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PROJECTIVE PLANE CURVES OVER A FINITE FIELD WITH CONICAL COMPONENTS

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ABSTRACT. We present the list of maximal projective plane curves containing conics and those which are arrangements of conics. The number of rational points and the corresponding polynomials are given. The third highest number of points of projective curves of degree d over a finite field $\mathbb{F}_q$ ($d < [q/3]$) is associated only to some linear curves. We show that for $q/2 + 2 < d < q$, this is no longer the case: the third highest number of points can also be obtained by some curves containing a conic. Throughout this work, we obtain some bounds concerning the number of $\mathbb{F}_q$-points of curves with linear, conic and cubic factors. these bounds apply (not sharply) to irreducible curves.

1. INTRODUCTION

The determination of some numbers of points in certain curves of degree d over a finite field $\mathbb{F}_q$ gives results on coding theory, such as for the weight distribution of the d-th order projective Reed-Muller codes, and for decoding with affine variety codes. The consideration of the first few weights gives rise to the distribution of the highest numbers of points of plane curves.

The second and the third highest numbers of points are computed in the case of hypersurfaces associated to homogeneous polynomials of degree d on a projective space with coefficients in $\mathbb{F}_q$. In [4, 5], it is shown that for $d \leq q/3+2$, the three first highest numbers of points are given by some hyperplane arrangements.

Consequently, the three first highest numbers of points of projective curves of degree d over a finite field $\mathbb{F}_q$, are given only by some linear curves when $d \leq q/3 + 2$.

In this paper, we will show that for $q/2 + 2 < d < q$, the third highest number can also be obtained by some curves which are a union of $d - 2$ lines and one conic (irreducible quadric in $PG(2, q)$). Finally, examples of corresponding polynomials are provided that illustrate the existence of curves containing only one conic, curves containing conics and only one line, and those which are completely arrangement of conics.

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2. CURVES SPLITTING TO LINEAR FACTORS (LINES)

Let us denote by $\mathbb{F}_q$ or $GF(q)$ a finite fields of q elements, and $\mathbb{P}^2(\mathbb{F}_q)$ or $PG(2,q)$ a projective plane over $\mathbb{F}_q$. By Π_2 we denote the number of points of $\mathbb{P}^2(\mathbb{F}_q)$, that is $\Pi_2 = \frac{q^3-1}{q-1} = q^2+q+1$.
Let d an integer ($\leq q$), an arrangement of d lines in $\mathbb{P}^2(\mathbb{F}_q)$ is a set of lines. We describe three types of arrangements of d lines in the projective plane $\mathbb{P}^2(\mathbb{F}_q)$, denoted by $\mathcal{A}_i^d$, $1 \leq i \leq 3$.

(1) Type $\mathcal{A}_1^d$: all the lines meet in a common point.
(2) Type $\mathcal{A}_2^d$: $(d-1)$ lines meet in a common point K and the d-th line meet the other lines in $(d-1)$ distinct points.
(3) Type $\mathcal{A}_3^d$: $(d-2)$ lines $L_1, ..., L_{d-2}$ meet in a common point K_1, the last two lines L_{d-1} and L_d meet in a point K_2 distinct from K_1, such that K_2 is contained in one L_i, for $1 \leq i \leq d-2$.

The number of points in the union of lines of an arrangement of type $\mathcal{A}_i^d$, $1 \leq i \leq 3$ was computed by Sboui in the general case of hyperplane arrangements [5].

Theorem 2.1. *The number of points N_i^ℓ in an $\mathcal{A}_i^d$-arrangement, $1 \leq i \leq 3$, with $2 < d \leq q$ and $n > 3$, is such that:*

$$\begin{aligned} N_1^\ell &= dq+1, \\ N_2^\ell &= N_1^\ell - (d-2), \\ N_3^\ell &= N_1^\ell - 2(d-3). \end{aligned}$$

Let us call N_1, N_2 and N_3 respectively the three maximal numbers of rational points of curves of degree d in $\mathbb{P}^2 = PG(2,q)$. The maximal number is given by Serre [7].

Theorem 2.2. *The maximal number of rational points of curves of degree d in $\mathbb{P}^2(\mathbb{F}_q)$ is $N_1 = N_1^\ell$.*

Sboui showed in the same paper [5] that some other weights of Reed Muller codes were associated to these line arrangements.

Theorem 2.3. *Let C be a curve, not union of linear factors, with $d > 4$.*

- *If $q \geq 2(d-1)$, then $\#C < N_2^\ell$, and one has $N_2 = N_2^\ell$.*
- *If $q \geq 3(d-2)$, then $\#C < N_3^\ell$, and one has $N_3 = N_3^\ell$.*

3. CURVES REACHING THE THIRD WEIGHT

3.1. **Configuration of Conics and tangents.** Let $\mathscr{C}$ be a conic (irreducible quadric in $\mathbb{P}^2(\mathbb{F}_q)$). Let us call $Ext(\mathscr{C})$ the union of all the tangent lines minus $\mathscr{C}$ and $Int(\mathscr{C})$ the complementary set of $Ext(\mathscr{C}) \cup \mathscr{C}$.

Lemma 3.1. *Let $\mathscr{C}$ be a conic in $\mathbb{P}^2(\mathbb{F}_q)$ and let M be a point.*

(1) *If q is odd and $M \in Ext(\mathscr{C})$, the number of tangent lines (unisecants) through M is 2, the number of lines containing M and external to $\mathscr{C}$ (0-secants) is $\frac{q-1}{2}$, which is equal to the number of bisecants to $\mathscr{C}$.*
(2) *If q is odd and $M \in Int(\mathscr{C})$, there is no tangent line to $\mathscr{C}$ through M. The number of lines containing M and externals to $\mathscr{C}$ is $\frac{q+1}{2}$, which is equal to the number of bisecants to $\mathscr{C}$.*

(3) *If q is even, M is not on $\mathscr{C}$ and is not the kernel of $\mathscr{C}$ then there is only one tangent line to $\mathscr{C}$ through M. The number of lines containing M and external to $\mathscr{C}$ is $\frac{q}{2}$, which is equal to the number of bisecants to $\mathscr{C}$.*

Proof. One considers the map from $\mathscr{C}$ to the set of lines going through M, which to $A \in \mathscr{C}$ associates the line through A and M. If $M \in Ext(\mathscr{C})$, there are two tangent lines and the other lines through A and M are secant to $\mathscr{C}$ in two points. So, their number is $\frac{q-1}{2}$. The proof of the other parts of the lemma are the same. □

Now, we can give the geometric description of a curve whose number of rational points is N_3^ℓ.

Theorem 3.2. *Let d be an integer such that $\frac{q+1}{2} + 2 \leq d < q$ and let C be a projective plane curve of degree d on $\mathbb{F}_q$, being the union of $d-2$ converging lines Δ_i to the same point M and a conic $\mathscr{C}$*

$$C = \cup_{i=1}^{d-2} \Delta_i \cup \mathscr{C}.$$

If we are in one of the following situation

(1) For q even, the point M does not belong to $\mathscr{C}$ and is distinct from the kernel of $\mathscr{C}$, among the $d-2$ lines containing M, $\frac{q}{2}$ do not intersect $\mathscr{C}$ and there is a tangent line to $\mathscr{C}$;
(2) For q odd, $M \in Int(\mathscr{C})$, and among the $d-2$ lines containing M, $\frac{q+1}{2}$ do not intersect $\mathscr{C}$;
(3) For q odd, $M \in Ext(\mathscr{C})$, and among the $d-2$ lines containing M, $\frac{q-1}{2}$ do not intersect $\mathscr{C}$ and two are tangent;

then one has $\#C = N_3^\ell$.

Proof. In the first case there are $\frac{q}{2}$ lines containing M and not intersecting $\mathscr{C}$ (it is the maximum possible), and one line containing M tangent to $\mathscr{C}$. The $d-2-(1+\frac{q}{2})$, other lines intersect $\mathscr{C}$ in two points. The number of rational points of C is

$$\begin{aligned} \#C &= \#\bigcup_{i=1}^{d-2} \Delta_i + \#\mathscr{C} - \#\left(\mathscr{C} \cap \bigcup_{i=1}^{d-2} \Delta_i\right) \qquad (3.1) \\ &= (d-2)q + 1 + q + 1 - (1 + 2(d-2-(1+q/2))) \\ &= N_3^\ell \end{aligned}$$

If $M \in Int(\mathscr{C})$, or $M \in Ext(\mathscr{C})$, we prove the result by the same way and one finds all the same $\#C = N_3^\ell$. □

4. Examples of highest curves of degree d arrangement of conics and lines

4.1. **Curves composed of $d-2$ lines and one conic reaching N_3^ℓ.** It is known that projective conics are classified in a single orbit under the action of the projective linear group. The equation $z^2 - xy = 0$ can be used as a single representation. Furthermore, the set of tangents to a conic in $\mathbb{P}^2(\mathbb{F}_q)$ (q odd) are $q+1$ lines in general position, they form a $(q+1)$-arc (no three of them are concurrent), which is a conic in the dual space of $\mathbb{P}^2(\mathbb{F}_q)$ (duality lines-points). Numerous tools can be found in [3, 6], along with some arithmetic and geometric details on conics and its tangents are discussed.
Consider the conic $\mathscr{C}$ defined as the projective variety in $\mathbb{P}^2(\mathbb{F}_q)$ of the quadratic form $z^2 - xy$, the projective points of $\mathscr{C}$ can be represented by the homogeneous coordinates: $\mathscr{C} = \{(1 : t^2 : t), t \in \mathbb{F}_q\} \cup \{M_\infty(0 : 1 : 0)\}$. A tangent δ_t to $\mathscr{C}$ at a point $M_t(1 : t^2 : t)$

has equation $\delta_t : t^2x + y - 2tz = 0$, $t \in \mathbb{F}_q$, and for the point at infinity $M_\infty(0 : 1 : 0)$ the tangent is $\delta_\infty : x = 0$. Hence, the set of the $q+1$ tangents to this conic can be written as $\{\delta_t : tx + t^{-1}y - 2z = 0,\ t \in \mathbb{F}_q^*\} \cup \{\delta_0 : y = 0\} \cup \{\delta_\infty : x = 0\}$.

Consider the pencil of $d-4$ lines $L_i : x - \lambda_i y = 0$, $1 \leq i \leq d-4$ these $d-4$ lines L_i with δ_0 and δ_∞ meet all in a common point $\delta_0 \cap \delta_\infty \cap_{i=1}^{d-4} L_i = \{w(0 : 0 : 1)\}$. For the position of L_i respect to $\mathscr{C}$, λ_i are chosen such that the lines L_i are external lines (0-secants) to $\mathscr{C}$. For that, we choose the cofficients λ_i a non squares in $\mathbb{F}_q$, which is possible because $\frac{q+1}{2} + 2 \leq d < q$ and there are $\frac{q-1}{2}$ nonsquares in $\mathbb{F}_q$.
The zero locus of the polynomial

$$P(x, y, z) = xy(z^2 - xy)\prod_{i=1}^{d-4}(x - \lambda_i y)$$

is the curve $C = \delta_0 \cup \delta_\infty \cup_{i=1}^{d-4} L_i \cup \mathscr{C}$. This curve satisfies the assumptions of situation (3) of theorem 3.2 $\{$the integer q is odd, $w \in Ext(\mathscr{C})$, and among the $d-2$ lines containing w, $d - 4 = \frac{q-1}{2}$ do not intersect $\mathscr{C}$ and two are tangents to $\mathscr{C}\}$. This curve reach the third highest number of rational points $\#C = N_3^\ell$.
By using the same method of building the curve factors, we may obtain polynomials corresponding to curves meeting the conditions of situation (1) and (2) of theorem 3.2.

4.2. **Curves composed of $\frac{d-1}{2}$ conics and one line.** For an odd prime power $q \equiv 1(mod4)$, select in $\mathbb{F}_q$ a fixed nonsquare α. The construction relies on the properties of a particular quadratic pencil. Let $(x : y : z)$ be homogeneous coordinates for $\mathbb{P}^2(\mathbb{F}_q)$, and define $V(P)$ to be the set of zeroes of a polynomial P in these three variables. Consider the quadratic pencil given by

$$\mathcal{P} = \{C_\beta : \beta \in \mathbb{F}_q \cup \infty\},$$

where $C_\beta = V(\beta x^2 + y^2 - \alpha z^2)$ for any $\beta \in \mathbb{F}_q$, and $C_\infty = V(x^2)$. It is not difficult to see that the elements of this pencil are pairwise disjoint, and that all of the pencil elements are conics except C_∞ (the line $[1, 0, 0]$) and C_0 (the point $(1, 0, 0)$). In our construction, we will take some components: the line C_∞ referring to it as l_∞ and $\frac{d-1}{2}$ conics (d odd). The zeros locus of the polynomial

$$P(x, y, z) = x\prod_{i=1}^{\frac{d-1}{2}}(\beta_i x^2 + y^2 - \alpha z^2)$$

is the curve $C = l_\infty \cup_{i=1}^{\frac{d-1}{2}} C_{\beta_i}$, It counts the most points for curves arrangements of $\frac{(d-1)}{2}$ conics.

$$\#C = \frac{(q+1)(d+1)}{2}.$$

For another alternative pencil of conics, we cane take the set:

$$C_\beta = V(x^2 - \alpha_i dy^2 + \alpha_i z^2),$$

where $\alpha_i \in \mathbb{F}_q^*$, $1 \leq i \leq \frac{d-1}{2}$ and d a nonsquare in $\mathbb{F}_q$.

5. Conclusions / Discussions

Several interesting problems remain to be addressed in this area. Constructing curves with more higher degree irreducible factors such as cubics will produce more outcomes on decoding process of affine variety codes. Moreover, the results on rational points of

curves, surfaces and hypersurfaces will award more results on the weight distribution of linear codes.
The characterization of projective curves with many points containing conics and cubics is insolved. Some interesting progress in this subject are given in [2].It would be interesting to find additional detailed results along this axis.
Following this work, a natural scheme is to focus on the axis of the highest curves containing linear factors and an irreducible cubic curve.
In order to examine the intersection locus between linear factors and irreducible cubics, we can start by improving the bounds given in this lemma:

Lemma 5.1. *Let $\mathcal{C}$ be an irreducible cubic in $\mathbb{P}^2(\mathbb{F}_q)$ containing λ rational points and let M be a point. If M does not belong to $\mathcal{C}$, the number of lines containing M and not intersecting $\mathcal{C}$ is lower than $q+1-\frac{\lambda}{3}$. If M belongs to $\mathcal{C}$ the number of lines containing M and not intersecting $\mathcal{C}$ in another point is lower than $q+1-\frac{\lambda}{2}$.*

Proof. If M does not belong to $\mathcal{C}$ the cardinal of the set of lines containing M and intersecting $\mathcal{C}$ is greater than $\frac{\lambda}{3}$ as a line intersect an irreducible cubic in three points at most. For the same reason, if M belongs to $\mathcal{C}$ the cardinal of the set of lines containing M and intersecting $\mathcal{C}$ in at least one point is greater than $\frac{\lambda}{2}$. $\square$

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