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TRIGONOMETRIC AND HYPERBOLIC POINCARÉ, SOBOLEV AND HILBERT-PACHPATTE TYPE INEQUALITIES

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ABSTRACT. In this article based on trigonometric and hyperbolic type Taylor formulae we establish Poincaré, Sobolev and Hilbert-Pachpatte type inequalities of different kinds specific and general.

1. MAIN RESULTS

Our motivation comes from the classic [2]. We start with a collection of Poincaré type inequalities.

Theorem 1.1. *Let $f \in C^2([a,b], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f(a) = f'(a) = 0$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|f\|_{L_q([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p \, dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \|f + f''\|_{L_q([a,b])} . \tag{1}$$

Proof. Since $f(a) = f'(a) = 0$, by Corollary 3.4 of [1] we have

$$f(x) = \int_a^x (f''(t) + f(t)) \sin(x-t) \, dt, \tag{2}$$

$\forall\, x \in [a,b]$.

It follows by Hölder's inequality that

$$|f(x)| \leq \int_a^x |f''(t) + f(t)| \, |\sin(x-t)| \, dt \leq$$
$$\left(\int_a^x |\sin(x-t)|^p \, dt \right)^{\frac{1}{p}} \left(\int_a^x |(f'' + f)(t)|^q \, dt \right)^{\frac{1}{q}} \leq \tag{3}$$
$$\left(\int_a^x |\sin(x-t)|^p \, dt \right)^{\frac{1}{p}} \|f'' + f\|_{L_q([a,b])} .$$

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Hence

$$|f(x)|^q \leq \left(\int_a^x |\sin(x-t)|^p dt\right)^{\frac{q}{p}} \|f''+f\|_{L_q([a,b])}^q, \tag{4}$$

and

$$\int_a^b |f(x)|^q dx \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p dt\right)^{\frac{q}{p}} dx\right) \|f''+f\|_{L_q([a,b])}^q. \tag{5}$$

We have proved that

$$\left(\int_a^b |f(x)|^q dx\right)^{\frac{1}{q}} \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \|f''+f\|_{L_q([a,b])}. \tag{6}$$

□

It follows

Theorem 1.2. *All as in Theorem 1.1. Then*

$$\|f\|_{L_q([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sinh(x-t)|^p dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \|f-f''\|_{L_q([a,b])}. \tag{7}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Corollary 3.5 of [1]. □

We continue with

Theorem 1.3. *Let $f \in C^4([a,b],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f(a)=f'(a)=f''(a)=f'''(a)=0$, and $p,q>1: \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{2}\left(\int_a^b \left(\int_a^x |\sinh(x-t)-\sin(x-t)|^p dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}} \|f-f''''\|_{L_q([a,b])}. \tag{8}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Corollary 3.6 of [1]. □

It follows

Theorem 1.4. *Let $f \in C^4([a,b],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f(a)=f'(a)=f''(a)=f'''(a)=0$, and $p,q>1: \frac{1}{p}+\frac{1}{q}=1$. Also let $\alpha,\beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{|\alpha||\beta||\beta^2-\alpha^2|}\left(\int_a^b \left(\int_a^x |\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))|^p dt\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}$$
$$\left\|f''''+\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right\|_{L_q([a,b])}. \tag{9}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Corollary 3.7 of [1]. □

Next comes

Theorem 1.5. *Let $f \in C^4([a,b],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f^{(i)}(a)=0$, $i=0,1,2,3$, and $p,q>1: \frac{1}{p}+\frac{1}{q}=1$; $\alpha\in\mathbb{R}$, $\alpha\neq 0$. Then*

$$\|f\|_{L_q([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3}\left(\int_a^b\left(\int_a^x|\sin(\alpha(x-t))-\alpha(x-t)\cos(\alpha(x-t))|^p dt\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}}$$
$$\left\|f^{(4)}+2\alpha^2 f''+\alpha^4 f\right\|_{L_q([a,b])}. \tag{10}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Corollary 3.8 of [1]. □

We continue with

Theorem 1.6. *All as in Theorem 1.4. Then*

$$\|f\|_{L_q([a,b])}\le$$
$$\frac{1}{|\alpha||\beta||\beta^2-\alpha^2|}\left(\int_a^b\left(\int_a^x|\alpha\sinh(\beta(x-t))-\beta\sinh(\alpha(x-t))|^p dt\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}}$$
$$\left\|f^{(4)}-\left(\alpha^2+\beta^2\right)f''+\alpha^2\beta^2 f\right\|_{L_q([a,b])}. \tag{11}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Corollary 3.9 of [1]. □

We also give

Theorem 1.7. *All as in Theorem 1.5. Then*

$$\|f\|_{L_q([a,b])}\le$$
$$\frac{1}{2|\alpha|^3}\left(\int_a^b\left(\int_a^x|\alpha(x-t)\cosh(\alpha(x-t))-\sinh(\alpha(x-t))|^p dt\right)^{\frac{q}{p}}dx\right)^{\frac{1}{q}}$$
$$\left\|f^{(4)}-2\alpha^2 f''+\alpha^4 f\right\|_{L_q([a,b])}. \tag{12}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Corollary 3.10 of [1]. □

We make

Remark. *The following come from* [1]. *Let K denote $\mathbb{R}$ or $\mathbb{C}$, $[a,b]\subset\mathbb{R}$. For $c=(c_0,...,c_n)\in K^{n+1}$ with $c_n=1$, let the n-th order linear differential operator*

$$D_c: C^n([a,b],K)\to C([a,b],K)$$

be defined by the formula

$$D_c(f):=c_n f^{(n)}+...+c_1 f'+c_0 f \quad (f\in C^n([a,b],K)).$$

Let $\omega_c\in C^n(\mathbb{R},\mathbb{C})$ denote the unique solution of the initial value problem

$$D_c(\omega_c)=0,\ \omega_c^{(l)}(0)=\delta_{l,n-1} \quad (l\in\{0,...,n-1\}). \tag{13}$$

The function ω_c will be called the characteristic solution of $D_c(\omega)=0$.

Define

$$(T_{a,c}f)(x):=\sum_{j=0}^{n-1}\left(f^{(j)}(a)\sum_{i=0}^{n-1-j}c_{i+j+1}\omega_c^{(i)}(x-a)\right), \tag{14}$$

if $f^{(j)}(a)=0$, $j=0,1,...,n-1$, then

$$(T_{a,c}f)(x)=0,\ \forall\, x\in[a,b].$$

Theorem 1.8. *([1]) Let $n \in \mathbb{N}$, $c = (c_0, ..., c_n) \in K^{n+1}$ with $c_n = 1$. Then for all $f \in C^n([a,b], K)$, $x \in [a,b]$, we have*

$$f(x) = (T_{a,c} f)(x) + \int_a^x D_c(f)(t)\,\omega_c(x-t)\,dt. \tag{15}$$

If $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, then

$$f(x) = \int_a^x D_c(f)(t)\,\omega_c(x-t)\,dt, \tag{16}$$

$\forall\, x \in [a,b]$.

We give the following general Poincaré type inequality.

Theorem 1.9. *All as in Remark 1 and $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\|f\|_{L_q([a,b])} \leq \left(\int_a^b \left(\int_a^x |\omega_c(x-t)|^p\,dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \|D_c(f)\|_{L_q([a,b])}. \tag{17}$$

Proof. As similar to Theorem 1.1 is omitted. It is based on Remark 1 and Theorem 1.8. □

Next follow Sobolev type inequalities.

Theorem 1.10. *All as in Theorem 1.1, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p\,dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \|f'' + f\|_{L_q([a,b])}. \tag{18}$$

Proof. As in (3) we have

$$|f(x)| \leq \left(\int_a^x |\sin(x-t)|^p\,dt \right)^{\frac{1}{p}} \|f'' + f\|_{L_q([a,b])}. \tag{19}$$

and (by $r > 0$)

$$|f(x)|^r \leq \left(\int_a^x |\sin(x-t)|^p\,dt \right)^{\frac{r}{p}} \|f'' + f\|_{L_q([a,b])}^r, \tag{20}$$

$\forall\, x \in [a,b]$.

Thus

$$\int_a^b |f(x)|^r\,dx \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p\,dt \right)^{\frac{r}{p}} dx \right) \|f'' + f\|_{L_q([a,b])}^r, \tag{21}$$

and

$$\left(\int_a^b |f(x)|^r\,dx \right)^{\frac{1}{r}} \leq \left(\int_a^b \left(\int_a^x |\sin(x-t)|^p\,dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \|f'' + f\|_{L_q([a,b])}, \tag{22}$$

proving the claim. □

Theorem 1.11. *All as in Theorem 1.2, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq \left(\int_a^b \left(\int_a^x |\sinh(x-t)|^p\,dt \right)^{\frac{r}{p}} dx \right)^{\frac{1}{r}} \|f - f''\|_{L_q([a,b])}. \tag{23}$$

Proof. As similar to Theorem 1.10 is omitted. □

Theorem 1.12. *All as in Theorem 1.3, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{2}\left(\int_a^b \left(\int_a^x |\sinh(x-t) - \sin(x-t)|^p\, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}} \|f - f''''\|_{L_q([a,b])}. \tag{24}$$

Proof. As similar to Theorem 1.10 is omitted. □

Theorem 1.13. *All as in Theorem 1.4, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{|\alpha||\beta||\beta^2-\alpha^2|}\left(\int_a^b \left(\int_a^x |\beta\sin(\alpha(x-t)) - \alpha\sin(\beta(x-t))|^p\, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f\right\|_{L_q([a,b])}. \tag{25}$$

Proof. As similar to Theorem 1.10 is omitted. □

Theorem 1.14. *All as in Theorem 1.5, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3}\left(\int_a^b \left(\int_a^x |\sin(\alpha(x-t)) - \alpha(x-t)\cos(\alpha(x-t))|^p\, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f^{(4)} + 2\alpha^2 f'' + \alpha^4 f\right\|_{L_q([a,b])}. \tag{26}$$

Proof. As similar to Theorem 1.10 is omitted. □

Theorem 1.15. *All as in Theorem 1.6, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{|\alpha||\beta||\beta^2-\alpha^2|}\left(\int_a^b \left(\int_a^x |\alpha\sinh(\beta(x-t)) - \beta\sinh(\alpha(x-t))|^p\, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f^{(4)} - \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f\right\|_{L_q([a,b])}. \tag{27}$$

Proof. As similar to Theorem 1.10 is omitted. □

Theorem 1.16. *All as in Theorem 1.7, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq$$

$$\frac{1}{2|\alpha|^3}\left(\int_a^b \left(\int_a^x |\alpha(x-t)\cosh(\alpha(x-t)) - \sinh(\alpha(x-t))|^p\, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}}$$
$$\left\|f^{(4)} - 2\alpha^2 f'' + \alpha^4 f\right\|_{L_q([a,b])}. \tag{28}$$

Proof. It is omitted. □

Theorem 1.17. *All as in Theorem 1.9, $r > 0$. Then*

$$\|f\|_{L_r([a,b])} \leq \left(\int_a^b \left(\int_a^x |\omega_c(x-t)|^p\, dt\right)^{\frac{r}{p}} dx\right)^{\frac{1}{r}} \|D_c(f)\|_{L_q([a,b])}. \tag{29}$$

Proof. It is omitted. □

We continue with a collection of Hilbert-Pachaptted inequalities.

Theorem 1.18. *Here $j = 1, 2$. Let $f_j \in C^2([a_j, b_j], K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, such that $f_j(a_j) = f_j'(a_j) = 0$, and $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{\int_{a_1}^{x_1} |\sin(x_1 - t_1)|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sin(x_2 - t_2)|^q dt_2}{q} \right]} \leq$$

$$(b_1 - a_1)(b_2 - a_2) \|f_1'' + f_1\|_{L_q([a_1,b_1])} \|f_2'' + f_2\|_{L_p([a_2,b_2])} . \tag{30}$$

Proof. As in (3) we have

$$|f_1(x_1)| \leq \left(\int_{a_1}^{x_1} |\sin(x_1 - t_1)|^p \, dt_1 \right)^{\frac{1}{p}} \|f_1'' + f_1\|_{L_q([a_1,b_1])} , \tag{31}$$

$\forall \, x_1 \in [a_1, b_1]$,

and

$$|f_2(x_2)| \leq \left(\int_{a_2}^{x_2} |\sin(x_2 - t_2)|^q \, dt_2 \right)^{\frac{1}{q}} \|f_2'' + f_2\|_{L_p([a_2,b_2])} . \tag{32}$$

$\forall \, x_2 \in [a_2, b_2]$.

Hence we have

$$|f_1(x_1)| |f_2(x_2)| \leq \left(\int_{a_1}^{x_1} |\sin(x_1 - t_1)|^p \, dt_1 \right)^{\frac{1}{p}} \left(\int_{a_2}^{x_2} |\sin(x_2 - t_2)|^q \, dt_2 \right)^{\frac{1}{q}}$$

$$\|f_1'' + f_1\|_{L_q([a_1,b_1])} \|f_2'' + f_2\|_{L_p([a_2,b_2])} \leq \tag{33}$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$)

$$\left[\frac{\int_{a_1}^{x_1} |\sin(x_1 - t_1)|^p \, dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sin(x_2 - t_2)|^q \, dt_2}{q} \right]$$

$$\|f_1'' + f_1\|_{L_q([a_1,b_1])} \|f_2'' + f_2\|_{L_p([a_2,b_2])} . \tag{34}$$

So far we have

$$\frac{|f_1(x_1)| |f_2(x_2)|}{\left[\frac{\int_{a_1}^{x_1} |\sin(x_1 - t_1)|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sin(x_2 - t_2)|^q dt_2}{q} \right]} \leq$$

$$\|f_1'' + f_1\|_{L_q([a_1,b_1])} \|f_2'' + f_2\|_{L_p([a_2,b_2])} , \tag{35}$$

$\forall \, (x_1, x_2) \in [a_1, b_1] \times [a_2, b_2]$.

The denominator in (35) can be zero only when both $x_1 = a_1$ and $x_2 = a_2$.

Therefore we obtain (30), by integrating (35) over $[a_1, b_1] \times [a_2, b_2]$. □

Theorem 1.19. *All as in Theorem 1.18. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(x_1)| |f_2(x_2)| \, dx_1 dx_2}{\left[\frac{\int_{a_1}^{x_1} |\sinh(x_1 - t_1)|^p dt_1}{p} + \frac{\int_{a_2}^{x_2} |\sinh(x_2 - t_2)|^q dt_2}{q} \right]} \leq$$

$$(b_1 - a_1)(b_2 - a_2) \|f_1 - f_1''\|_{L_q([a_1,b_1])} \|f_2 - f_2''\|_{L_p([a_2,b_2])} . \tag{36}$$

Proof. As similar to Theorem 1.18 is omitted. □

Theorem 1.20. *Here $j=1,2$. Let $f_j \in C^4([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f_j^{(i)}(a_j)=0$, $i=0,1,2,3$, and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\sinh(x_1-t_1)-\sin(x_1-t_1)|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\sinh(x_2-t_2)-\sin(x_2-t_2)|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{4}\left\|f_1-f_1^{(4)}\right\|_{L_q([a_1,b_1])}\left\|f_2-f_2^{(4)}\right\|_{L_p([a_2,b_2])}. \tag{37}$$

Proof. As similar to Theorem 1.18 is omitted. □

Theorem 1.21. *Here $j=1,2$. Let $f_j \in C^4([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f_j^{(i)}(a_j)=0$, $i=0,1,2,3$, and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Alsi let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\beta\sin(\alpha(x_1-t_1))-\alpha\sin(\beta(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\beta\sin(\alpha(x_2-t_2))-\alpha\sin(\beta(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\alpha^2\beta^2\left(\beta^2-\alpha^2\right)^2}\left\|f_1^{(4)}+\left(\alpha^2+\beta^2\right)f_1''+\alpha^2\beta^2f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}+\left(\alpha^2+\beta^2\right)f_2''+\alpha^2\beta^2f_2\right\|_{L_p([a_2,b_2])}. \tag{38}$$

Proof. It is omitted. □

Theorem 1.22. *Here $j=1,2$. Let $f_j \in C^4([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, such that $f_j^{(i)}(a_j)=0$, $i=0,1,2,3$, and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $\alpha\in\mathbb{R}$, $\alpha\neq 0$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\sin(\alpha(x_1-t_1))-\alpha(x_1-t_1)\cos(\alpha(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\sin(\alpha(x_2-t_2))-\alpha(x_2-t_2)\cos(\alpha(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{4\alpha^6}\left\|f_1^{(4)}+2\alpha^2f_1''+\alpha^4f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}+2\alpha^2f_2''+\alpha^4f_2\right\|_{L_p([a_2,b_2])}. \tag{39}$$

Proof. It is omitted. □

Theorem 1.23. *All as in Theorem 1.21. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\alpha\sinh(\beta(x_1-t_1))-\beta\sinh(\alpha(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\alpha\sinh(\beta(x_2-t_2))-\beta\sinh(\alpha(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\alpha^2\beta^2\left(\beta^2-\alpha^2\right)^2}\left\|f_1^{(4)}-\left(\alpha^2+\beta^2\right)f_1''+\alpha^2\beta^2f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}-\left(\alpha^2+\beta^2\right)f_2''+\alpha^2\beta^2f_2\right\|_{L_p([a_2,b_2])}. \tag{40}$$

Proof. It is omitted. □

Theorem 1.24. *All as in Theorem 1.22. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\alpha(x_1-t_1)\cosh(\alpha(x_1-t_1))-\sinh(\alpha(x_1-t_1))|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\alpha(x_2-t_2)\cosh(\alpha(x_2-t_2))-\sinh(\alpha(x_2-t_2))|^q dt_2}{q}\right]}\leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{4\alpha^6}\left\|f_1^{(4)}-2\alpha^2 f_1''+\alpha^4 f_1\right\|_{L_q([a_1,b_1])}$$
$$\left\|f_2^{(4)}-2\alpha^2 f_2''+\alpha^4 f_2\right\|_{L_p([a_2,b_2])}. \tag{41}$$

Proof. It is omitted. □

We finish with

Theorem 1.25. *Let $j=1,2$. Here $f_j\in C^4([a_j,b_j],K)$, where $K=\mathbb{R}$ or $\mathbb{C}$, and $f_j^{(i)}(a_j)=0$, $i=0,1,...,n-1$. All the rest are as in Remark 1 and $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(x_1)|\,|f_2(x_2)|\,dx_1dx_2}{\left[\frac{\int_{a_1}^{x_1}|\omega_c(x_1-t_1)|^p dt_1}{p}+\frac{\int_{a_2}^{x_2}|\omega_c(x_2-t_2)|^q dt_2}{q}\right]}\leq$$

$$(b_1-a_1)(b_2-a_2)\|D_c(f_1)\|_{L_q([a_1,b_1])}\|D_c(f_2)\|_{L_p([a_2,b_2])}. \tag{42}$$

Proof. It is omitted. □

Conclusion: We presented a set of very important analytical inequalities generated by some new trigonometric and hyperbolic Taylor formulae. That is a new important way in establishing inequalities.

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