deeper understanding of the properties and behaviour of BCI-algebras in the presence of uncertain information.

Kim and Kim introduced the concept of a BE-algebra to generalize a BCK-algebra and studied its numerous properties [12]. In an attempt to further develop the concept of BE-algebras, Meng introduced CI-algebras as a generalization [16]. Subsequently, Kim examined the ideal theory and attributes of CI-algebras [11].

This research builds upon previous work in the fields of soft sets and BCI-algebras. Aktas and Cagman [1] introduced the concept of soft sets and their applications in various

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mathematical structures. Huang [5] provided a comprehensive study of BCI-algebras and their properties. Jun [6] extended the concept of soft sets to BCK/BCI-algebras. Jun and Park [9] explored the applications of soft sets in the ideal theory of BCK/BCI-algebras.

Fuzzy soft sets were introduced by Maji et al. [14, 15], who also explored their application in decision-making problems. Since then, research on the theory of soft sets has progressed rapidly. For instance, Jun et al. [7] studied intersection-soft filters in R_0-algebras, while Roh and Jun [26] investigated positive implicative ideals in BCK-algebras using intersectional soft sets. Also, Akram [2] introduced the notion of fuzzy soft Lie algebras. Similarly, Roy et al. [27] applied fuzzy soft sets to decision-making problems, and Aygünoğlu et al. [4] proposed and studied the concept of a fuzzy soft group. Moreover, Jun et al. [8] introduced the notion of fuzzy soft BCK/BCI-algebras (also known as FSB-algebras) by applying the theory of fuzzy soft sets to BCK/BCI-algebras. Additionally, numerous studies have been conducted by Muhiuddin et al. that explore the application of soft set theory to various algebraic structures [3, 10, 18, 19, 20, 21, 22, 23, 24, 25].

In this paper, we build upon the existing literature by focusing specifically on the relationship between soft sets and subalgebra in CI-algebras. We aim to provide a comprehensive analysis of this relationship and establish important results that contribute to the understanding of soft set theory in the context of CI-algebras. The applicability of soft set theory to CI-algebraic subalgebras is examined in this study. Specifically, by utilising the idea of soft sets, we investigate several subalgebraic qualities like their intersection and union. Additionally, we define a soft ideal as a soft set of a CI-algebra. It is our intention to demonstrate how soft set theory can be a useful tool for studying and comprehending subalgebras of CI-algebras.

Following is how the current paper is structured: The terms and characteristics relevant to soft sets and CI-algebras are covered in Section 2 in detail. Soft set operations are used in Section 3 to examine the intersection, union and other results based on soft CI-algebras. Our conclusions and possible future study directions are summarised in Section 4 for the conclusion.

2. Preliminaries

A type $(2, \theta)$ algebra $(L_0; *, 1)$ is referred to as a $wwwCI$-algebra (briefly, ***CI-A***) if it fulfills the following criteria:

(CI1) $m_1 * m_1 = 1$,
(CI2) $1 * m_1 = m_1$,
(CI3) $m_1 * (m_2 * m_3) = m_2 * (m_1 * m_3)$,

for all $m_1, m_2, m_3 \in L_0$. A ***CI-A*** $(L_0; *, 1)$ is said to be *transitive* if it satisfies:

$$(\forall m_1, m_2, m_3 \in L_0)\left((m_2 * m_3) * ((m_1 * m_2) * (m_1 * m_3)) = 1\right). \tag{2.1}$$

A ***CI-A*** $(L_0; *, 1)$ is said to be *self-distributive* if it satisfies:

$$(\forall m_1, m_2, m_3 \in L_0)\left(m_1 * (m_2 * m_3) = (m_1 * m_2) * (m_1 * m_3)\right). \tag{2.2}$$

Note that every self-distributive ***CI-A*** is a transitive ***CI-A*** (see [11]).

A non-empty subset $\hat{I}$ of a ***CI-A*** $(L_0; *, 1)$ is called an *ideal* of L_0 (see [11]) if it satisfies:

(I1) $(\forall m_1, m_2 \in L_0)\left(m_2 \in \hat{I} \Rightarrow m_1 * m_2 \in \hat{I}\right)$,
(I2) $(\forall m_1, r_0, s_0 \in L_0)\left(r_0, s_0 \in \hat{I} \Rightarrow (r_0 * (s_0 * m_1)) * m_1 \in \hat{I}\right)$.

Molodtsov [17] presented the following definition of a soft set. Consider an initial universe set $\widehat{U}$ and a set of parameters E, with $\mathscr{Y}(\widehat{U})$ denoting the power set of $\widehat{U}$. Let $\widehat{J_1} \subset E$.

Definition 2.1. A pair $(\tilde{\rho}, \widehat{J_1})$ is defined as a *soft set* over $\widehat{U}$, where $\tilde{\rho}$ is a mapping given by $\tilde{\rho} : \widehat{J_1} \to \mathscr{Y}(\widehat{U})$.

Consider two soft sets over a common universe $\widehat{U}$, namely $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$. The intersection of these two soft sets is defined as the soft set $(\tilde{h}, \widehat{W})$, which satisfies the following conditions:

Definition 2.2. [13] The *intersection* of $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ is given by the soft set $(\tilde{h}, \widehat{W})$, where:

(i) $\widehat{W} = \widehat{J_1} \cap \widehat{J_2}$, and
(ii) For every $e \in \widehat{W}$, we have ($\tilde{h}(e) = \tilde{\rho}(e)$ or $\tilde{\sigma}(e)$), since both are the same set.

This intersection is denoted as $(\tilde{\rho}, \widehat{J_1}) \tilde{\cap} (\tilde{\sigma}, \widehat{J_2}) = (\tilde{h}, \widehat{W})$.

Definition 2.3. [13] The *union* of $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ is given by the soft set $(\tilde{h}, \widehat{W})$, where:

(i) $\widehat{W} = \widehat{J_1} \cup \widehat{J_2}$,
(ii) For all $e \in \widehat{W}$,

$$\tilde{h}(e) = \begin{cases} \tilde{\rho}(e) & \text{if } e \in \widehat{J_1} \setminus \widehat{J_2}, \\ \tilde{\sigma}(e) & \text{if } e \in \widehat{J_2} \setminus \widehat{J_1}, \\ \tilde{\rho}(e) \cup \tilde{\sigma}(e) & \text{if } e \in \widehat{J_1} \cap \widehat{J_2}. \end{cases}$$

This union is denoted as $(\tilde{\rho}, \widehat{J_1}) \tilde{\cup} (\tilde{\sigma}, \widehat{J_2}) = (\tilde{h}, \widehat{W})$.

Definition 2.4. If $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ are two soft sets over a common universe $\widehat{U}$, then the operation "AND" between them, denoted by $(\tilde{\rho}, \widehat{J_1}) \tilde{\wedge} (\tilde{\sigma}, \widehat{J_2})$, is defined as $(\tilde{\rho}, \widehat{J_1}) \tilde{\wedge} (\tilde{\sigma}, \widehat{J_2}) = (\tilde{h}, \widehat{J_1} \times \widehat{J_2})$, where $\tilde{h}(\beta, \beta) = \tilde{\rho}(\beta) \cap \tilde{\sigma}(\beta)$ for all $(\beta, \beta) \in \widehat{J_1} \times \widehat{J_2}$.

Definition 2.5. [13] If $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ are two soft sets over a common universe $\widehat{U}$, then the operation "OR" between them, denoted as $(\tilde{\rho}, \widehat{J_1}) \tilde{\vee} (\tilde{\sigma}, \widehat{J_2})$, is defined as $(\tilde{h}, \widehat{J_1} \times \widehat{J_2})$, where $\tilde{h}(\beta, \beta) = \tilde{\rho}(\beta) \cup \tilde{\sigma}(\beta)$ for all $(\beta, \beta) \in \widehat{J_1} \times \widehat{J_2}$.

Definition 2.6. [13] Two soft sets $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ over a common universe $\widehat{U}$ are said to have a soft subset relationship, denoted by $(\tilde{\rho}, \widehat{J_1}) \tilde{\subset} (\tilde{\sigma}, \widehat{J_2})$, if they satisfy the following conditions:

(i) $\widehat{J_1} \subset \widehat{J_2}$,
(ii) For every $\varepsilon \in \widehat{J_1}$, $\tilde{\rho}(\varepsilon)$ and $\tilde{\sigma}(\varepsilon)$ are identical approximations.

For a soft set $(\tilde{\rho}, L_0)$ over $\widehat{U}$ and a subset γ of $\widehat{U}$, the *γ-inclusive set* of $(\tilde{\rho}, L_0)$, denoted by $(\tilde{\rho}; \gamma)^{\supseteq}$, is defined to be the set

$$(\tilde{\rho}; \gamma)^{\supseteq} := \{m_1 \in L_0 \mid \gamma \subseteq \tilde{\rho}(m_1)\}.$$

3. Soft CI-algebras

In the subsequent discussion, we consider a *CI-A* denoted as L_0 and a nonempty set denoted as $\widehat{J_1}$. We use the symbol R to represent an arbitrary binary relation between an element of $\widehat{J_1}$ and an element of L_0. Specifically, R is a subset of the Cartesian product

$\widehat{J_1} \times L_0$, unless stated otherwise. A set-valued function $\tilde{\rho} : \widehat{J_1} \to \mathscr{Y}(L_0)$ can be formally defined as $\tilde{\rho}(m_1) = \{m_2 \in L_0 \mid m_1 R m_2\}$ for all $m_1 \in \widehat{J_1}$. The pair $(\tilde{\rho}, \widehat{J_1})$ can be considered as a soft set over L_0. The order of an element m_1 in a *CI-A* L_0 is defined as $o(m_1)$ and is given by $o(m_1) = \min\{n \in \mathbb{N} \mid 0 * m_1^n = 0\}$.

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Definition 3.1. $(\tilde{\rho}, \widehat{J_1})$ is an soft *CI-A* over L_0 if $\tilde{\rho}(m_1)$ is a subalgebra of L_0 for every $m_1 \in \widehat{J_1}$.

Let's illustrate this definition with the examples below.

Example 3.2. Let $X = \{\theta, r_0, s_0, t_0, p_0\}$ be a BCK-algebra with the following Cayley table:

$*$	θ	r_0	s_0	t_0	p_0
θ	θ	θ	θ	θ	θ
r_0	r_0	θ	r_0	r_0	r_0
s_0	s_0	s_0	θ	s_0	s_0
t_0	t_0	t_0	t_0	θ	t_0
p_0	p_0	p_0	p_0	p_0	θ

Let $(\tilde{\rho}, \widehat{J_1})$ be a soft set over L_0, where $\widehat{J_1} = L_0$ and $\tilde{\rho} : \widehat{J_1} \to \mathscr{Y}(L_0)$ is a set-valued function defined by

$$\tilde{\rho}(m_1) = \{m_2 \in L_0 \mid m_1 R m_2 \Leftrightarrow m_2 \in m_1^{-1} I\}$$

for all $m_1 \in \widehat{J_1}$ where $\hat{I} = \{\theta, r_0\}$ and $m_1^{-1}\hat{I} = \{m_2 \in L_0 \mid m_1 \wedge m_2 \in \hat{I}\}$. Then $\tilde{\rho}(\theta) = \tilde{\rho}(r_0) = L_0$, $\tilde{\rho}(s_0) = \{\theta, r_0, t_0, p_0\}$, $\tilde{\rho}(t_0) = \{\theta, r_0, s_0, p_0\}$, and $\tilde{\rho}(p_0) = \{\theta, r_0, s_0, t_0\}$ are subalgebras of L_0. Therefore $(\tilde{\rho}, \widehat{J_1})$ is a soft *CI-A* over L_0.

Let $\widehat{J_1}$ be a fuzzy *CI-SubA* of L_0 with membership function $\mu_{\widehat{J_1}}$. Let us consider the family of β-level sets for the function $\mu_{\widehat{J_1}}$ given by

$$\tilde{\rho}(\beta) = \{m_1 \in L_0 \mid \mu_{\widehat{J_1(x)}} \geq \beta\},\ \beta \in [0,1].$$

Then $\tilde{\rho}(\beta)$ is a *CI-SubA* of L_0. If we know the family $\tilde{\rho}$, we can find the functions $\mu_{\widehat{J_1(x)}}$ by means of the following formula:

$$\mu_{\widehat{J_1(m_1)}} = \sup\{\beta \in [0,1] \mid m_1 \in \tilde{\rho}(\beta)\}.$$

Thus, every fuzzy *CI-SubA* $\widehat{J_1}$ may be considered as the soft *CI-A* $(\tilde{\rho}, [0,1])$.

Theorem 3.1. *Let $(\tilde{\rho}, \widehat{J_1})$ be a soft CI-A over L_0. If $\widehat{J_2}$ is a subset of $\widehat{J_1}$, then $(\tilde{\rho}|_{\widehat{J_2}}, \widehat{J_2})$ is a soft CI-A over L_0.*

Proof. Straightforward. □

The following example shows that there exists a soft set $(\tilde{\rho}, \widehat{J_1})$ over L_0 such that

(i) $(\tilde{\rho}, \widehat{J_1})$ is not a soft *CI-A* over L_0.
(ii) there exists a subset $\widehat{J_2}$ of $\widehat{J_1}$ such that $(\tilde{\rho}|_{\widehat{J_2}}, \widehat{J_2})$ is a soft *CI-A* over L_0.

Theorem 3.2. *If $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ are two soft CI-As over L_0 with a non-empty intersection $\widehat{J_1} \cap \widehat{J_2}$, then their intersection $(\tilde{\rho}, \widehat{J_1}) \tilde{\cap} (\tilde{\sigma}, \widehat{J_2})$ is also a soft CI-A over L_0.*

Proof. Let $(\tilde{h}, \widehat{W})$ be the intersection of $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$, where $\widehat{W} = \widehat{J_1} \cap \widehat{J_2}$ and $\tilde{h}(m_1) = \tilde{\rho}(m_1)$ or $\tilde{\sigma}(m_1)$ for all $m_1 \in \widehat{W}$. Since $\tilde{h} : \widehat{W} \to \mathscr{Y}(L_0)$ is a mapping,

$(\tilde{h}, \widehat{W})$ is a soft set over L_0. As $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ are soft *CI-A*s over L_0, we have $\tilde{h}(m_1) = \tilde{\rho}(m_1)$ or $\tilde{h}(m_1) = \tilde{\sigma}(m_1)$ is a *CI-SubA* of L_0 for all $m_1 \in \widehat{W}$. Therefore, $(\tilde{h}, \widehat{W}) = (\tilde{\rho}, \widehat{J_1}) \tilde{\cap} (\tilde{\sigma}, \widehat{J_2})$ is a soft *CI-A* over L_0. □

Corollary 3.3. *Let $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_1})$ be two soft CI-As over L_0. Then their intersection $(\tilde{\rho}, \widehat{J_1}) \tilde{\cap} (\tilde{\sigma}, \widehat{J_1})$ is a soft CI-A over L_0.*

Proof. Straightforward. □

Theorem 3.4. *If $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_1})$ are two soft CI-As over L_0 with disjoint sets $\widehat{J_1}$ and $\widehat{J_2}$, then their union $(\tilde{\rho}, \widehat{J_1}) \tilde{\cup} (\tilde{\sigma}, \widehat{J_1})$ is a soft CI-A over L_0.*

Proof. Let $(\tilde{h}, \widehat{W})$ be the union of $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$, where $\widehat{W} = \widehat{J_1} \cup \widehat{J_2}$ and for every $e \in \widehat{W}$,

$$\tilde{h}(e) = \begin{cases} \tilde{\rho}(e) & \text{if } e \in \widehat{J_1} \setminus \widehat{J_2}, \\ \tilde{\sigma}(e) & \text{if } e \in \widehat{J_2} \setminus \widehat{J_1}, \\ \tilde{\rho}(e) \cup \tilde{\sigma}(e) & \text{if } e \in \widehat{J_1} \cap \widehat{J_2}. \end{cases}$$

Since $\widehat{J_1}$ and $\widehat{J_2}$ are disjoint sets, for every $x \in \widehat{W}$ we have that either $m_1 \in \widehat{J_1} \setminus \widehat{J_2}$ or $m_1 \in \widehat{J_2} \setminus \widehat{J_1}$. If $m_1 \in \widehat{J_1} \setminus \widehat{J_2}$, then $\tilde{h}(m_1) = \tilde{\rho}(m_1)$ is a *CI-SubA* of L_0 since $(\tilde{\rho}, \widehat{J_1})$ is a soft *CI-A* over L_0. Similarly, if $m_1 \in \widehat{J_2} \setminus \widehat{J_1}$, then $\tilde{h}(m_1) = \tilde{\sigma}(m_1)$ is a *CI-SubA* of L_0 since $(\tilde{\sigma}, \widehat{J_2})$ is a soft *CI-A* over L_0. Hence, $(\tilde{h}, \widehat{W}) = (\tilde{\rho}, \widehat{J_1}) \tilde{\cup} (\tilde{\sigma}, \widehat{J_1})$ is a soft *CI-A* over L_0. □

Theorem 3.5. *If $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ are both soft CI-As over L_0, then $(\tilde{\rho}, \widehat{J_1}) \tilde{\wedge} (\tilde{\sigma}, \widehat{J_2})$ is also soft.*

Proof. By means of Definition 2.4, we know that

$$(\tilde{\rho}, \widehat{J_1}) \tilde{\wedge} (\tilde{\sigma}, \widehat{J_2}) = (\tilde{h}, \widehat{J_1} \times \widehat{J_2}),$$

where $\tilde{h}(m_1, m_2) = \tilde{\rho}(m_1) \cap \tilde{\sigma}(m_2)$ for all $(m_1, m_2) \in \widehat{J_1} \times \widehat{J_2}$. Since $\tilde{\rho}(m_1)$ and $\tilde{\sigma}(m_2)$ are *CI-SubA*s of L_0, the intersection $\tilde{\rho}(m_1) \cap \tilde{\sigma}(m_2)$ is also a *CI-SubA* of L_0. Hence $\tilde{h}(m_1, m_2)$ is a *CI-SubA* of L_0 for all $(m_1, m_2) \in \widehat{J_1} \times \widehat{J_2}$, and therefore $(\tilde{\rho}, \widehat{J_1}) \tilde{\wedge} (\tilde{\sigma}, \widehat{J_2}) = (\tilde{h}, \widehat{J_1} \times \widehat{J_2})$ is a soft *CI-A* over L_0. □

Definition 3.3. In the context of soft *CI-A*s over L_0, a soft *CI-A* $(\tilde{\rho}, \widehat{J_1})$ is considered *trivial* if $\tilde{\rho}(m_1) = 0$ for all $m_1 \in \widehat{J_1}$, and it is considered *whole* if $\tilde{\rho}(m_1) = L_0$ for all $m_1 \in \widehat{J_1}$.

Example 3.4. Consider the BCI-algebra $L_0 = \{\theta, r_0, s_0, t_0\}$ introduced in Example 3.2. For $\widehat{J_1} = L_0$, we define $\tilde{\rho} : \widehat{J_1} \to \mathscr{Y}(L_0)$ as follows:

$$\tilde{\rho}(m_1) = \{\theta\} \cup \{m_2 \in L_0 \mid m_1 R m_2, \Leftrightarrow, o(m_1) = o(m_2)\}$$

for every $m_1 \in \widehat{J_1}$. It can be observed that $\tilde{\rho}(x) = L_0$ for all $m_1 \in \widehat{J_1}$, indicating that $(\tilde{\rho}, \widehat{J_1})$ forms a whole soft BCI-algebra over L_0.

Consider a mapping, denoted as $\bar{\eta} : L_0 \to Q_0$, which maps *CI-A*s. Now, suppose we have a soft set $(\tilde{\rho}, \widehat{J_1})$ over L_0. In this case, $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ represents a soft set over Q_0, where $\bar{\eta}(\tilde{\rho}) : \widehat{J_1} \to \mathscr{Y}(Q_0)$ is defined as $\bar{\eta}(\tilde{\rho})(m_1) = \bar{\eta}(\tilde{\rho}(m_1))$ for any m_1 belonging to $\widehat{J_1}$.

Lemma 3.6. *Let $\bar{\eta} : L_0 \to Q_0$ be a homomorphism of CI-As. If $(\tilde{\rho}, \widehat{J_1})$ is a soft CI-A over L_0, then $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ is a soft CI-A over Q_0.*

Proof. For every $m_1 \in \widehat{J_1}$, we have $\bar{\eta}(\tilde{\rho})(m_1) = \bar{\eta}(\tilde{\rho}(m_1))$ is a *CI-SubA* of Q_0 since $\tilde{\rho}(m_1)$ is a *CI-SubA* of L_0 and its homomorphic image is also a *CI-SubA* of Q_0. Hence $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ is a soft *CI-A* over Q_0. □

Theorem 3.7. *Let $\bar{\eta} : L_0 \to Q_0$ be a homomorphism of CI-As and let $(\tilde{\rho}, \widehat{J_1})$ be a soft CI-A over L_0.*

(i) *If $\tilde{\rho}(m_1) = \ker(\bar{\eta})$ for all $m_1 \in \widehat{J_1}$, then $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ is the trivial soft CI-A over Q_0.*

(ii) *If $\bar{\eta}$ is onto and $(\tilde{\rho}, \widehat{J_1})$ is whole, then $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ is the whole soft CI-A over Q_0.*

Proof. (i) Assume that $\tilde{\rho}(m_1) = \ker(\bar{\eta})$ for all $m_1 \in \widehat{J_1}$. Then $\bar{\eta}(\tilde{\rho})(m_1) = \bar{\eta}(\tilde{\rho}(m_1)) = \{0_{Q_0}\}$ for all $m_1 \in \widehat{J_1}$. Hence $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ is the trivial soft *CI-A* over Q_0 by Lemma 3.6 and Definition 3.3.

(ii) Suppose that $\bar{\eta}$ is onto and $(\tilde{\rho}, \widehat{J_1})$ is whole. Then $\tilde{\rho}(m_1) = L_0$ for all $m_1 \in \widehat{J_1}$, and so $\bar{\eta}(\tilde{\rho})(m_1) = \bar{\eta}(\tilde{\rho}(m_1)) = \bar{\eta}(L_0) = Q_0$ for all $m_1 \in \widehat{J_1}$. It follows from Lemma 3.6 and Definition 3.3 that $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})$ is the whole soft *CI-A* over Q_0. □

Definition 3.5. Let $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ be two soft *CI-A*s over L_0. Then $(\tilde{\rho}, \widehat{J_1})$ is called a *soft subalgebra* (briefly, *S-SubA*) of $(\tilde{\sigma}, \widehat{J_2})$, denoted by $(\tilde{\rho}, \widehat{J_1}) \tilde{<} (\tilde{\sigma}, \widehat{J_2})$, if it satisfies:

(i) $\widehat{J_1} \subset \widehat{J_2}$,

(ii) $\tilde{\rho}(m_1)$ is a *CI-SubA* of $\tilde{\sigma}(m_1)$ for all $m_1 \in \widehat{J_1}$.

Example 3.6. Let $(\tilde{\rho}, \widehat{J_1})$ be a soft BCK-algebra over L_0 which is given in Example 3.4. Let $\widehat{J_2} = \{r_0, t_0, p_0\}$ be a subset of $\widehat{J_1}$ and let $G : \widehat{J_2} \to \mathscr{Y}(L_0)$ be a set-valued function defined by

$$\tilde{\sigma}(m_1) = \{m_2 \in L_0 \mid m_1 R m_2 \Leftrightarrow m_2 \in m_1^{-1}\hat{I}\}$$

for all $m_1 \in \widehat{J_2}$, where $\hat{I} = \{\theta, r_0\}$ and $m_1^{-1}\hat{I} = \{m_2 \in L_0 \mid m_1 \wedge m_2 \in \hat{I}\}$. Then $\tilde{\sigma}(r_0) = L_0$, $\tilde{\sigma}(t_0) = \{\theta, r_0, s_0, p_0\}$ and $\tilde{\sigma}(p_0) = \{\theta, r_0, s_0, t_0\}$ are BCK-subalgebras of $\tilde{\rho}(r_0)$, $\tilde{\rho}(t_0)$ and $\tilde{\rho}(p_0)$, respectively. Hence $(\tilde{\sigma}, \widehat{J_2})$ is a *S-SubA* of $(\tilde{\rho}, \widehat{J_1})$.

Theorem 3.8. *Let $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_1})$ be two soft CI-As over L_0.*

(i) *If $\tilde{\rho}(m_1) \subset \tilde{\sigma}(m_1)$ for all $m_1 \in \widehat{J_1}$, then $(\tilde{\rho}, \widehat{J_1}) \tilde{<} (\tilde{\sigma}, \widehat{J_1})$.*

(ii) *If $\widehat{J_2} = \{\theta\}$ and $(\tilde{h}, \widehat{J_2})$, $(\tilde{\rho}, L_0)$ are soft CI-As over L_0, then $(\tilde{h}, \widehat{J_2}) \tilde{<} (\tilde{\rho}, L_0)$.*

Proof. Straightforward. □

Theorem 3.9. *Let $(\tilde{\rho}, \widehat{J_1})$ be a soft CI-A over L_0 and let $(\tilde{\sigma}_1, \widehat{J_{21}})$ and $(\tilde{\sigma}_2, \widehat{J_{22}})$ be S-SubAs of $(\tilde{\rho}, \widehat{J_1})$. Then*

(i) $(\tilde{\sigma}_1, \widehat{J_{21}}) \tilde{\cap} (\tilde{\sigma}_2, \widehat{J_{22}}) \tilde{<} (\tilde{\rho}, \widehat{J_1})$.

(ii) $\widehat{J_{21}} \cap \widehat{J_{22}} = \emptyset \Rightarrow (\tilde{\sigma}_1, \widehat{J_{21}}) \tilde{\cup} (\tilde{\sigma}_2, \widehat{J_{22}}) \tilde{<} (\tilde{\rho}, \widehat{J_1})$.

Proof. (i) Using Definition 2.2, we can write

$$(\tilde{\sigma}_1, \widehat{J_{21}}) \tilde{\cap} (\tilde{\sigma}_2, \widehat{J_{22}}) = (\tilde{\sigma}, \widehat{J_2}),$$

where $\widehat{J_2} = \widehat{J_{21}} \cap \widehat{J_{22}}$ and $\tilde{\sigma}(m_1) = \tilde{\sigma}_1(m_1)$ or $\tilde{\sigma}_2(m_1)$ for all $m_1 \in \widehat{J_2}$. Obviously, $\widehat{J_2} \subset \widehat{J_1}$. Let $m_1 \in \widehat{J_2}$. Then $m_1 \in \widehat{J_{21}}$ and $m_1 \in \widehat{J_{22}}$. If $x \in \widehat{J_{21}}$, then $\tilde{\sigma}(m_1) = \tilde{\sigma}_1(m_1)$ is a *CI-SubA* of $\tilde{\rho}(x)$ since $(\tilde{\sigma}_1, \widehat{J_{21}}) \tilde{<} (\tilde{\rho}, \widehat{J_1})$. If $m_1 \in \widehat{J_{22}}$, then $\tilde{\sigma}(m_1) = \tilde{\sigma}_2(m_1)$ is a *CI-SubA* of $\tilde{\rho}(m_1)$ since $(\tilde{\sigma}_2, \widehat{J_{22}}) \tilde{<} (\tilde{\rho}, \widehat{J_1})$. Hence $(\tilde{\sigma}_1, \widehat{J_{21}}) \tilde{\cap} (\tilde{\sigma}_2, \widehat{J_{22}}) = (\tilde{\sigma}, \widehat{J_2}) \tilde{<} (\tilde{\rho}, \widehat{J_1})$.

(ii) Assume that $\widehat{J_{21}} \cap \widehat{J_{22}} = \emptyset$. We can write $(\tilde{\sigma}_1, B_1) \tilde{\cup} (\tilde{\sigma}_2, B_2) = (\tilde{\sigma}, \widehat{J_2})$ where $\widehat{J_2} = B_1 \cup \widehat{J_{22}}$ and

$$\tilde{\sigma}(m_1) = \begin{cases} \tilde{\sigma}_1(m_1) & \text{if } m_1 \in \widehat{J_{21}} \setminus \widehat{J_{22}}, \\ \tilde{\sigma}_2(m_1) & \text{if } m_1 \in \widehat{J_{22}} \setminus \widehat{J_{21}}, \\ \tilde{\sigma}_1(m_1) \cup \tilde{\sigma}_2(m_1) & \text{if } m_1 \in \widehat{J_{21}} \cap \widehat{J_{22}} \end{cases}$$

for all $m_1 \in \widehat{J_2}$. Since $(\tilde{\sigma}_i, \widehat{J_{2i}}))\tilde{<}(\tilde{\rho}, \widehat{J_1})$ for $i = 1, 2$, $\widehat{J_2} = \widehat{J_{21}} \cup \widehat{J_{22}} \subset \widehat{J_1}$ and $\tilde{\sigma}_i(m_1)$ is a *CI-SubA* of $\tilde{\rho}(m_1)$ for all $m_1 \in \widehat{J_{2i}}$, $i = 1, 2$. Since $\widehat{J_{21}} \cap \widehat{J_{22}} = \emptyset$, $\tilde{\sigma}(m_1)$ is a *CI-SubA* of $\tilde{\rho}(m_1)$ for all $m_1 \in B$. Therefore $(\tilde{\sigma}_1, \widehat{J_{21}}) \tilde{\cup} (\tilde{\sigma}_2, \widehat{J_{22}}) = (\tilde{\sigma}, \widehat{J_2})\tilde{<}(\tilde{\rho}, \widehat{J_1})$. □

Theorem 3.10. *Let $\bar{\eta} : L_0 \to Q_0$ be a homomorphism of CI-As and let $(\tilde{\rho}, \widehat{J_1})$ and $(\tilde{\sigma}, \widehat{J_2})$ be soft CI-As over L_0. Then*

$$(\tilde{\rho}, \widehat{J_1})\tilde{<}(\tilde{\sigma}, \widehat{J_2}) \Rightarrow (\bar{\eta}(\tilde{\rho}), \widehat{J_1})\tilde{<}(\bar{\eta}(\tilde{\sigma}), \widehat{J_2}).$$

Proof. Assume that $(\tilde{\rho}, \widehat{J_1})\tilde{<}(\tilde{\sigma}, \widehat{J_2})$. Let $m_1 \in \widehat{J_1}$. Then $\widehat{J_1} \subset \widehat{J_2}$ and $\tilde{\rho}(m_1)$ is a *CI-SubA* of $\tilde{\sigma}(m_1)$. Since $\bar{\eta}$ is a homomorphism, $\bar{\eta}(\tilde{\rho})(m_1) = \bar{\eta}(\tilde{\rho}(m_1))$ is a *CI-SubA* of $\bar{\eta}(\tilde{\sigma}(m_1)) = \bar{\eta}(\tilde{\sigma})(m_1)$, and therefore $(\bar{\eta}(\tilde{\rho}), \widehat{J_1})\tilde{<}(\bar{\eta}(\tilde{\sigma}), \widehat{J_2})$. □

4. CONCLUSION

This investigation successfully applies soft set theory to CI-algebraic subalgebras and offers new insights into their properties. We can investigate the intersection and union of subalgebras and the notion of a soft ideal formed by a soft set of a CI-algebra, with the help of soft set operations. Our findings create the framework for further research in this area and demonstrate the value of using soft set theory to analyse CI-algebraic subalgebras. The results of this work may have an impact on the development of new algorithms and techniques for handling ambiguous data in CI-algebras. Overall, this work extends our knowledge of soft set theory and its applications in CI-algebras and paves the way for future research in the field.

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