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ALGEBRAICNESS PROPERTIES ON TRANSITIVE BINARY RELATIONAL SETS AND A CHARACTERIZATION OF CONTINUOUS DIRECTED COMPLETE POSETS

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ABSTRACT. In this work we introduce and study the concepts of algebraicness, and continuity on transitive binary relational sets (so called TRS). Some interactions between these concepts are investigated. Further more a characterization of continuous directed complete posets and Algebraic TRS are studied. Our results are generalizations of corresponding results in posets.

1. INTRODUCTION

It is interset to mention that in 1994 [1], S. Abramsky and A. Jung considered the concepts of continuous directed complete posets (continuous domain) and algebraic domains. R. Hekmann considered and studies these concepts in detail in this paper [5]. Continuous posets were introduced and studied independently by R. E. Hoffmann [2,6,7,8], J. D. Lawson [11,12] and in more fashion by G. Markowsky [14] and M. Erne [3]. It is worth to mention that J. Nino-Salcedo, considered and studied in moredetails the concepts of continuous posets and algebraic posets in this paper [15]. In fact the concept of continuous poset (resp., algebraic poset) in the sense of J. Nino-Salcedo [15] and the concep of continuous. In [18], H. Zhang studied a type of continuous poset which is a generalization of continuous poset in the sense of J. Nino-Salcedo [15]. In [5], the concepts of bounded complete posets, bounded complete domains, finitely complete posets, finitely complete domains are studied. It worth to mention that H. Zhang [18] studied some interactions between bounded complete domains and scott-topology and lawson topology. Our aims here is devoted to introduce and study the continuity and algebraicness properties of TRS. Our results extended the corresponding results in posets and domains [5,9,15,18]. In section 2 we explain the concepts of a continuous TRS, the relations between lower (resp. upper) closure in X of A and when x below (resp. y is way above) y (resp. x) directed subset of X.Also, prove that a if A TRS $(X, \leq)$ is continuous poset in the sence of H. Zhang (resp., continuous poset in the sence of J. Nino-Salcedo, continuous domain in the sence of R. Hekmann), then the way relation below ' $\ll'$is continuous information system, Finally take about when the way relation below ' $\ll'$ is interpolative and a subset B of a poset $(X, \leq)$ is called a base for X.In this section 3 we prove if TRS equivalent domain, i.e., a poset in which

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every directed subset has a supremum, then a continuous domain in the sence of in the sence of R. Hekmann and continuous TRS are identical.

Definition 1.1. Let $A \subseteq X$. Then:

(1) A subset A of the domain [5] (resp. Poset) X is called directed closed (d-closed for short) iff $\forall$ directed subset D of A, $\bigvee(D) \in A$;

(2) A subset A of the Poset X is called Scott-closed iff A is d-closed lower subset of X [15] ;

(3) A is called d-(resp. Scott-) open iff A^c d-(resp. Scott-) closed [5,15];

(4) Let $x, y \in X$. We say x below (resp. y is way above) y (resp. x), denoted by $x \ll y$ iff $\forall\, D \subseteq X,\ s.t.\ D$ is directed subset of X with $\bigvee D$ exists and $y \leq \bigvee D$, $\exists\, d \in D \quad s.t.\ x \leq d$. The family of all elements in X each of which way above (resp. way below) x is denoted and defined as follows: $\Uparrow x = \{y \in X : x \ll y\}$ (resp. $\Downarrow x = \{y \in X : y \ll x\}$ [15];

(5) Let $x \in X$. If $x \ll x$, then x is said to be isolated. The family of all isolated points above (resp. below) $x \in X$ is denoted and defined by: $\uparrow^{\circ} x = \{y \in X : y \ll y$ and $x \leq y\}$ (resp. $\downarrow_{\circ} x = \{y \in X : y \ll y$ and $y \leq x\}$ [15].

Proposition 1.1 (Proposition 6.4.7 [5], Lemma 2.15(i)[16]) Let X be a continuous domain, then for every two points $x, z \in X$ with $x \ll z$ there is a point $y \in X$ s.t., $x \ll y \ll z$.

Proposition 1.2 (Proposition 1.17 [18]). Let X be a continuous poset in the sence of Zhang, then for every two points $x, z \in X$ with $x \ll z$ there is a point $y \in X$ s.t., $x \ll y \ll z$.

Proposition 1.3 (Proposition 6.7.2 [5] and Lemma 2.14[15]) If $(X, \leq)$ is continuous domain in the sence of R. Hekmann iff X is a continuous poset in the sence of J. Nino-Salcedo.

Proposition 1.4 (Proposition 6.2.2 [5]) A domain X is an algebraic in the sence of R. Hekmann iff it is algebraic poset in the sence of J. Nino-Salcedo.

Theorem 1.1 (Proposition 6.7.3 [5]). If X is an algebraic domain then, it is a continuous domain.

Definition 1.2. Let $A \subseteq X$. Then:

(1) A is called directed subset of X iff $A \neq \phi$ and $\forall\, x, y \in A$, $\exists\, z \in A \quad s.t.\ x \leq z$ and $y \leq z$ [5];

(2) The lower (resp. upper) closure in X of A is denoted by $\downarrow A$ (resp. $\uparrow A$) and defined as follows: $\downarrow A = \{x \in X : \exists\ y \in A\ \ s.t.\ x \leq y\ \}$(resp. $\uparrow A = \{x \in X : \exists\ y \in A\ \ s.t.\ y \leq x\ \}$)[5]

(3) The convex hull A is denoted by $\updownarrow A$ and defined as follows: $\updownarrow A = \downarrow A \cap \uparrow A$ [5];

(4) Let $A, B \subseteq X$. B is called cofinal in A iff $B \subseteq A \subseteq \downarrow(B)$[5].

Definition 1.3. Let ' $\leq'$ be a binary relation set on $X \neq \phi$. Then;

(1) ' $\leq'$ is called reflexive iff $\forall\, x \in X,\ x \leq x$ [13];

(2) ' $\leq'$ is called antisymetric iff $\forall\, x, y \in X,\ x \leq y$ and $y \leq x \Rightarrow x = y$ [13];

(3) ' $\leq'$ is called transitive iff $\forall\, x, y, z \in X,\ x \leq y$ and $y \leq z \Rightarrow x = z$[13];

(4) ' $\leq'$ is called symetric iff $\forall\, x, y \in X,\ x \leq y \Rightarrow y \leq x$ [13];

(5) ' $\leq'$ is called interpolative iff $\forall\, x, z \in X$, with $x \leq z$, $\exists\ y \in X\ \ s.t.\ x \leq y \leq z$ $[5, 16]$.

(6) if ' $\leq'$satisfies the conditions (1), (2) and (3), then $(X, \leq)$ is called Partialy order set (Poset) [13];

(7) if ' $\leq'$satisfies the conditions (1), and (3), then $(X, \leq)$ is called pre-orderd set (Quasi set)[13];

(8) if ' $\leq'$satisfies the conditions (1), (2), (3) and (4), then $(X, \leq)$ is called an equvalence set,

(9) if ' $\leq'$satisfies the conditions (3) and (5), then $(X, \leq)$ is a continuous information system [10,16].

(10) if ' $\leq'$satisfies the conditions (3), and $\forall\ x \in X$, and for every finite subset A of X the following axiom holds: if $\forall\, y \in A,\ y \leq x$ then $\exists\ z \in X\ \ s.t.\ \forall\, y \in A,\ y \leq z$ and $z \leq x$, then $(X, \leq)$ is abstract basis [17].

2. **Continuous TRS**

Here in this section explain the concepts of a continuous TRS, the relations between lower (resp. upper) closure in X of A and when x below (resp. y is way above) y (resp. x), directed subset of X.Also, prove that a if A $TRS\ \ (X, \leq)$ is continuous poset in the sence of H. Zhang (resp., continuous poset in the sence of J. Nino-Salcedo, continuous domain in the sence of R. Hekmann), then the way relation below ' $\ll'$is continuous information system, Finally take about when the way relation below ' $\ll'$ is interpolative and a subset B of a poset $(X, \leq)$ is called a base for X.

Definition 2.1. A $TRS\ \ (X, \leq)$ is continuous TRS iff $\forall\, x \in X$, the following conditions are satisfied:

(1) $\bigvee(\{x\}) \neq \phi$
(2) $\Downarrow\ x$ be a directed subset of X , and
(3) $x \in\downarrow (\bigvee(\bigcup\{\bigvee\ (\Downarrow\ a) :\ a\ \in \Downarrow\ x\}))$.

Theorem 2.1. Let $(X, \leq)$ be a TRS. let $x, y, z \in X$. Then :

(1) If $x \leq y$ and $y \ll z$, then $x \ll z$;
(2) If $x \ll y$ and $y \leq z$, then $x \ll z$;
(3) If $\bigvee(\{y\}) \neq \phi$. and $x \ll y$, then $x \leq y$;
(4) If $\bigvee(\{y\}) \neq \phi$. or $\bigvee(\{z\}) \neq \phi$, $x \ll y$ and $y \ll z$, then $x \ll z$.

Proof (1) Let D be a directed subset of X s.t. $z \in\downarrow \bigvee(D)$. Then $\exists\ d \in D\ \ s.t.\ y \leq d$.Then $x \leq d$ and hence $x \ll z$.

(2) Let D be a directed subset of X s.t. $z \in\downarrow \bigvee(D)$. Then $\exists\ k \in \bigvee(D)\ \ s.t.\ z \leq k$.Thus $y \leq k$ and so $y \in\downarrow \bigvee(D)$. Therefore $\exists\, l \in D\ \ s.t.\ x \leq l$. hence $x \ll z$.

(3) Let $D = \{y\}$ and assume that $x \ll y$. Then $\exists\ d \in D\ \ s.t.\ x \leq d$ but $y = d$. Thus $x \leq y$;

(4) The proof follow directly From (1) and (3) above.

Lemma 2.1 Let $(X,\ \leq)$ be a TRS, let $\Downarrow\ \ x$ be a directed subset of $\forall\ x \in X$. Then $\forall\, z \in X,\ \ D = \bigcup\{\Downarrow\ a : a \in \Downarrow\ z\}$ is called directed subset.

Proof Let $\lambda, \mu \in D$ s.t., $\lambda \neq \mu$. Then $a_1,\ a_2 \in \Downarrow\ z$ s.t., $\lambda \in \Downarrow\ a_1$ and $\mu \in \Downarrow\ a_2$. If $a_1 = a_2$, the result hlolds. the otherwise let $a \in ub(\{a_1,\ a_2\}) \cap \Downarrow\ z$ so that Theorem 2.1 (2). $\lambda, \mu \in \Downarrow a$. Since $\Downarrow a$ is called directed subset, then $\exists \rho \in ub(\{\lambda, \mu\}) \cap \Downarrow\ a \subseteq ub(\{\lambda, \mu\}) \cap\ D$. Thus D is called directed subset.

Lemma 2.2. Let $(X, \leq)$ be a TRS, let $\forall\, x \in X$. Then $\forall\, x \in X,\ ub(\bigcup\{\Downarrow\ a : a \in \Downarrow x) = ub(\bigcup\{\bigvee(\Downarrow\ a) : a \in \Downarrow x)$. Thus $\bigvee(\bigcup\{(\Downarrow\ a) : a \in \Downarrow x) = \bigvee(\bigcup\{\bigvee(\Downarrow\ a) : a \in \Downarrow x)$.

Proof.

$$\begin{array}{lcl} \lambda \in ub(\bigcup\{(\Downarrow\ a) : a \in \Downarrow x) & \Leftrightarrow & \forall \mu \in \bigcup\{(\Downarrow\ a) : a \in \Downarrow x\}, \\ \lambda \geq \mu & \Leftrightarrow & \forall \mu \in (\Downarrow\ a),\ a \in (\Downarrow\ x), \\ \lambda \geq \mu & \Leftrightarrow & \forall \rho \in \bigvee(\Downarrow\ a),\ a \in (\Downarrow\ x), \\ \lambda \geq \rho & \Leftrightarrow & \lambda \in ub(\bigcup \bigvee(\Downarrow\ a) : a \in \Downarrow x). \end{array}$$

The following theorem is a generalization of the corresponding result in Proposition 1.1 and Proposition 1.2 (Proposition 6.4.7 [5], Lemma 2.15(i)[16] and Proposition 1.17 [18]) .

Theorem 2.2. If $(X, \leq)$ is continuous TRS, then the way relation below $`\ll'$is interpolative, i.e., $\forall\, x, z \in X$, with $x \leq z$, implies that $\exists\ y \in X\ \ s.t.\ x \ll y \ll z$.

Proof. From Lemma 2.1 and 2.2, $z \in \downarrow (\bigvee(\bigcup\{\Downarrow\ y : y \in \Downarrow z))$ and $\bigcup\{\Downarrow\ y : y \in \Downarrow z)$ is directed. Then $\exists\, d \in \Downarrow y$ for some $y \in \Downarrow z$ s.t., $x \leq d$. From Theorem 2.1.(1), we have $x \ll y$. Hence $x \ll y \ll z$.

Theorem 2.3. If $(X, \leq)$ is continuous TRS, then the way relation below $`\ll'$is continuous information system.

Proof. From Theorem 2.1.(4). and Theorem 2.2.

Corollary 2.1 If $(X, \leq)$ is continuous poset in the sence of H. Zhang (resp., continuous poset in the sence of J. Nino-Salcedo, continuous domain in the sence of R. Hekmann), then the way relation below $`\ll'$is continuous information system.

Lemma 2.3. For any $(X, \leq)$ TRS. if $\forall\, x \in X,\ \bigvee(\{x\}) \neq \phi$, and the way relation below $`\ll'$ is interpolative, then $\forall\, x \in X,\ \Downarrow\ x = \bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$.

Proof Let $z \in \bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$. Then $\exists\, a \in \Downarrow\ x$ s.t., $z \ll a$. From From Theorem 2.1.(4), $z \ll x$, i.e., $z \in \Downarrow\ x$. Second, let $z \in \Downarrow\ x$. Then $z \ll x$. $Since$ $`\ll'$ is interpolative, $\exists\, a \in X$ s.t., $z \ll a \ll x$, i.e., $z \in \bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$.

Applying Lemma 2.3., Theorem 2.2. and Theorem 2.3. one have the following theorem.

Theorem 2.4. A TRS $(X, \leq)$ is continuous iff the following conditions are satisfied:
(1) The way relation below $`\ll'$ is interpolative
(2) $\forall\, x \in X,\ \bigvee(\{x\}) \neq \phi$;
(3) $\forall\, x \in X,\ \Downarrow\ x$ is directed ;

(4) $\forall\, x \in X,\ \ x \in \downarrow \bigvee(\Downarrow\ x)$.

Proof First of all, we note that conditions (2) and (3) above are common.

$\Longrightarrow$: From Theorem 2.2., ‘ $\ll'$ is interpolative so that conditions (1) above is satisfied. From Lemma 2.3., conditions (4) above is satisfied.

$\Longleftarrow$: From Lemma 2.3., one can have that conditions (3) in Definition 2.1

Corollary 2.2 A TRS $(X, \leq)$ is continuous domain in the sence of R. Hekmann (resp., continuous poset in the sence of H. Zhang), if the following conditions are satisfied:

(1) The way relation below ‘ $\ll'$ is interpolative

(2) $\forall\, x \in X,\ \Downarrow\ x$ is directed ;

(3) $\forall\, x \in X,\ \ x \in \downarrow \bigvee(\Downarrow\ x)$.

Proposition 2.1. Let $\lambda, \mu \in X$. If μ directed subset and cofinal in λ, then λ is directed subset and $\bigvee(\lambda) = \bigvee(\mu)$.

Theorem 2.5.A TRS $(X, \leq)$ is continuous if the following conditions are satisfied:

(1) The way relation below ‘ $\ll'$ is interpolative

(2) $\forall\, x \in X,\ \bigvee(\{x\}) \neq \phi$;

(3) $\forall\, x \in X,\ \exists$ a directed subset D of $\Downarrow\ x$ s.t., $x \in \downarrow\ (\bigvee(D))$.

Proof $\Longrightarrow$: From Theorem 2.4., conditions (1) and (2) are satisfied. The condition (3) is satisfied if we put $D = \Downarrow\ x$.

$\Longleftarrow$: Now conditions (1) and (2) are satisfied in Theorem 2.4. are given above. We need to prove that D is cofinal in $\Downarrow\ x$. First $D \subseteq \Downarrow\ x$ and D is directed . Second, let $y \in \Downarrow\ x.\ Since\ \ x \in \downarrow\ (\bigvee(D))$, then $\exists\, d \in D$ s.t., $y \leq d$. So, $y \in \downarrow\ (D)$. Then from Proposition 2.1, $\Downarrow\ x$ is directed and $\bigvee(\Downarrow\ x) = \bigvee(D)$. Hence conditions (3) and (4) in Theorem 2.4**.** are satisfied.

Corollary 2.3 A TRS $(X, \leq)$ is continuous domain in the sence of R. Hekmann (resp., continuous poset in the sence of H. Zhang), if the following conditions are satisfied:

(1) The way relation below ‘ $\ll'$ is interpolative

(2) $\forall\, x \in X,\ \exists$ a directed subset D of $\Downarrow\ x$ s.t., $x = \downarrow\ (\bigvee(D))$.

The following theorem is a generalization of the corresponding result in Proposition 1.3 (Proposition 6.7.2 [5] and Lemma 2.14[15]).

Theorem 2.6.A TRS $(X, \leq)$ is continuous if the following conditions are satisfied:

(1) $\forall\, x \in X,\ \bigvee(\{x\}) \neq \phi$;

(2) $\forall\, x \in X,\ \exists$ a directed subset D of $\bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$. s.t., $x \in \downarrow\ (\bigvee(D))$.

Proof $\Longrightarrow$: From Theorem 2.5., and Lemma 2.3. on can have that $\Downarrow\ x = \bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$ so that from conditions (3) in Theorem 2.5 on have directly conditions (2) in above.

$\Longleftarrow$: $Since\ \ D$ is cofinal in $\bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$ (Indeed, $D \subseteq \bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$ let $z \in \bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$. $z \ll a$ from some $a \in \Downarrow\ x$ so that from From Theorem 2.1.(4).$z \ll x$ Since D is directed and $x \in \downarrow\ (\bigvee(D))$, then $\exists\ d \in D$ s.t., $z \leq d$. i.e.,, $z \in \downarrow\ (D)$.), then from Proposition 2.1, $\bigcup\{\Downarrow\ a : a \in \Downarrow\ x\}$ is directed and $\bigvee(D) = \bigvee(\bigcup\{\Downarrow\ a : a \in \Downarrow\ x\})$. Hence conditions (3) in Theorem 2.5 is satisfied. Also, one can prove that ' $\ll'$ is interpolative (Indeed, $x \ll z$ and from conditions (2) above $z \in \downarrow (\bigvee(\bigcup\{\Downarrow a : a \in \Downarrow x\}))$. Thus $\exists\, d \in \bigcup\{\Downarrow a : a \in \Downarrow x\}$ s.t., $x \le d \ll a \ll z$. .So, From From Theorem 2.1.(1), $x \ll a$. Then $x \ll a \ll z$.). Then conditions (2) in Theorem 2.5 is satisfied. Hence A $TRS\ (X, \le)$ is continuous.

Corollary 2.4 A $TRS\ (X, \le)$ is continuous poset in the sence of in the sence of H. Zhang iff $\forall\, x \in X,\ \exists$ directed subset D of $\bigcup\{\Downarrow a : a \in \Downarrow x\}$ s.t., $x = \bigvee(D)$.

Definition 2.2. A subset B of X is called a base for X if the following conditions are satisfied:

1) $\forall\, x \in X,\ \bigvee(\{x\}) \ne \phi$;
(2) $\forall\, x \in X,\ \exists$ a directed subset D of B s.t., $D \subseteq \bigcup\{\Downarrow a : a \in \Downarrow x\}$. and $x \in \downarrow (\bigvee(D))$

Theorem 2.7.A $TRS\ (X, \le)$ is continuous iff it has a base.

Proof $\Longrightarrow$: From Theorem 2.6, $B = \bigcup_{x\in X}(\bigcup\{\Downarrow a : a \in \Downarrow x\}$;
$\Longleftarrow$: Conditions (2) in Theorem 2.6 is satisfied directly from the definition of the base of a A $TRS\ (X, \le)$.

Corollary 2.5 Let $TRS\ (X, \le)$ be a domain. X is a continuous domain in the sence of in the sence of R. Hekmann (continuous poset in the sence of J. Nino-Salcedo,)

Definition 2.3. A subset B of a poset $(X, \le)$ is called a base for X iff $\forall\, x \in X,\ \exists$ a directed subset D of $\bigcup\{\Downarrow a : a \in \Downarrow x\}$. s.t., $\bigvee(D)$ exists and $x = \bigvee(D)$.

Corollary 2.6 A poset $(X, \le)$ is a continuous domain in the sence of in the sence of H. Zhang iff it has a base (as in Definition 2.3.).

3. A characterization of continuous directed complete posets and Algebraic TRS.

In this section we prove if $TRS \iff$ domain, i.e., a poset in which every directed subset has a supremum, then a continuous domain in the sence of in the sence of R. Hekmann and continuous TRS are identical.

Theorem 3.1. Let $(X, \le)$ be a TRS. let $x, y, z \in X$. Then :

(1) If $x \le y$ and $y \ll z$, then $x \ll z$;
(2) If $x \ll y$ and $y \le z$, then $x \ll z$;
(3) If $\bigvee(\{y\}) \ne \phi$. and $x \ll y$, then $x \le y$;
(4) If $\bigvee(\{y\}) \ne \phi$. or $\bigvee(\{z\}) \ne \phi,\ x \ll y$ and $y \ll z$, then $x \ll z$.

Proof (1) Let D be a directed subset of X s.t. $z \in \downarrow \bigvee(D)$. Then $\exists\, d \in D\ \ s.t.\ y \le d$.Then $x \le d$ and hence $x \ll z$.
(2) Let D be a directed subset of X s.t. $z \in \downarrow \bigvee(D)$. Then $\exists\, k \in \bigvee(D)\ \ s.t.\ z \le k$.Thus $y \le k$ and so $y \in \downarrow \bigvee(D)$. Therefore $\exists\, l \in D\ \ s.t.\ x \le l$. hence $x \ll z$.

(3) Let $D = \{y\}$ and assume that $x \ll y$. Then $\exists\, d \in D \;\; s.t.\; x \leq d$ but $y = d$. Thus $x \leq y$;

(4) The proof follow directly From (1) and (3) above.

Lemma 3.1 If $(X, \leq)$ is a continuous TRS and '$\leq$'is antisymmetric, then:

(1) The supremum of any set is unique whenever it exist; and
(2) $\forall\, x \in X,\; x = \bigvee((\Downarrow x))$.

Proof (1) Obvious;

(2) From Theorem 2.4(4), $x \in \downarrow\; (\bigvee(\Downarrow\; x))$. so that $x \leq \bigvee(\{\Downarrow x\})$. Conversely, $\forall\, \lambda \in (\Downarrow x),\; \lambda \ll x$. From Theorem 3.1(3), $\forall\, \lambda \in (\Downarrow x),\; \lambda \leq x$ so that $\bigvee((\Downarrow x)) \leq x$

Lemma 3.2 If $TRS\; (X, \leq)$ is a continuous and if '$\leq$'is reflexive, then a subset D of X is a directed subset iff $\forall\, x, y \in D,\; \exists\, z \in ub(\{x, y\}) \cap D$.

Proof $\Longrightarrow$:Let $\{x, y\}) \subseteq D$ s.t., $x = y$. $Since$ '$\leq$'is reflexive, then $x \in ub(\{x, y\}) \cap D$,
$\Longleftarrow$: Obvious.

Theorem 3.2. Let A $TRS\;\; (X, \leq)$ is a continuous domain in the sence of in the sence of R. Hekmann (continuous poset in the sence of J. Nino-Salcedo,) iff A $TRS\;\; (X, \leq)$ is a continuous domain with '$\leq$'is reflexive and antisymmetric.

Proof From Lemma 3.1, Lemma 3.2 and Definition 2.1**.** the proof is obtained.

Definition 3.1.A $TRS\;\; (X, \leq)$ is algebraic iff the following conditions are satisfied:
(1) $\forall\, x \in X,\; \downarrow_\circ x$ is directed subset of X , and
(2) $\forall\, x \in X,\;\; x \in \downarrow\; (\bigvee(\downarrow_\circ x))$

Theorem 3.3. If a $TRS\;\; (X, \leq)$ is algebraic, then '$\leq$'is interpolative.

Proof Let $x, z \in X$ s.t., $x \ll z$. Then condition (2) above implies that from fact $z \in \downarrow$ $(\bigvee(\downarrow_\circ z)),\; \exists\;\; l \in \downarrow_\circ z$ s.t., $x \leq l$. Since $l \ll l$ and $x \leq l$, then from From Theorem 3.1(1), (2), $\exists\;\; l \in X$ s.t., $x \ll l \ll z$.

Theorem 3.4. If a $TRS\;\; (X, \leq)$ is algebraic, then $(X, \ll)$ is continuous information system.

Proof Applying Theorem 3.3**,** it rests to prove that '$\ll$'is transitive. Let $x, y, z \in X$ s.t., $x \ll y$ and $y \ll z$. Since $y \ll z$ and $z \in \downarrow\; (\bigvee(\downarrow_\circ z))$, then $\exists\;\; l \in \downarrow_\circ z$ s.t., $y \ll l$. Since $l \in\; (\downarrow_\circ z)$ then $l \leq z$ so that $y \leq z$. Now, $x \ll y$ and $y \leq z$, then from From Theorem 3.1(2), we have that $x \ll z$.

Remark3.1. In from Theorem 3.1(4), we need the property that $\forall\, x \in X,\; \bigvee(\{x\}) \neq \phi$, to prove that '$\ll$'is transitive for a continuous.TRS but as we illustrate in the proof of Theorem 3.4 we do not need this property to prove that '$\ll$'is transitive for an a TRS algebraic.

Corollary 3.1 If a TRS $(X, \leq)$ is algebraic domain in the sence of in the sence of R. Hekmann (resp., algebraic domain in the sence of J. Nino-Salcedo,), then $(X, \ll)$ is continuous information system.

The following theorem is a generalization of the corresponding result in Proposition 1.4 (Proposition 6.2.2 [5]).

Theorem 3.4. A TRS $(X, \leq)$ is algebraic, iff $\forall\ x \in X$, $\exists$ a directed subset D of $\downarrow_\circ x$ s.t., $x \in \downarrow (\bigvee(D))$.

Proof $\Longleftarrow$:$\forall\ x \in X$, take $D = \downarrow_\circ x$. Then the result holds.

$\Longrightarrow$: Let $x \in X$. We need to prove that D is cofinal in $\downarrow_\circ x$. Now , $D \subseteq \downarrow_\circ x$. Let $z \in \downarrow_\circ x$. So that from Theorem 3.1(2) $z \ll x$. Since $x \in \downarrow (\bigvee(D))$, $\exists\ d \in D$ s.t., $z \leq d$. Thus $z \in \downarrow D$. Hence from Proposition 2.1the result holds..

Theorem 3.5. A TRS $(X, \leq)$ is algebraic, then

(1) $x \leq y \Rightarrow \downarrow_\circ x \subseteq \downarrow_\circ y$

(2) $\downarrow_\circ x \subseteq \downarrow_\circ y \Rightarrow \forall\ z \in (\bigvee(\downarrow_\circ y)),\ x \leq z.$

Proof (1) Let $x \leq y$ and let $z \in \downarrow_\circ x$. Then $z \ll z$ and $z \leq x$ so that $z \ll z$ and $z \leq y$ so that $z \in \downarrow_\circ y$

(2) Suppose $\downarrow_\circ x \subseteq \downarrow_\circ y$. Then $\forall\ l \in (\bigvee(\downarrow_\circ x))$, $\forall\ z \in (\bigvee(\downarrow_\circ y)),\ l \leq z. Since x \in (\bigvee(\downarrow_\circ x))$, then $\exists\ l_\circ \in (\bigvee(\downarrow_\circ x))$ s.t., $x \in l_\circ$. So, $\forall\ z \in (\bigvee(\downarrow_\circ y)),\ x \leq z.$

Corollary 3.2 If a TRS $(X, \leq)$ is algebraic domain and $` \leq'$ is antisymmetric, then $x \leq y$ iff $\downarrow_\circ x \subseteq \downarrow_\circ y.$

Proof $\Longleftarrow$: Since $` \leq'$ is antisymmetric and X is domain , then $\bigvee(\downarrow_\circ y)$ exists and unique, say equal z. Since X is algebraic, then $z = y$. Thus from Theorem 3.5 (2), the results holds.

$\Longrightarrow$: It follows fr (1) in Theorem 3.5.

The corollary 3.2 is a generalization of the corresponding result in Proposition 6.2.3 [5] since the reflexivity of $' \leq'$ is not assumed.

Theorem 3.6. If TRS $(X, \leq)$ is algebraic, then:

$\forall\ x \in X$, $\exists$ a directed subset D of $\Downarrow x$ s.t., $x \in \downarrow (\bigvee(D))$.

Proof If we prove that $\forall x \in X$, $\downarrow_\circ x \subseteq \Downarrow x$. The result holds. So let $z \in \downarrow_\circ x$. Then $z \ll z$ and $z \leq x$. Then from Theorem 3.1 (2), $z \in \Downarrow x$.

The Theorem 3.6 is a generalization of the corresponding result in Theorem 1.1 (Proposition 6.7.3 [5]).

Corollary 3.3 If a TRS $(X, \leq)$ is algebraic and $\forall x \in X$, $\bigvee(\{x\}) \neq \phi$, then $(X, \leq)$ is continuous

Definition 3.2. In TRS $(X, \leq)$, $\forall\ x \in X$, $x \ll x$, then x is said to be isolated and we can write $x \in \mathbf{K}(X)$.

Theorem 3.7. For TRS $(X, \leq)$ assume that the following conditions are satisfied:

(1) $\forall\, x \in X,\ \Downarrow x$ is directed subset ;
(2) $\forall\, x \in X,\ x \in \downarrow (\bigvee(\Downarrow x))$;
(3) $\forall\, x \in X,\ x \ll y,\ \exists\, k \in \mathbf{K}(X)$ s.t., $x \leq k \leq y$. Then $(X, \leq)$ is algebraic.

Proof We need to prove that $\forall\ x \in X,\ \downarrow_\circ x$ is cofinal in $\Downarrow x$. Let $z \in \downarrow_\circ x$. Then $z \ll z$ and $z \leq x$. from Theorem 3.1 (2) $z \ll x$, i.e., $z \in \Downarrow x$. Hence $\downarrow_\circ x \subseteq \Downarrow x$. Let $l \in \Downarrow x$. So $l \leq x$. Thus $\exists\ k \in \mathbf{K}(X)$ with $l \leq k \leq x$. from Theorem 3.1 (2), one can deduce that $k \in \downarrow_\circ x$. Hence $l \in \downarrow (\downarrow_\circ x)$. Then Proposition 2.1, $\downarrow_\circ x$ is directed and $\bigvee(\Downarrow x) = \bigvee(\downarrow_\circ x)$.

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